Multi-Sensor Kalman Filtering With Intermittent Measurements

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***Abstract*—In this paper, we extend the stability theory on Kalman filtering with intermittent measurements from the scenario of one single sensor to the one of multiple sensors. Consider that a group of sensors take measurement of the states of a process and then send the data to a remote estimator. The estimator receives the measurements intermittently, which may be caused by the fact that the channels have packet dropouts or that the sensors schedule the data transmission stochastically. Based on the received measurements, the estimator computes the estimates of the process states by multi-sensor Kalman filtering. Because of the intermittent measurements, the estimator may be unstable. This stability issue is mainly investigated in this paper. A notion of transmission capacity, which is related to the communication rates of sensors, is proposed. It is shown that the expected estimation error covariance diverges for all feasible communication rates collections of the sensors when the transmission capacity is below a certain value; meanwhile, when the transmission capacity is above another certain value, there exists a feasible communication rates collection such that the expected estimation error covariance is bounded.**

***Index Terms*—Intermittent measurements, Kalman filtering, modified algebraic Riccati equation (MARE), multi-sensor.**

I. Introduction

In networked control systems, intermittent communication is frequently involved, which may result from unreliable channels or the stochastic manner of data transmission. The intermittent communication may influence the performance of the components in a networked control system; for example, if the estimator receives measurements from the sensors intermittently, it may be unstable.

The stability issue of remote state estimation caused by intermittent measurements has been investigated by many researchers. Sinopoli *et al.* [1] studied the stability of Kalman filtering with intermittent measurements from one single sensor. They showed the existence of a critical arrival rate below which the estimation error may diverge; they also provided lower and upper bounds of the critical arrival rate. This result was further developed by [2] and [3]. Liu and Goldsmith [4] considered a system with two sensors and provided one form of the lower bound of the expected estimation error covariance. Rong [5] extended the result of the lower bound in [4] to the system with multiple sensors, and proposed an explicit form of the upper bound of the critical arrival rate for the system with a single sensor, which is an improvement of [1].

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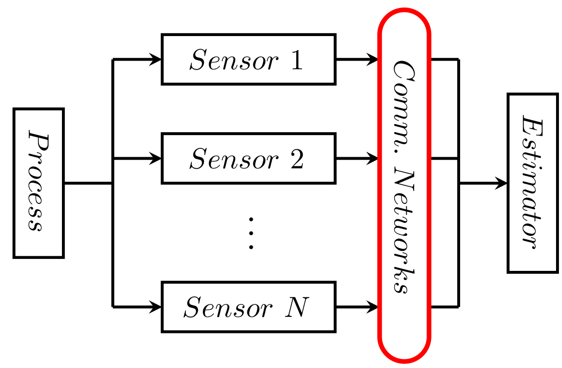

Fig. 1. Structure of the centralized sensor network.

In this paper, the stability of centralized state estimation with intermittent measurements from multiple sensors is studied. A group of sensors observe the states of a process and then send the measurements to a remote estimator; the estimator receives intermittent measurements and computes the state estimates (see Fig. 1). The estimator applies multi-sensor Kalman filtering to compute the state estimates.

In the existing literature, the stability problem of Kalman filtering caused by intermittent measurements from one single sensor is well studied in [1] and [5]. Meanwhile, the study for multiple sensors has limited results and is still open. In the study for multiple sensors, the search for the lower and upper bounds of the expected estimation error covariance plays a crucial role, which, however, has few results. In the existing literature, [4] and [5] only give the lower bound. The upper bound proposed in [6] only works in the particular case where just one sensor is chosen at each time.

It is found that the optimal control via fading channels is the dual problem of the optimal estimation with intermittent measurements; hence, the research on the former problem contributes to the study in this paper. Elia [7] studied the mean square stability of a system whose control inputs are sent via an erasure channel. Xiao *et al.* [8] considered the feedback control for a system with fading channels and studied the stabilization problem. Zheng *et al.* [9] considered the optimization problem of feedback control via fading channels; they proposed an associated novel modified algebraic Riccati equation (MARE) and provided a necessary and sufficient condition for the unique existence of a positive semidefinite fixed point of this MARE.

The main contribution of this paper is that it develops the stability theory of Kalman filtering caused by intermittent measurements in the scenario of multiple sensors. The novelty and contributions are summarized as follows.

1) A novel upper bound of the expected estimation error covariance is provided (Theorem 3).
2) The conditions on the divergence of the lower bound and the convergence of the upper bound are given, respectively (Theorems 1 and 4).
3) A notion of *transmission capacity* is proposed to relate the communication rates for multiple channels. It is shown that when the capacity is below a certain value, the expected estimation error covariance diverges for all feasible communication rates collection of the sensors (Theorem 2). Meanwhile, when the capacity is above some value, there exists a feasible communication rates collection such that the expected estimation error covariance is bounded (Theorem 6).

The remainder of the paper is organized as follows. Section II introduces the mathematical setup. Section III is the main analysis. Examples are in Section IV, and Section V concludes this paper.

Notation: $\mathbb{Z}_+$ is the set of nonnegative integers. $k \in \mathbb{Z}_+$ is the time index. $\mathbb{R}$ is the set of real numbers. $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{S}_+^n$ (and $\mathbb{S}_{++}^n$) is the set of $n \times n$ positive semidefinite matrices (and positive definite matrices). When $X \in \mathbb{S}_+^n$ (and $\mathbb{S}_{++}^n$), it is written as $X \succeq 0$ (and $X \succ 0$). $X \succeq Y$ if $X - Y \in \mathbb{S}_+^n$. $\boldsymbol{E}[\cdot]$ or $\boldsymbol{E}X$ is the expectation of a random variable; $\boldsymbol{E}[\cdot|\cdot]$ is the conditional expectation. $\mathrm{Tr}(\cdot)$ is the trace of a matrix. For functions f, f_1, and f_2 with appropriate domains, $f_1 f_2(x)$ stands for the function composition $f_1\big(f_2(x)\big)$, and $f^n(x) \triangleq f\big(f^{n-1}(x)\big)$ with $f^0(x) \triangleq x$. δ function is defined as $\delta_{ij} = 1$ if $i = j$; otherwise $\delta_{ij} = 0$.

II. Problem Setup

The system studied in this paper is shown in Fig. 1. In this section, the models of the system components are provided and the stability problem is proposed.

A. Process and Sensors

Consider a single process whose states are observed by a total number of N sensors

$$x_{k+1} = Ax_k + w_k \tag{1}$$

$$y_k^i = C_i x_k + v_k^i, \quad i = 1, 2, \ldots, N. \tag{2}$$

In the equations above, $x_k \in \mathbb{R}^n$ is the process state at time k and $y_k^i \in \mathbb{R}^{p_i}$ is the measurement taken by sensor i, where n and p_i are the corresponding dimensions. The sequences $\{w_k\}$ and $\{v_k^i\}$ are zero-mean white Gaussian noise processes with $\boldsymbol{E}[w_k w_j'] = \delta_{kj} Q$ ($Q \succeq 0$) and $\boldsymbol{E}[v_k^i (v_j^i)'] = \delta_{kj} R_i$ ($R_i \succ 0$), respectively. Meanwhile, $\{w_k\}$ and $\{v_k^i\}$ are mutually independent processes, i.e., $\boldsymbol{E}[w_k (v_j^i)'] = 0$, $\forall j, k$. The initial state x_0 is a Gaussian random vector with distribution $\mathcal{N}(0, \Pi)$ and is uncorrelated with w_k and v_k^i, $\forall k, i$. Assume that C_i has full row rank.

B. Intermittent Communication

The sensors send their measurements to a remote estimator via wireless communication channels, while the estimator receives the measurements intermittently. The intermittent measurements may result from varies scenarios: the sensors do the transmission constantly but the channels are unreliable, or the channels are reliable but the sensors do the transmission only at scheduled time instants, or so on. Eventually, these scenarios can be modeled as intermittent communication.

Define the *communication variable* γ_k^i as follows:

$$\gamma_k^i = \begin{cases} 1, & y_k^i \text{ is received} \\ 0, & y_k^i \text{ is not received.} \end{cases}$$

In this paper, we consider the scenario that $\{\gamma_k^i\}$ is a Bernoulli process with mean $\boldsymbol{E}[\gamma_k^i] = \lambda_i$, which is the communication rate of sensor i. Define the *communication rates collection* for the group of sensors as $\boldsymbol{\lambda} \triangleq \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$.

C. Estimation Process

The remote estimator uses multi-sensor Kalman filtering to calculate the minimum mean-squared error estimate of the state x_k based on the received measurements. Define

$$\tilde{y}_k \triangleq \Big(\gamma_k^1 (y_k^1)', \gamma_k^2 (y_k^2)', \ldots, \gamma_k^N (y_k^N)'\Big)' \tag{3}$$

and let $\tilde{\boldsymbol{Y}}_k \triangleq \{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k\}$. Define $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$

$$\hat{x}_{k|k-1} \triangleq \boldsymbol{E}[x_k | \tilde{\boldsymbol{Y}}_{k-1}]$$

$$\hat{x}_{k|k} \triangleq \boldsymbol{E}[x_k | \tilde{\boldsymbol{Y}}_k].$$

Let $P_{k|k-1}$ and $P_{k|k}$ be the estimation error covariance matrices associated with $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$, respectively

$$P_{k|k-1} \triangleq \boldsymbol{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})' | \tilde{\boldsymbol{Y}}_{k-1}]$$

$$P_{k|k} \triangleq \boldsymbol{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})' | \tilde{\boldsymbol{Y}}_k].$$

At time k, the estimator first calculates $\hat{x}_{k|k-1}$ and $P_{k|k-1}$

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} \tag{4}$$

$$P_{k|k-1} = AP_{k-1|k-1}A' + Q \tag{5}$$

where the recursion starts from $\hat{x}_{0|0} = 0$ and $P_{0|0} = \Pi$. After receiving available measurements of time k, the estimator fuses them to obtain $\tilde{y}_k$ and computes the following quantities:

$$\tilde{C}_k \triangleq (\gamma_k^1 C_1', \gamma_k^2 C_2', \ldots, \gamma_k^N C_N')' \tag{6}$$

$$\tilde{R}_k \triangleq \mathrm{diag}\{\gamma_k^1 R_1, \gamma_k^2 R_2, \ldots, \gamma_k^N R_N\}. \tag{7}$$

Then, it computes $\hat{x}_{k|k}$ and $P_{k|k}$ as follows:

$$P_{k|k} = \left(P_{k|k-1}^{-1} + \sum_{i=1}^{N} \gamma_k^i C_i' R_i^{-1} C_i\right)^{-1} \tag{8}$$

$$K_k = P_{k|k} \tilde{C}_k' \tilde{R}_k^{\dagger} \tag{9}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (\tilde{y}_k - \tilde{C}_k \hat{x}_{k|k-1}) \tag{10}$$

where $\dagger$ represents the Moore–Penrose pseudoinverse.

D. Stability Problem

When $\lambda_1, \lambda_2, \ldots, \lambda_N$ are all close to zero, the estimator will receive few measurements and the estimation error is likely to be unbounded. Namely, the communication rates influence the stability of the estimator. The main objective of this paper is to find the relationship between $\boldsymbol{\lambda}$ and the performance of the estimation error.

III. Stability Analysis

In this section, we investigate the stability problem of the multi-sensor Kalman filtering with intermittent measurements.

A. Transmission Capacity

When involving a single sensor, the stability is related to the communication rate of this sensor alone. For multiple channels, we introduce the concept of *transmission capacity* to represent the overall effect of the communication rates of multiple sensors.

For sensor i, denote the variance of the random process $\{\gamma_k^i\}$ as σ_i^2. Define

$$q_i \triangleq \frac{\lambda_i^2}{\sigma_i^2} \tag{11}$$

and the *individual transmission capacity* as

$$\mathfrak{C}_i \triangleq \frac{1}{2}\ln(1+q_i).$$

Moreover, define the *overall transmission capacity* as

$$\mathfrak{C} \triangleq \sum_{i=1}^{N} \mathfrak{C}_i.$$

Remark 1: We have $q_i = \frac{\lambda_i}{1-\lambda_i}$. It can be calculated that

$$\mathfrak{C} = -\sum_{i=1}^{N} \frac{1}{2}\ln(1-\lambda_i)$$

which shows the connection between the transmission capacity and the multiple communication rates of the Bernoulli processes. Moreover, the transmission capacity also reflects the communication resources distributed or assigned in the corresponding channels.

B. Preliminaries

In this section, we present necessary preliminaries. We study the *a priori* error covariance $P_{k|k-1}$ for stability analysis. In the following part, P_k^- is used for short of $P_{k|k-1}$.

We introduce an alternative formulation for estimation error covariances. Define the *sensing precision matrix* [10] S_i for sensor i as

$$S_i \triangleq C_i' R_i^{-1} C_i.$$

Let

$$H_i = \sqrt{R_i^{-1}} C_i.$$

Then, S_i can be factorized as

$$S_i = H_i' H_i.$$

Define

$$H \triangleq (H_1', H_2', \ldots, H_N')'.$$

Let

$$\Gamma_k \triangleq \operatorname{diag}\{\gamma_k^1 I_{p_1}, \gamma_k^2 I_{p_2}, \ldots, \gamma_k^N I_{p_N}\}$$

where I_{p_i} is the identity matrix with order p_i, i.e., the order of y_k^i. The recursive update equation of P_k^- is as follows:

$$P_{k+1}^- = A\Big((P_k^-)^{-1} + H'\Gamma_k H\Big)^{-1} A' + Q. \tag{12}$$

Since P_k^- depends on Γ_k and is, thus, stochastic, only statistical properties can be deduced. Alternatively, we study $\boldsymbol{E}[P_k^-]$ to eliminate the uncertainty caused by Γ_k.

In [1], Sinopoli *et al.* studied the one single sensor case. They proved the existence of a critical arrival rate below which the estimation error may diverge. They also proposed lower and upper bounds of the estimation error covariance, and further based on them they provided lower and upper bounds of the critical arrival rate.

The analysis for multiple sensors follows a parallel way. In Section III-C, we study the lower bound of $\boldsymbol{E}[P_k^-]$. We first present a lower bound (Proposition 1). Then, we give a necessary and sufficient condition on the convergence and divergence of this lower bound (Theorem 1). Furthermore, we present a critical overall transmission capacity $\underline{\mathfrak{C}}$: when $\mathfrak{C} < \underline{\mathfrak{C}}$, the lower bound diverges for any feasible communication rates collection $\boldsymbol{\lambda}$, which leads to that $\liminf_{k\to\infty} \boldsymbol{E}[P_k^-]$ diverges (Theorem 2). In Section III-D, we study the upper bound of $\boldsymbol{E}[P_k^-]$. We first give an upper bound (Theorem 3). We further present a necessary and sufficient condition on the convergence of the upper bound (Theorem 4), and a sufficient one on that in terms of transmission capacities (Theorem 5). Then, we give a critical overall transmission capacity $\overline{\mathfrak{C}}$: above $\overline{\mathfrak{C}}$, there exists a feasible communication rates collection $\boldsymbol{\lambda}$, which guarantees the convergence of the upper bound and, hence, that of $\limsup_{k\to\infty} \boldsymbol{E}[P_k^-]$ (Theorem 6). Section III-E presents the results for some special cases.

C. Lower Bound

The proposition below gives a lower bound of $\{\boldsymbol{E}[P_k^-]\}$.

Proposition 1: Define

$$m(X) \triangleq \prod_{i=1}^{N}(1-\lambda_i)AXA' + Q. \tag{13}$$

Let the sequence $\{G_k\}$ be constructed as follows: $G_1 = \mathbf{0}$ and $G_{k+1} = m(G_k)$. Then

$$G_k \preceq \boldsymbol{E}[P_k^-] \quad \forall k.$$

Proof: [4] provides the proof when $N = 2$ and the general case is stated by [5, Corollary 4.3.1]. ■

The next theorem gives a necessary and sufficient condition on the convergence and divergence of the lower bound $\{G_k\}$.

Theorem 1: If A is unstable and $(A, \sqrt{Q})$ is controllable, a necessary and sufficient condition for $\{G_k\}$ to converge is

$$\mathfrak{C} > \ln \rho(A) \tag{14}$$

where $\rho(A)$ denotes the spectrum radius of A. Furthermore, $\{G_k\}$ diverges if and only if

$$\mathfrak{C} \le \ln \rho(A).$$

Proof: Define

$$\tilde{A} \triangleq \left(\prod_{i=1}^{N}(1-\lambda_i)\right)^{\frac{1}{2}} A.$$

Then

$$m(X) = \tilde{A}X\tilde{A}' + Q.$$

Since $(A, \sqrt{Q})$ is controllable, $(\tilde{A}, \sqrt{Q})$ is also controllable. Then, $X = m(X)$ has a unique strictly positive definite solution if and only if $\rho(\tilde{A}) < 1$, which leads to

$$\left(\prod_{i=1}^{N}(1-\lambda_i)\right)^{\frac{1}{2}} \rho(A) < 1. \tag{15}$$

Equation (15) yields

$$\ln \rho(A) < -\sum_{i=1}^{N} \frac{1}{2} \ln(1-\lambda_i)$$

i.e.,

$$\mathfrak{C} > \ln \rho(A).$$

Assume that (14) holds. Then, $\tilde{A}$ is stable, and $X = m(X)$ has a unique positive definite solution $\overline{G}$, which is given by

$$\overline{G} = \sum_{k=0}^{\infty} \tilde{A}^k Q (\tilde{A}')^k.$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} G_k &= \tilde{A}^{k+1} G_1 (\tilde{A}')^{k+1} + \sum_{k=0}^{\infty} \tilde{A}^k Q (\tilde{A}')^k \\ &= \sum_{k=0}^{\infty} \tilde{A}^k Q (\tilde{A}')^k = \overline{G}.\end{aligned}$$

Therefore, it is proved that $\{G_k\}$ converges. On the other hand, if $\{G_k\}$ converges, since

$$\lim_{k\to\infty} G_k = \sum_{k=0}^{\infty} \tilde{A}^k Q (\tilde{A}')^k$$

the series on the right-hand side converges. We denote the series by $\overline{G}$. Notice that $\overline{G}$ is the solution to $X = m(X)$. Since $(\tilde{A}, \sqrt{Q})$ is controllable, according to the property of controllability, $\overline{G}$ has full rank. Hence, it is positive definite. The fact that $\overline{G} = m(\overline{G})$ and $\overline{G} \succ 0$ leads to that $\tilde{A}$ is stable, i.e., $\mathfrak{C} > \ln \rho(A)$. Consequently, it is proved that $\{G_k\}$ converges if and only if $\mathfrak{C} > \ln \rho(A)$.

Furthermore, $\mathfrak{C} \le \ln \rho(A)$ is equivalent to that $\{G_k\}$ does not converge. Notice that $\{G_k\}$ is monotonically increasing. Then, the case that $\{G_k\}$ does not converge is equivalent to that $\{G_k\}$ is unbounded. Hence, $\{G_k\}$ diverges if and only if $\mathfrak{C} \le \ln \rho(A)$. ∎

The following theorem gives a condition on the critical transmission capacity regarding the divergence of $\{\boldsymbol{E}[P_k^-]\}$.

Theorem 2: Assume that $(A, \sqrt{Q})$ is controllable. Let

$$\underline{\mathfrak{C}} \triangleq \ln \rho(A). \tag{16}$$

When the overall transmission capacity $\mathfrak{C} \le \underline{\mathfrak{C}}$, $\{\boldsymbol{E}[P_k^-]\}$ diverges for all initial values. Moreover, when $\mathfrak{C} > \underline{\mathfrak{C}}$, there exists a unique positive definite $\overline{G}$ satisfying $\overline{G} = m(\overline{G})$, and

$$\liminf_{k\to\infty} \boldsymbol{E}[P_k^-] \succeq \overline{G}. \tag{17}$$

Proof: From Proposition 1, we have $G_k \preceq \boldsymbol{E}[P_k^-], \forall k$. When $\mathfrak{C} \le \underline{\mathfrak{C}}$, $\lim_{k\to\infty} G_k = \infty$ according to Theorem 1. Then

$$\lim_{k\to\infty} \boldsymbol{E}[P_k^-] \succeq \lim_{k\to\infty} G_k = \infty$$

i.e., $\{\boldsymbol{E}[P_k^-]\}$ diverges. When $\mathfrak{C} > \underline{\mathfrak{C}}$, according to the proof of Theorem 1, $X = m(X)$ has a unique positive definite solution $\overline{G}$. Since $G_k \preceq \boldsymbol{E}[P_k^-]$, we have

$$\liminf_{k\to\infty} \boldsymbol{E}[P_k^-] \succeq \lim_{k\to\infty} G_k = \overline{G}.$$

∎

Remark 2: When A is stable, we have

$$\overline{G} = \begin{cases} Q, & \lambda_i \to 1, \ \exists i \\ \Sigma, & \lambda_i \to 0 \ \ \forall i \end{cases}$$

where Σ satisfies $\Sigma - A\Sigma A' = Q$.

Remark 3: It is worth to mention that (8) implies another lower bound for $\boldsymbol{E}[P_k^-]$. Define an operator $\bar{g} : \mathbb{S}_+^n \times \mathbb{S}_+^n \to \mathbb{S}_+^n$

$$\bar{g}(X; S) \triangleq A(X^{-1} + S)^{-1} A' + Q.$$

Construct a sequence $\{F_k\}$, where $F_1 = P_1^-$, and

$$F_{k+1} = \bar{g}\left(F_k; \sum_{i=1}^{N} \lambda_i S_i\right).$$

Then

$$F_k \preceq \boldsymbol{E}[P_k^-] \quad \forall k$$

which results from the convexity of $\bar{g}(X; S)$ in S [10]. This bound is shown by [11]. $\{F_k\}$ is a loose lower bound and always converges under the condition that $(A, \sqrt{Q})$ is controllable. Consequently, it is not able to provide a certain critical transmission capacity.

D. Upper Bound

We study the upper bound of $\boldsymbol{E}[P_k^-]$ in this part. Define the MARE associated with the multi-sensor Kalman filter with intermittent measurements as follows:

$$\begin{aligned}g_{\boldsymbol{\lambda}}(X) \triangleq\ & AXA' + Q \\ & - AXH'(W \odot (HXH' + I))^{-1} HXA'\end{aligned} \tag{18}$$

where $\odot$ is the Hadamard product denoting the elementwise matrix multiplication, $W = \mathbf{1}\mathbf{1}' + D_{\text{SNR}}^{-1}\mathcal{I}$, $\mathbf{1}$ is the column vector with all components equal to 1, and

$$D_{\text{SNR}} \triangleq \operatorname{diag}\{q_1 I_{p_1}, q_2 I_{p_2}, \ldots, q_N I_{p_N}\} \tag{19}$$

$$\mathcal{I} \triangleq \operatorname{diag}\{\mathbf{1}_{p_1}\mathbf{1}'_{p_1}, \mathbf{1}_{p_2}\mathbf{1}'_{p_2}, \ldots, \mathbf{1}_{p_N}\mathbf{1}'_{p_N}\}. \tag{20}$$

Since $q_i = \frac{\lambda_i}{1-\lambda_i}$, W is the function of λ_is. The subscript $\boldsymbol{\lambda}$ in $g_{\boldsymbol{\lambda}}(X)$ indicates that $g_{\boldsymbol{\lambda}}(X)$ depends on $\boldsymbol{\lambda}$.

Some properties of $g_{\boldsymbol{\lambda}}(X)$ are presented in Lemma 1.

Lemma 1: The following results hold:

1) if $Y \succeq X \succ 0$, then $g_{\boldsymbol{\lambda}}(Y) \succeq g_{\boldsymbol{\lambda}}(X)$;
2) $g_{\boldsymbol{\lambda}}(X)$ is concave with respect to X.

Proof: See the appendix. ∎

Next theorem presents an upper bound of $\{\boldsymbol{E}[P_k^-]\}$.

Theorem 3: Construct a sequence $\{V_k\}$ as follows: $V_1 = P_1^-$ and $V_{k+1} = g_{\boldsymbol{\lambda}}(V_k)$. Then

$$\boldsymbol{E}[P_k^-] \preceq V_k \quad \forall k.$$

Proof: Define an operator

$$\Phi(L, X, \Gamma) \triangleq (A - L\Gamma H)X(A - L\Gamma H)' + Q + L\Gamma I \Gamma L'.$$

From (12), we have

$$\begin{aligned}P_{k+1}^- &= A\Big((P_k^-)^{-1} + H'\Gamma_k I \Gamma_k H\Big)^{-1} A' + Q \\ &= AP_k^- A' + Q \\ &\quad - AP_k^- H'\Gamma_k(\Gamma_k H P_k^- H'\Gamma_k + I)^{-1}\Gamma_k H P_k^- A' \\ &= AP_k^- A' + Q \\ &\quad - AP_k^- H'\Gamma_k(\Gamma_k H P_k^- H'\Gamma_k + \Gamma_k)^{\dagger}\Gamma_k H P_k^- A'\end{aligned}$$

where the second equality follows by Matrix Inversion Lemma and the third one can be verified by simple calculation. This is the conventional expression for the *a priori* error covariance P_k^-. Then, P_{k+1}^- satisfies

$$P_{k+1}^- = \Phi(\bar{K}_k, P_k^-, \Gamma_k)$$

where $\bar{K}_k = AP_k^- H'\Gamma_k(\Gamma_k HP_k^- H'\Gamma_k + \Gamma_k)^\dagger$. According to the theory of Kalman filtering, $\bar{K}_k$ is the gain that minimizes $\Phi(L, P_k^-, \Gamma_k)$. Hence, we have

$$P_{k+1}^- = \min_L \Phi(L, P_k^-, \Gamma_k).$$

Define

$$\Lambda \triangleq \operatorname{diag}\{\lambda_1 I_{p_1}, \lambda_2 Ip_2, \ldots, \lambda_N I_{p_N}\} \tag{21}$$

and

$$L_k \triangleq AP_k^- H'(W \odot (HP_k^- H' + I))^{-1}\Lambda^{-1}.$$

We use mathematical induction to prove the argument. When $k = 0$, the argument holds. Assume when $k = l$, $\boldsymbol{E}P_l^- \preceq V_l$ holds. For time $k = l + 1$, we have

$$P_{l+1}^- \preceq \Phi(L_l, P_l^-, \Gamma_l).$$

Take expectation of both sides with respect to $\{\Gamma_k\}$

$$\begin{aligned}
&\boldsymbol{E}[P_{l+1}^-] \\
&\preceq \boldsymbol{E}[\Phi(L_l, P_l^-, \Gamma_l)] \\
&= \boldsymbol{E}\Big[(A - L_l\Gamma_l H)P_l^-(A - L_l\Gamma_l H)' + Q + L_l\Gamma_l I\Gamma_l L_l'\Big] \\
&= \boldsymbol{E}\Big[AP_l^- A' + Q + L_l\Gamma_l HP_l^- H'\Gamma_l L_l' + L_l\Gamma_l I\Gamma_l L_l' \\
&\quad - AP_l^- H'\Gamma_l L_l' - L_l\Gamma_l HP_l^- A'\Big].
\end{aligned}$$

Since P_l^- depends on $\Gamma_1, \Gamma_2, \ldots, \Gamma_{l-1}$, it is independent from Γ_l. Hence, we first take expectation of the right-hand side with respect to Γ_l. We have $\boldsymbol{E}[\Gamma_l] = \Lambda$ and $\boldsymbol{E}[\Gamma_l X\Gamma_l] = \Lambda\big(W \odot X\big)\Lambda$ for a constant matrix X. Then

$$\begin{aligned}
\boldsymbol{E}[P_{l+1}^-] \preceq\ &\boldsymbol{E}\Big[AP_l^- A' + Q + L_l\Lambda\big(W \odot (HP_l^- H' + I)\big)\Lambda L_l' \\
&- AP_l^- H'\Lambda L_l' - L_l\Lambda HP_l^- A'\Big].
\end{aligned}$$

The expectation on the right-hand side is with respect to $\Gamma_1, \Gamma_2, \ldots, \Gamma_{l-1}$. Take the expression of L_l into the right-hand side and it turns to be $\boldsymbol{E}[g_{\boldsymbol{\lambda}}(P_l^-)]$. Hence,

$$\boldsymbol{E}[P_{l+1}^-] \preceq \boldsymbol{E}[g_{\boldsymbol{\lambda}}(P_l^-)] \preceq g_{\boldsymbol{\lambda}}(\boldsymbol{E}[P_l^-]) \preceq g_{\boldsymbol{\lambda}}(V_l) = V_{l+1}$$

where the second and third inequalities follow from the concavity and the increasing monotonicity of $g_{\boldsymbol{\lambda}}(X)$, respectively. From mathematical induction, $\boldsymbol{E}[P_k^-] \preceq V_k$ holds for all $k \geq 0$. ∎

The theorem below provides a necessary and sufficient condition for the convergence of the upper bound $\{V_k\}$.

Theorem 4: A necessary and sufficient condition for $\{V_k\}$ to converge for an arbitrary initial value, namely $X = g_{\boldsymbol{\lambda}}(X)$ has one unique positive definite solution, is given as follows.

1) $\exists\, P \succ 0$, L, and $\boldsymbol{\lambda}$, such that

$$\begin{aligned}
P \succ\ &(A - L\Lambda H)P(A - L\Lambda H)' \\
&+ L\Big((\Sigma^2\mathcal{I}) \odot (HPH')\Big)L'
\end{aligned} \tag{22}$$

where Λ is defined in (21), and

$$\Sigma \triangleq \operatorname{diag}\{\sigma_1 I_{p_1}, \sigma_2 I_{p_2}, \ldots, \sigma_N I_{p_N}\}. \tag{23}$$

2) $[\,A - e^{j\omega}I\ \ B\,]$ has full row rank for all $\omega \in \mathbb{R}$, where $BB' = Q$.

Proof: According to [9], the MARE (18) has one unique positive semidefinite solution if and only if

1) for the stochastic system

$$e_{k+1} = (A - L\Gamma_k H)' e_k \tag{24}$$

there exists a static L such that $\lim_{k\to\infty} \boldsymbol{E}[e_k e_k'] = 0$;

2) the system

$$\begin{aligned}
x_{k+1} &= A' x_k \\
y_k &= B' x_k
\end{aligned} \tag{25}$$

has no unobservable eigenvalues on the unit circle.

For system (24), let $\Pi_k \triangleq \boldsymbol{E}[e_k e_k']$. We have

$$\begin{aligned}
\Pi_k =\ &(A - L\Lambda H)'\Pi_{k-1}(A - L\Lambda H) \\
&+ H'\Big((\Sigma^2\mathcal{I}) \odot (L'\Pi_{k-1}L)\Big)H.
\end{aligned}$$

Similar to the result in [12, ch. 9], $\lim_{k\to\infty} \Pi_k = 0$ is equivalent to that $\exists\, P \succ 0$, L, and $\boldsymbol{\lambda}$, such that (22) is satisfied.

It is simple to verify that condition 2) is equivalent to that system (25) has no unobservable eigenvalues on the unit circle. ∎

Remark 4: If the system model is generalized to that $\{w_k\}$ and $\{v_k\}$ are correlated with $\boldsymbol{E}[w_k v_k'] = S$, the corresponding MARE is

$$\begin{aligned}
g_{\boldsymbol{\lambda}}(X) \triangleq\ &AXA' + Q - (AXC' + S) \\
&\cdot \Big(W \odot (CXC' + R)\Big)^{-1}(CXA' + S')
\end{aligned}$$

which is studied in [9].

In Theorem 4, condition 2) is simple to check, while condition 1) is still not straightforward. In the following part, we relate the convergence of $\{V_k\}$ to the conditions on scheduling capacities.

Define the Mahler measure [13] of A

$$\mathcal{M}(A) \triangleq \prod_{i=1}^{n} \max\{1, |\lambda_i(A)|\}$$

where $\lambda_i(A)$ are the eigenvalues of A. Define the topological entropy [14] of A

$$\mathfrak{h}(A) = \ln \mathcal{M}(A).$$

Assume that the pair (H, A) is observable and H has row rank m. Then, one can choose from H_i, $i = 1, 2, \ldots, N$, a subset of rows, denoted as $h_{i1}, h_{i2}, \ldots, h_{it_i}$, $0 \leq t_i \leq p_i$, and let $\mathcal{H}_i = (h_{i1}', h_{i2}', \ldots, h_{it_i}')'$, $\mathcal{H} = (\mathcal{H}_1', \mathcal{H}_2', \ldots, \mathcal{H}_N')'$, such that $\mathcal{H}$ has m rows with full row rank and $(\mathcal{H}, A)$ is observable. According to Wonham decomposition [15], A and $\mathcal{H}$ can be transformed to $\tilde{A}$ and $\tilde{\mathcal{H}}$ by some similarity transformation $\mathfrak{T}$ with the following form:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & \ldots & 0 \\ \star & \tilde{A}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \ldots & \tilde{A}_N \end{bmatrix}, \tilde{\mathcal{H}} = \begin{bmatrix} \tilde{\mathcal{H}}_1 & 0 & \ldots & 0 \\ 0 & \tilde{\mathcal{H}}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{\mathcal{H}}_N \end{bmatrix} \tag{26}$$

in which

$$\tilde{A}_i = \begin{bmatrix} \tilde{A}_{i1} & 0 & \ldots & 0 \\ \star & \tilde{A}_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \ldots & \tilde{A}_{it_i} \end{bmatrix}, \tilde{\mathcal{H}}_i = \begin{bmatrix} \tilde{h}_{i1} & 0 & \ldots & 0 \\ 0 & \tilde{h}_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{h}_{it_i} \end{bmatrix}$$

where $\tilde{A}_{ij} \in \mathbb{R}^{n_{ij}\times n_{ij}}$, $\tilde{h}_{ij} \in \mathbb{R}^{n_{ij}\times 1}$, $\sum n_{ij} = n$, and pair $(\tilde{h}_{ij}, \tilde{A}_{ij})$ is observable. The following theorem provides sufficient conditions on the convergence of $\{V_k\}$ in terms of scheduling capacities.

Theorem 5: Assume that the pair (H, A) is observable and condition 2) in Theorem 4 is satisfied. If

$$\mathfrak{C} > \sum_{i=1}^{N} \max_{j}\{\mathfrak{h}(\tilde{A}_{ij})\} \tag{27}$$

where $\tilde{A}_{ij}$ is given in (26), there exists a communication rates collection $\boldsymbol{\lambda}$, such that $\{V_k\}$ converges for all initial values. In particular, if

$$\mathfrak{C} > \mathfrak{h}(A) \tag{28}$$

$\{V_k\}$ converges.

Proof: See the appendix. ■

The following theorem gives a condition on the critical overall capacity related to the convergence of $\{\boldsymbol{E}[P_k^-]\}$.

Theorem 6: Assume that condition 2) in Theorem 4 is satisfied. If $\mathfrak{C} > \overline{\mathfrak{C}}$, where

$$\overline{\mathfrak{C}} \triangleq \mathfrak{h}(A) \tag{29}$$

there exists a communication rates collection $\boldsymbol{\lambda}$ such that $\limsup_{k\to\infty} \boldsymbol{E}[P_k^-]$ converges for an arbitrary initial value. Moreover, let $\overline{V}$ satisfy $\overline{V} = g_{\boldsymbol{\lambda}}(\overline{V})$. Then, $\overline{V}$ exists and

$$\limsup_{k\to\infty} \boldsymbol{E}[P_k^-] \preceq \overline{V}. \tag{30}$$

Proof: The arguments are verified simply by combining the result of Theorems 3 and 5.

E. Special Cases

In this section, we present further results on some special cases. Theorem 5 only provides sufficient conditions on the convergence of $\{V_k\}$. For some special systems, they become necessary and sufficient conditions.

Theorem 7: Assume that A has no zero eigenvalues and the pair (H, A) is observable. If $\{V_k\}$ converges for an arbitrary initial value, then

$$\sum_{i=1}^{N} p_i \mathfrak{C}_i > \ln\det(A). \tag{31}$$

Proof: The proof is similar to the one of [8, Th. 1]. ■

Corollary 1: Assume that condition 2) in Theorem 4 is satisfied. When all the eigenvalues of A are unstable and the measurements of all sensors are scalar valued, there exists a communication rates collection $\boldsymbol{\lambda}$ such that $\{V_k\}$ converges if and only if

$$\mathfrak{C} > \mathfrak{h}(A).$$

Proof: Notice that when all the eigenvalues of A are unstable, $\det(A) = \mathcal{M}(A)$. Meanwhile, $p_i = 1, \forall i$. Hence, (31) coincides with (28). ■

When $N = 1$, the system is reduced to the one considered in [1]. Denote the mean of the packet arrival rate of the single channel by λ. The MARE (18) is reduced to

$$g_\lambda(X) = AXA' + Q - \lambda AXH'(HXH' + I)^{-1}HXA'. \tag{32}$$

Notice that [1] only gives a sufficient condition on the unique existence of a positive definite solution to (32) in terms of an LMI. Theorems 4 and 5 improve the results. Sinopoli *et al.* [1] prove the existence of a critical packet arrival rate λ_c upon which the expected estimated error covariance of the filter converges. It also gives the lower and upper bounds of λ_c, where the lower bound $\underline{\lambda}$ is stated by a closed form while the upper bound $\overline{\lambda}$ is provided by solving a linear matrix inequality (LMI). According to Theorem 5, the upper bound can be improved as a closed form, i.e., $\overline{\lambda} = 1 - \frac{1}{\mathcal{M}^2(A)}$. This is also obtained by [5].

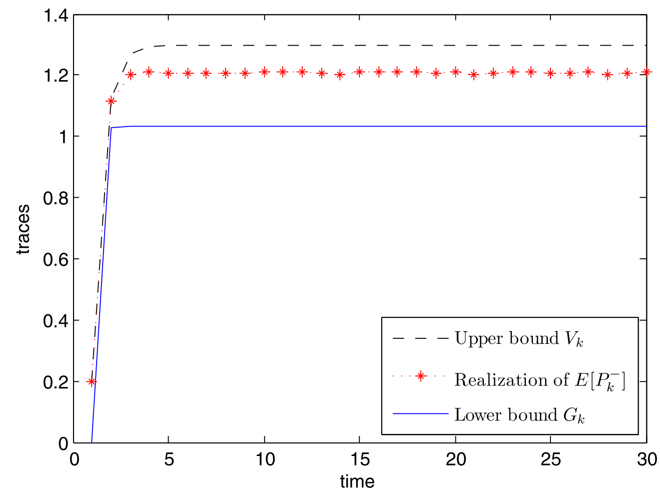

Fig. 2. Lower and upper bounds of $\boldsymbol{E}[P_k^-]$.

In some special scenarios, the two capacity bounds $\overline{\mathfrak{C}}$ and $\underline{\mathfrak{C}}$ meet.

Lemma 2: When A has only one unstable eigenvalue, we have $\underline{\mathfrak{C}} = \overline{\mathfrak{C}}$.

Proof: Since A has only one unstable eigenvalue, the Mahler measure coincides with the spectrum radius

$$\mathcal{M}(A) = \rho(A).$$

Thus, the argument follows. ■

Corollary 2: For a first-order system, when A is unstable, we have $\underline{\mathfrak{C}} = \overline{\mathfrak{C}}$.

IV. Examples

In this section, we provide an example to demonstrate the lower and upper bounds of $\boldsymbol{E}[P_k^-]$.

Example 1: Consider a system with $n = 2$, $N = 8$. We plot the lower bound G_k, upper bound V_k, and a realization of $\boldsymbol{E}[P_k^-]$ by Monte Carlo method. We randomly choose $\boldsymbol{\lambda} = (0.2443, 0.4572, 0.7435, 0.6174, 0.1747, 0.6024, 0.8031, 0.3663)$. The system has the following matrices, which are generated randomly:

$$A = \begin{bmatrix} -0.82 & 0.53 \\ 0.34 & 0.78 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.53 & 0.21 \\ 0.21 & 0.50 \end{bmatrix}$$

$$H_1 = [-0.69, 0.67], \qquad H_2 = \begin{bmatrix} 1.18 & -0.36 \\ 0.70 & -0.38 \end{bmatrix}$$

$$H_3 = [-1.58, 1.43], \qquad H_4 = [-1.71, -2.12]$$

$$H_5 = \begin{bmatrix} -0.65 & 0.49 \\ -2.01 & 0.46 \end{bmatrix}, \qquad H_6 = \begin{bmatrix} 1.79 & 2.2 \\ 0.59 & 0.31 \end{bmatrix}$$

$$H_7 = \begin{bmatrix} 1.28 & 2.07 \\ -1.00 & -0.92 \end{bmatrix}, \qquad H_8 = \begin{bmatrix} 0.18 & 0.21 \\ 2.04 & 0.91 \end{bmatrix}.$$

The plot is presented in Fig. 2 . The plot shows that the two bounds are not tight.

V. Conclusion

This paper studies the stability issue of the multi-sensor Kalman filtering with intermittent measurements. It provides lower and upper bounds of the expected estimation error covariance, and gives conditions on the divergence and convergence of them. Based on those results, the boundness of the expected estimation error covariance is analyzed.

This paper only considers communication controlled by Bernoulli random variables. More advanced communication types, such as Markov processes, can be left as a future study.

APPENDIX

A. Proof of Lemma 1

Let

$$\begin{aligned}\psi_{\boldsymbol\lambda}(L,X) =\ & AXA' + Q + L\Lambda\left(W\odot(HXH'+I)\right)\Lambda L'\\ & -AXH'\Lambda L' - L\Lambda HXA'\end{aligned}$$

where Λ is defined in (21), and

$$L(X) = AXH'\big(W\odot(HXH'+I)\big)^{-1}\Lambda^{-1}.$$

Notice that

$$g_{\boldsymbol\lambda}(X) = \psi_{\boldsymbol\lambda}\big(L(X),X\big) = \min_L \psi_{\boldsymbol\lambda}(L,X). \tag{33}$$

We first prove the monotonicity.

$$\begin{aligned}\psi_{\boldsymbol\lambda}(L,X) =\ & AXA' + L\Lambda HXH'\Lambda L' - AXH'\Lambda L'\\ & - L\Lambda HXA' + L\Lambda\Big((D_{\mathrm{SNR}}^{-1}\mathcal{I})\odot(HXH')\Big)\Lambda L'\\ & + L\Lambda(W\odot I)\Lambda L' + Q\\ =\ & (A-L\Lambda H)X(A-L\Lambda H)' + L\Lambda(W\odot I)\Lambda L'\\ & + Q + \sum_{i=1}^{N} L\Lambda\left(\frac{1}{q_i}\overline{H}_i X\overline{H}_i'\right)\Lambda L'\end{aligned}$$

where $\overline{H}_i = [\mathbf{0}, H_i', \mathbf{0}]'$, which is obtained by replacing in H all the blocks by $\mathbf{0}$ matrices except H_i. Hence, $\psi_{\boldsymbol\lambda}(L,X)$ is quadratic in X. For X,Y satisfying $Y\succeq X\succ 0$, it is easy to show that $\psi_{\boldsymbol\lambda}(L,X)\preceq\psi_{\boldsymbol\lambda}(L,Y)$. Then

$$\begin{aligned}g_{\boldsymbol\lambda}(X) =\ & \psi_{\boldsymbol\lambda}\big(L(X),X\big)\preceq\psi_{\boldsymbol\lambda}\big(L(Y),X\big)\\ & \preceq \psi_{\boldsymbol\lambda}\big(L(Y),Y\big) = g_{\boldsymbol\lambda}(Y)\end{aligned}$$

which shows the monotonicity. To prove the concavity, let $Z=\alpha X+\beta Y$, where X,Y are positive definite matrices, and α,β are positive real numbers with $\alpha+\beta=1$. Then

$$\begin{aligned}g_{\boldsymbol\lambda}(Z) =\ & g_{\boldsymbol\lambda}(\alpha X+\beta Y) = \psi_{\boldsymbol\lambda}\big(L(Z),Z\big)\\ =\ & AZA' + Q + L(Z)\Lambda\big(W\odot(HZH'+I)\big)\Lambda L(Z)'\\ & - AZH'\Lambda L(Z)' - L(Z)\Lambda HZA'\\ =\ & \alpha\Big[AXA' + Q + L(Z)\Lambda\big(W\odot(HXH'+I)\big)\Lambda L(Z)'\\ & - AXH'\Lambda L(Z)' - L(Z)\Lambda HXA'\Big]\\ & +\beta\Big[AYA' + Q + L(Z)\Lambda\big(W\odot(HYH'+I)\big)\Lambda L(Z)'\\ & - AYH'\Lambda L(Z)' - L(Z)\Lambda HYA'\Big]\\ =\ & \alpha\psi_{\boldsymbol\lambda}\big(L(Z),X\big) + \beta\psi_{\boldsymbol\lambda}\big(L(Z),Y\big)\\ \succeq\ & \alpha\psi_{\boldsymbol\lambda}\big(L(X),X\big) + \beta\psi_{\boldsymbol\lambda}\big(L(Y),Y\big)\\ =\ & \alpha g_{\boldsymbol\lambda}(X) + \beta g_{\boldsymbol\lambda}(Y)\end{aligned}$$

which shows the concavity.

B. Proof of Theorem 5

We only need to show that condition 1) in Theorem 4 holds. To prove this, we use the similar method in the proof of [8, Th. 3.1]. Since (27) holds, one can find a communication rates collection $\boldsymbol\lambda$ with associated $\mathfrak{C}_i$ satisfying $\mathfrak{C}_i > \max_j\{\mathfrak{h}(\tilde A_{ij})\}$, i.e., $1+\frac{\lambda_i^2}{\sigma_i^2} > \left[\mathcal{M}(\tilde A_{ij})\right]^2, \forall i,j$. According to [7, Th. 6.4 and Corollary 8.4], for all i,j, there exists $\tilde P_{ij}\succ 0$ and $\tilde l_{ij}\in\mathbb{R}^{n_{ij}\times 1}$, such that

$$\tilde P_{ij}\succ(\tilde A_{ij}-\tilde l_{ij}\lambda_i\tilde h_{ij})\tilde P_{ij}(\tilde A_{ij}-\tilde l_{ij}\lambda_i\tilde h_{ij})' + \sigma_i^2\tilde l_{ij}\tilde h_{ij}\tilde P_{ij}\tilde h_{ij}'\tilde l_{ij}'.$$

First, we prove that for all i, there exist $\tilde P_i$ and $\tilde{\mathcal{L}}_i$, such that

$$\tilde P_i\succ(\tilde A_i-\tilde{\mathcal{L}}_i\lambda_i\tilde{\mathcal{H}}_i)\tilde P_i(\tilde A_i-\tilde{\mathcal{L}}_i\lambda_i\tilde{\mathcal{H}}_i)' + \sigma_i^2\tilde{\mathcal{L}}_i\tilde{\mathcal{H}}_i\tilde P_i\tilde{\mathcal{H}}_i'\tilde{\mathcal{L}}_i' \tag{34}$$

holds, where $\tilde A_i$ and $\tilde{\mathcal{H}}_i$ are given in (26). Denote

$$\tilde A_i^s = \begin{bmatrix}\tilde A_{i1} & & 0\\ & \ddots & \\ \star & & \tilde A_{is}\end{bmatrix},\quad \tilde{\mathcal{H}}_i^s = \begin{bmatrix}\tilde h_{i1} & & 0\\ & \ddots & \\ 0 & & \tilde h_{is}\end{bmatrix}$$

$s=1,2,\ldots,t_i$. We show that there exist $\tilde P_i^s$ and $\tilde{\mathcal{L}}_i^s$, such that

$$\begin{aligned}\tilde P_i^s\succ\ & (\tilde A_i^s-\tilde{\mathcal{L}}_i^s\lambda_i\tilde{\mathcal{H}}_i^s)\tilde P_i^s(\tilde A_i^s-\tilde{\mathcal{L}}_i^s\lambda_i\tilde{\mathcal{H}}_i^s)'\\ & +\sigma_i^2\tilde{\mathcal{L}}_i^s\tilde{\mathcal{H}}_i^s\tilde P_i^s(\tilde{\mathcal{H}}_i^s)'(\tilde{\mathcal{L}}_i^s)'\end{aligned} \tag{35}$$

holds for $s=1,2,\ldots,t_i$. We prove this by mathematical induction. When $s=1$, the case is proved, where $\tilde P_i^1=\tilde P_{i1}, \tilde{\mathcal{L}}_i^1=\tilde l_{i1}$. Assume the case $s=m-1$ is verified. When $s=m$, let

$$\tilde P_i^m = \begin{bmatrix}\tilde P_i^{m-1} & 0\\ & \alpha\tilde P_{im}\end{bmatrix},\quad \tilde{\mathcal{L}}_i^m = \begin{bmatrix}\tilde{\mathcal{L}}_i^{m-1} & 0\\ 0 & \tilde l_{im}\end{bmatrix} \tag{36}$$

where α is to be given later. Denote

$$\begin{aligned}\delta_{ij} =\ & \tilde P_{ij} - (\tilde A_{ij}-\tilde l_{ij}\lambda_i\tilde h_{ij})\tilde P_{ij}(\tilde A_{ij}-\tilde l_{ij}\lambda_i\tilde h_{ij})'\\ & -\sigma_i^2\tilde l_{ij}\tilde h_{ij}\tilde P_{ij}\tilde h_{ij}'\tilde l_{ij}',\\ \Delta_i^s =\ & \tilde P_i^s - (\tilde A_i^s-\tilde{\mathcal{L}}_i^s\lambda_i\tilde{\mathcal{H}}_i^s)\tilde P_i^s(\tilde A_i^s-\tilde{\mathcal{L}}_i^s\lambda_i\tilde{\mathcal{H}}_i^s)'\\ & -\sigma_i^2\tilde{\mathcal{L}}_i^s\tilde{\mathcal{H}}_i^s\tilde P_i^s(\tilde{\mathcal{H}}_i^s)'(\tilde{\mathcal{L}}_i^s)'\end{aligned}$$

and

$$\tilde A_i^m = \begin{bmatrix}\tilde A_i^{m-1} & 0\\ A_L & \tilde A_{im}\end{bmatrix}.$$

Notice that there exists $\beta>0$ such that $\delta_{im}\succ\beta I$. Choose sufficiently large α such that

$$\alpha\beta I\succ\mathcal{K}\Delta_i^{m-1}\mathcal{K}' + A_L\tilde P_i^{m-1}A_L'$$

holds, where $\mathcal{K} = A_L\tilde P_i^{m-1}(\tilde A_i^{m-1}-\tilde{\mathcal{L}}_i^{m-1}\lambda_i\tilde{\mathcal{H}}_i^{m-1})'$. Straightforward calculation shows that $\tilde P_i^m$ and $\tilde{\mathcal{L}}_i^m$ given by (36) satisfy (35). By mathematical induction, there exist $\tilde P_i^s$ and $\tilde{\mathcal{L}}_i^s$ such that (35) holds for $s=1,2,\ldots,t_i$. When $s=t_i$, (35) is exactly (34), with $\tilde P_i=\tilde P_i^{t_i}$ and $\tilde{\mathcal{L}}_i=\tilde{\mathcal{L}}_i^{t_i}$. From (34), in a similar way one can further show that $\tilde{\mathcal{L}}=\mathrm{diag}\{\tilde{\mathcal{L}}_1,\tilde{\mathcal{L}}_2,\ldots,\tilde{\mathcal{L}}_N\}$ and $\tilde P=\mathrm{diag}\{\tilde P_1,\alpha_1\tilde P_2,\ldots,\alpha_{N-1}\tilde P_N\}$ with sufficiently large $\alpha_1,\alpha_2,\ldots,\alpha_N$ satisfying

$$\tilde P\succ(\tilde A-\tilde{\mathcal{L}}\Lambda\tilde{\mathcal{H}})\tilde P(\tilde A-\tilde{\mathcal{L}}\Lambda\tilde{\mathcal{H}})' + \tilde{\mathcal{L}}\Big((\overline{\Sigma}^2\overline{\mathcal{I}})\odot(\tilde{\mathcal{H}}\tilde P\tilde{\mathcal{H}}')\Big)\tilde{\mathcal{L}}'$$

where $\overline{\Sigma}$ and $\overline{\mathcal{I}}$ have proper dimensions. By the inverse transformation $\mathfrak{T}^{-1}$ ($\mathfrak{T}$ is the one in Wonham decomposition), we have

$$P\succ(A-\mathcal{L}\Lambda\mathcal{H})P(A-\mathcal{L}\Lambda\mathcal{H})' + \mathcal{L}\Big((\overline{\Sigma}^2\overline{\mathcal{I}})\odot(\mathcal{H}P\mathcal{H}')\Big)\mathcal{L}'.$$

Without loss of generality, we assume that H has the form of

$$H = (\mathcal{H}_1', \bar{\mathcal{H}}_1', \mathcal{H}_2', \bar{\mathcal{H}}_2', \ldots, \mathcal{H}_N', \bar{\mathcal{H}}_N')'.$$

Correspondingly, let

$$L = (\mathcal{L}_1, 0, \mathcal{L}_2, 0, \ldots, \mathcal{L}_N, 0).$$

It is easy to see that P and L satisfies

$$P \succ (A - L\Lambda H)P(A - L\Lambda H)' + L\Big((\Sigma^2 \mathcal{I}) \odot (HPH')\Big)L'.$$

By Theorem 4, the convergence of $\{V_k\}$ is guaranteed.

Since

$$\mathfrak{h}(A) = \sum_{i=1}^{N} \sum_{j=1}^{t_i} \{\mathfrak{h}(\tilde{A}_{ij})\} > \sum_{i=1}^{N} \max_j \{\mathfrak{h}(\tilde{A}_{ij})\}$$

(28) also leads to the convergence of $\{V_k\}$.

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