INTRODUCTION TO GROMOV–WITTEN THEORY AND QUANTUM COHOMOLOGY

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Abstract. We recollect in this note the topics discussed in the lectures. We give references for the results that we are describing.

Preliminary draft version: please do not circulate

Contents

Program of the course:

(1) Enumeration of rational curves in $\mathbb{P}^2$;
(2) Stable maps and GW-theory;
(3) Quantum Cohomology;
(4) Virtual fundamental class;
(5) Orbifold cohomology (Chen–Ruan cohomology).

This is a series of lectures, devoted to a smooth introduction into the world of Gromov–Witten theory and Quantum Cohomology, from the point of view of algebraic geometry. We will be focusing on the enumerative questions, and on their respective intersection-theoretic answers.

The generality we adopt is: we always build on the category of schemes of finite type over $\mathbb{C}$. Almost all of the time we will be focusing in particular on smooth, projective varieties.

We will always take singular cohomology and Chow groups with rational coefficients. For many of the varieties that will be object of study, we shall moreover make the assumption that the cycle map is an isomorphism, that the variety is convex (to be explained).

The first lecture is addressed to a general audience, so we work somewhat intuitively. The first lecture provides motivation for the following ones. Moreover some key proofs in the following lectures will borrow ideas from the main construction/result of the first lecture.

References for the course (books):

(1) The book of Kock-Vaisencher: [KV07]. This is a gentle invitation to quantum cohomology. It mainly works out the case of the projective space $\mathbb{P}^r$.
(2) The fourth chapter in the bible-type book edited by Vafa: [HKKPTVVZ03].
(3) A nice introduction to moduli of curves and Gromov–Witten theory from the point of view of undergraduate algebraic geometry can be found in [V08].
(4) A series of lectures on Quantum Cohomology and related topics were given by a series of top experts at the Mittag–Leffler institut, Stockholm, in 1996. Paolo Aluffi recollected them all in [A96].
(5) Some notes written by Fulton and Pandharipande settled down the fundations of the moduli spaces of stable maps from the point of view of algebraic geometry: [FP97].
(6) The book of Cox and Katz on Mirror Symmetry, has a chapter devoted to Gromov–Witten theory [CK99].
(7) A more specific book on Frobenius Manifolds and Quantum Cohomology, written by Manin [M99]. We will follow this book to describe equivalent formulations of the theory of quantum cohomology.

Some research articles (mainly for the last lecture):

(1) [KM94]
(2) [K92], [K95]
(3) [W91]
(4) [B97], [B99], [BF97], [BM96]
(5) [CR01],[CR04], [R02], [A08], [AV02], [AGV02], [AGV08], [JKK05], [JKK07], [JK01], [K08], [FG03].

In this note we recollect the results discussed in the lectures. For the results, we try to cite the precise original reference with detailed proofs, and (if available), the more intuitive and sketchy explanations available in the literature.

1. Enumeration of rational curves in $\mathbb{P}^2$

In this section we work somewhat informally. Our presentation owes a lot to [KV07]. The main result we explain here appeared first in [KM94], and is due to Kontsevich.

We want to give an answer to:

Q_d : “How many rational curves of degree d pass through 3d-1 points in general position in the plane?”

We call $N_d \in \mathbb{N}$ the answer to such question.

With a rational curve of degree d, we mean the image of a degree d map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ (in the following, we will always work on $\mathbb{C}$, and omit the subscript).

Exercise 1.1. *The space of such maps of degree d has dimension $3d+2$. Deduce that the space of rational curves of degree d in $\mathbb{P}^2$ has dimension $3d-1$ from the fact that $\dim(\mathrm{Aut}(\mathbb{P}^1)) = 3$.*

A degree d curve in the projective plane is given as the zero locus of an homogeneous, degree d polynomial in 3 variables. The space of such hypersurfaces is projective of dimension $\frac{(d+2)(d+1)}{2} - 1$. Inside this space, there is a Zariski open subset U^{sm} where the resulting curve will be smooth and have genus $\frac{(d-1)(d-2)}{2}$. Another Zariski open subset U^{nod}, is made of those hypersurfaces that correspond to curves with at worst nodal singularities. This last space can be stratified by the number of nodes that occur (0 nodes corresponds to U^{sm}). The formula for the genus of one such curve will then be $\frac{(d-1)(d-2)}{2} - n$, where n is the number of nodes. Note that the locus of such hypersurfaces that have k nodes will have codimension k.

Exercise 1.2. *Let us consider the space of hypersurfaces of degree d in $\mathbb{P}^2$. To pass through a given point P corresponds to a linear condition on such space.*

If $d = 1, 2$, the numbers $\frac{(d+2)(d+1)}{2} - 1$ and $3d-1$ coincide, and moreover each degree d curve in this range is rational. Thus we find that $N_1 = N_2 = 1$. This numbers were known to Euclid ($\sim$ 300 B.C.) and Apollonious ($\sim$ 200 B.C.).

We sketch now the computation of N_3. A general degree 3 plane curve is a smooth elliptic curve (genus 1). There is an hypersurface in the space of degree 3 hypersurfaces, made of curves with 1 node and genus 0. We want to understand the degree of such an hypersurface, so we call k the degree. Then, since we have that the space of hypersurfaces has dimension 9, we are imposing 8 linear condition, and a condition of degree k, we will have that the number N_3 equals k. So let us consider a line in the space of degree 3 curves in $\mathbb{P}^2$: let F and G be two homogeneous, degree 3 polynomials, and consider the line in the space of homogeneous degree 3 polynomials: $F + tG$. The zero locus of $F + tG$ in $\mathbb{P}^1 \times \mathbb{P}^2$ is a surface S, with two projections $p_1 : S \to \mathbb{P}^1$ and $p_2 : S \to \mathbb{P}^2$. The first map p_1 exhibits S as a surface fibered in tori over $\mathbb{P}^1$. Note that k such tori are in fact rational curves with one node. Using the multiplicativity of the Euler characteristic, we have that $\chi(S) = k$ (remember that each torus have Euler characteristic 0 and each rational curve with one node has Euler characteristic 1). On the other hand, the map p_2 exhibits S as the blow-up of $\mathbb{P}^2$ in 9 points (the points where both F and G vanish). So using the additivity of the

Euler characteristic, we have that $\chi(S) = \chi(\mathbb{P}^2) - 9\chi(\text{point}) + 9\chi(\mathbb{P}^1) = 12$. Therefore we obtain that $k = 12$, and so that $N_3 = 12$ as well.

The number N_3 was computed by Chasles/Steiner in the early XIX century, while $N_4 = 620$ was computed in the late XIX century by Schubert/Zeuthen. The number $N_5 = 87304$ was also computed in the mid XX century.

In 1993, Kontsevich found the following beautiful recursive formula for N_d:

$$N_d = \sum_{d_1+d_2=d, d_i>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) \tag{1.3}$$

Note that, once one proves the formula, it is enough to compute N_d for $d = 1$ to derive all the other numbers! From now on, this section will be devoted to explaining how one derives this formula. Let us start in translating our enumerative problem, in an intersection-theoretic problem on a moduli space.

Definition 1.4. Let us define the *moduli space of maps of degree d from a genus 0 curve*:

$$\mathcal{M}_{0,n}(\mathbb{P}^2, d) := \{(C, x_1, \dots, x_n, \phi)\}$$

where C is a smooth genus 0 curve, $x_i \in C$ are pairwise distinct points of it, and $\phi : C \to \mathbb{P}^2$ is a degree d map.

Exercise 1.5. *Understand why the dimension of such space is* $3d - 1 + n$.

Exercise 1.6. *Let* $P \in \mathbb{P}^2$ *be a point. Figure out why the condition that* $\phi(x_i) = P$ *defines a codimension* 2 *condition on* $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$.

There are well-defined evaluation maps $ev_i : \mathcal{M}_{0,n}(\mathbb{P}^2, d) \to \mathbb{P}^2$:

$$ev_i(C, x_1, \dots, x_n, \phi) := \phi(x_i)$$

Our enumerative question Q_d is then translated into the following problem. Let $P_1, \dots, P_{3d-1}$ be general points in $\mathbb{P}^2$. Then we want to compute the number of points:

$$N_d := \#\{ev_1^{-1}(P_1) \cap \dots \cap ev_{3d-1}^{-1}(P_{3d-1})\}$$

As a matter of fact, intersection theory works much better for compact spaces, and our moduli spaces are not. Thus we aim at compactifying them:

Definition 1.7. A *prestable* $n-$*pointed rational curve* is $(C, x_1, \dots, x_n)$ where C is a nodal curve of arithmetic genus 0, and x_i are distinct points in the smooth locus of C.

A *special point* of a prestable n-pointed rational curve is a node, or one of the points x_i.

A prestable curve is *stable*, if every irreducible component of it contains hat least 3 special points.

Exercise 1.8. *Figure out why stable is equivalent to the fact that the automorphism group of the curve is trivial.*

Definition 1.9. We define the moduli space of genus 0 pointed stable curves:

$$\overline{\mathcal{M}}_{0,n} := \{(C, x_1, \dots, x_n)\}$$

It is a fact that we will assume that this is a smooth, compact (in fact, projective) variety.

Exercise 1.10. *The moduli space $\overline{\mathcal{M}}_{0,4}$. Understand that:*

$$\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

In the compactification $\overline{\mathcal{M}}_{0,4}$ we are adding three points, corresponding to the rational curves made of two irreducible components, and the four points $\{1, 2, 3, 4\}$ distributed on them two–by–two. Identify this 3 points as the limit points $0, 1, \infty$ in the isomorphism $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

From this point on, we shall use the singular cohomology theory. Let $l \in H^2(\mathbb{P}^2)$ be the class of a line.

Definition 1.11. A *rational $n-$pointed prestable map of degree d* is the datum of $(C, x_1, \dots, x_n, \phi)$, where $(C, x_1, \dots, x_n)$ is a prestable n-pointed curve, and $\phi : C \to \mathbb{P}^2$ is a morphism such that $\phi_*([C]) = dl$. The map is *stable* if all the irreducible components of C that ϕ contract have at least 3 special points.

Definition 1.12. We define the moduli space of rational stable $n-$pointed degree d maps as:

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) := \{(C, x_1, \dots, x_n, \phi)\}$$

where $(C, x_1, \dots, x_n, \phi)$ is a rational $n-$pointed stable map of degree d.

We have the following result (which shall be discussed in later sections with some more detail):

Theorem 1.13. *The space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ is a smooth compactification of $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$.*

We also have some natural maps:

(1) An evaluation map: $ev_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to \mathbb{P}^2$: $ev_i(C, x_i, \phi) := \phi(x_i)$.
(2) A forgetful map that forgets the point x_i (and if stability is lost, contracts the unstable component): $forg_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to \overline{\mathcal{M}}_{0,n-1}(\mathbb{P}^2, d)$.
(3) An other forgetful map that remembers only the prestable rational curve (C, x_i) (and if necessary, stabilizes it): $f : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to \overline{\mathcal{M}}_{0,n}$.

We want to have some grasp on the boundary components of $\partial\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ for the derivation of Formula 1.3.

Definition 1.14. We define the codimension 1 boundary component $D(N_1, d_1|N_2, d_2)$ for every decomposition $[n] = N_1 \sqcup N_2$ and $d = d_1 + d_2$:

$D(N_1, d_1|N_2, d_2) := \{(C, \phi, x_1, \dots x_n)|\ C$ has two irred. comp. C_1 and $C_2,\ N_i \subset C_i,\ f_*([C_i]) = d_i l\}$

and $\overline{D}(N_1, d_1|N_2, d_2)$ the closure of it (the curves C_1 and C_2 can degenerate further).

Observe that, with this position, we have that:

$$\partial\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) = \bigcup_{[n]=N_1\sqcup N_2, d_1+d_2=d} \overline{D}(N_1, d_1|N_2, d_2)$$

Moreover, we can describe the spaces $\overline{D}(N_1, d_1|N_2, d_2)$ in a nice way in terms of other moduli spaces of stable maps. Let $*$ and $\bullet$ be two symbols. Then we have the evaluation map:

$$ev_*^{N_1,d_1} \times ev_\bullet^{N_2,d_2} : \overline{\mathcal{M}}_{0,N_1\sqcup\bullet}(\mathbb{P}^2, d_1) \times \overline{\mathcal{M}}_{0,N_2\sqcup *}(\mathbb{P}^2, d_2) \to \mathbb{P}^2 \times \mathbb{P}^2$$

and if $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the class of the diagonal, we have:

$$\overline{D}(N_1, d_1|N_2, d_2) = (ev_*^{N_1,d_1} \times ev_\bullet^{N_2,d_2})^{-1}(\Delta) \tag{1.15}$$

Exercise 1.16. *Figure out this last equality.*

Let now $P \in H^4(\mathbb{P}^2)$ be the class of a point, and $1 \in H^0(\mathbb{P}^2)$ be the fundamental class.

Exercise 1.17. *Understand that the class $[\Delta] \in H^4(\mathbb{P}^2 \times \mathbb{P}^2)$ is equivalent (under Künneth decomposition) to $P \otimes 1 + l \otimes l + 1 \otimes P$.*

Let now introduce the following notation, for $\alpha_1, \dots, \alpha_n \in H^*(\mathbb{P}^2)$:

$$\langle \alpha_1, \dots, \alpha_n \rangle_d := \left(\prod ev_i^*(\alpha_i)\right) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)] = \int_{[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)]} \prod ev_i^*(\alpha_i)$$

With this position, our number N_d is simply equal[1] to:

$$N_d = \langle P, \dots, P \rangle_d$$

(where the class of P appears $3d - 1$ times).

The symbol just introduced $\langle\rangle_d$ has some nice properties, that can be easily explained geometrically:

A) Fundamental class insertion:

$$\langle \alpha_1, \dots, \alpha_{n-1}, 1 \rangle_d = 0$$

unless $n = 3, d = 0$: in this case $\langle \alpha_1, \alpha_2, 1 \rangle_0 = \alpha_1 \cup \alpha_2$.

[1]This fact, despite being intuitive, must be checked. We will prove the enumerativity in a later section

B) Degree 0 class:

$$\langle \alpha_1, \ldots, \alpha_n, 1\rangle_0 = 0$$

unless $n = 3$: in this case $\langle \alpha_1, \alpha_2, \alpha_3\rangle_0 = \alpha_1 \cup \alpha_2 \cup \alpha_3$.

C) Divisor insertion. Let $D \in H^2(\mathbb{P}^2)$ be equal to el, for $e \in \mathbb{N}$. Then:

$$\langle \alpha_1, \ldots, \alpha_{n-1}, D\rangle_d = de\langle \alpha_1, \ldots, \alpha_{n-1}\rangle_d$$

Exercise 1.18. *Translate each of the last properties into its geometric meaning, and figure out why it should be true.*

Now, if one believes all the results explained in this chapter, it is easily possible to prove Kontsevich's recursion formula.

Proof. (of Formula 1.3) Let us consider $\overline{\mathcal{M}}_{0,3d}(\mathbb{P}^2, d)$, and then:

$$\mathcal{C} := \langle l, l, P, P \ldots, P\rangle$$

Exercise 1.19. *The dimension of $\mathcal{C}$ is 1: it is the class of a curve in $\overline{\mathcal{M}}_{0,3d}(\mathbb{P}^2, d)$.*

Let $F : \overline{\mathcal{M}}_{0,3d}(\mathbb{P}^2, d) \to \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ be the map that forgets everything but the first four points:

$$F(C, x_i, \phi) := (C, x_1, x_2, x_3, x_4)$$

If Q is the class of a point in $\overline{\mathcal{M}}_{0,4}$ the idea is now to compute:

$$\mathcal{C} \cup F^*(Q) \tag{1.20}$$

in two different ways. Since all the points on $\mathbb{P}^1$ are homologically equivalent, we can compute 1.20 in two different ways. We can take $F^*(Q)$ as class of $F^{-1}(Q_1)$, where Q_1 is the point of $\overline{\mathcal{M}}_{0,4}$ that corresponds to the rational nodal curve with two irreducible components and the points $1, 2$ on one, and the points $3, 4$ on the other. This is equivalent to taking the class of $F^{-1}(Q_2)$, where Q_2 is the point corresponding to the rational nodal curve with two twigs and the points $1, 3$ on one, and the points $2, 4$ on the other. Using 1.15 we obtain:

$$F^{-1}(Q_1) = \bigcup_{\substack{N_1 \sqcup N_2 = [n], \\ \{1,2\} \in N_1, \{3,4\} \in N_2, \\ d_1 + d_2 = d}} (ev_*^{N_1,d_1} \times ev_\bullet^{N_2,d_2})^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2})$$

$$F^{-1}(Q_2) = \bigcup_{\substack{N_1 \sqcup N_2 = [n], \\ \{1,3\} \in N_1, \{2,4\} \in N_2, \\ d_1 + d_2 = d}} (ev_*^{N_1,d_1} \times ev_\bullet^{N_2,d_2})^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2})$$

Using the fact that the two classes $[F^{-1}(Q_i)]$ are cohomologically equivalent, and the formula derived in 1.17 for the diagonal, we obtain $\mathcal{C} \cap F^*(Q)$ in two different ways, which produce the same result:

$$\sum_{\substack{N_1 \sqcup N_2 = [n], \\ \{1,2\} \in N_1, \{3,4\} \in N_2, \\ d_1 + d_2 = d}} \langle l, l, \dots, P \rangle_{d_1} \langle P, P, \dots, 1 \rangle_{d_2} + \langle l, l, \dots, l \rangle_{d_1} \langle P, P \dots, l \rangle_{d_2} + \langle l, l \dots, 1 \rangle_{d_1} \langle P, P \dots, P \rangle_{d_2} =$$

$$\sum_{\substack{N_1 \sqcup N_2 = [n], \\ \{1,3\} \in N_1, \{2,4\} \in N_2, \\ d_1 + d_2 = d}} \langle l, P, \dots, P \rangle_{d_1} \langle l, P, \dots, 1 \rangle_{d_2} + \langle l, P, \dots, l \rangle_{d_1} \langle l, P \dots, l \rangle_{d_2} + \langle l, P \dots, 1 \rangle_{d_1} \langle l, P \dots, P \rangle_{d_2}$$

Note that the dots all correspond to classes of points P, and that the $3d - 4$ classes of points P ought to be distributed according to the subdivision $N_1 \sqcup N_2 = [n]$ for all the possible partitions of $[n]$.

Let us now use the properties A, B, C to simplify the LHS of the equality. The first summand vanishes by the property A. Again using the same property, the third summand reduces to the number N_d. For the second summand we can use the property B and C, and paying attention to how many of the classes P can be plugged in the $\langle \rangle$ in such a way that it becomes a term of the right dimension. So it becomes:

$$\sum_{d_1 + d_2 = d, d_1 > 0, d_2 > 0} d_1^3 d_2 N_{d_1} N_{d_2} \binom{d-4}{d_1 - 3}$$

Analogously, on the RHS the first and the third term vainsh by the property A, and the second term reduces to:

$$\sum_{d_1 + d_2 = d, d_1 > 0, d_2 > 0} d_1^2 d_2^2 N_{d_1} N_{d_2} \binom{d-4}{d_1 - 2}$$

and from the equality LHS=RHS, we obtain Formula 1.3. $\square$

Exercise 1.21. *Work out the computation in detail.*

Exercise 1.22. *Repeat the same argument and find a "Kontsevich's formula" to enumerate the rational curves in $\mathbb{P}^1 \times \mathbb{P}^1$ (instead of $\mathbb{P}^2$).*

Exercise 1.23. *What are the actual hypothesis on the space $X = \mathbb{P}^2$ that we used to derive Kontsevich formula?*

2. Introduction to the course

The topic of this course is the study of curves in smooth projective varieties X, using the techniques of algebraic geometry and in particular intersection theory, with the enumerative geometric meaning in mind.

We want to study a moduli space whose points parametrize embeddings of Riemann surfaces into some target space X. We will want to be able to speak about how many curves in X hit certain subvarieties. For this we need to introduce in the moduli space marked points on each Riemann surface.

Guiding examples for X will be smooth algebraic varieties such as a point, $\mathbb{P}^1$, $\mathbb{P}^2$, a Calabi–Yau threefold, ...

The moduli space, if exists, has different components that are distinguished by discrete invariants such as the genus of the curve, the number of marked points, and the class $\beta \in H_2(X, \mathbb{Z})$ that the curve C realizes as a cycle in X once it is embedded. Once these three parameters are fixed, we can ideally construct a moduli space $\mathcal{M}_{g,n}(X, \beta)$.

Nice things we we will be able to obtain from this moduli spaces:

(1) The datum of a family over a space S of embeddings of such curves into X corresponds exactly to a map from S to the moduli space,
(2) The moduli spaces $\mathcal{M}_{g,n}(X, \beta)$ are connected by natural maps with each other,
(3) There are evaluation maps $ev_i : \mathcal{M}_{g,n}(X, \beta) \to X$ defined by the following prescription: given an embedding $f : (C, x_1, \ldots, x_n) \to X$, $ev_i(f) := f(x_i)$.
(4) Given cycles $\alpha_1, \ldots, \alpha_n \in Z^*(X)^n$ we can get cycles in $H^*(\mathcal{M}_{g,n}(X, \beta))$ by taking:
$$ev_1^*(\alpha_1) \cup \ldots \cup ev_n^*(\alpha_n)$$
if the class thus obtained is a top degree class, using Poincaré duality we can obtain a number: the number of genus g curves whose associated cycle realize a given class β in X, and that pass through $\alpha_1, \ldots, \alpha_n$. Ideally, this number should be a natural number,
(5) These numbers can be organized in certain ways in generating functions (Gromov–Witten potentials). They satisfy integrable hyerarchies. For example even the trivial case where X is just a point corresponds to the integrable hyerarchy of the KDV equation, a differential equation that describes the diffusion of waves in shallow water.
(6) These numbers are also relevant in string theory, as they are a realization of a particular case of Feynmann integral. Instead of integrating over the space of all possible paths here we restrict to the space of all possible holomorphic (algebraic) maps.
(7) The intersection of two subvarieties of X can be rewritten fixing the moduli space $\mathcal{M}_{0,3}(X, \beta = 0)$, and observing that:
$$\alpha_1 \cup \alpha_2 = ev_{3*}(ev_1^*(\alpha_1) \cup ev_2^*(\alpha_2) \cup 1)$$
By taking a general β this will allow us to generalize intersection theory on X.

Problems:

(1) These moduli spaces are not compact. As we will want to do intersection theory on them, we we will introduce a compactification (Kontsevich' compactification),
(2) They are not fine moduli spaces: for this we will be using the language of stacks,
(3) They are not smooth, and when they are they are often not of the expected dimension. This is probably the biggest issue, and we will address it in two ways: by introducing the theory first in the case when it is smooth, generalizing it axiomatically to the general case by means of some desired properties. We will finally solve the problems by means of the construction of a virtual fundamental class: a class in the homology of the moduli spaces that loosely speaking will allow us to do intersection theory on the moduli spaces as if they were smooth.

3. Moduli of Stable Maps

The purpose of this section is to give the definition of the Kontsevich compactification of $\mathcal{M}_{g,n}(X, \beta)$ for a fixed X, and to study its first properties. We start by recalling the Deligne–Mumford–Knudsen compactification of $\mathcal{M}_{g,n}$.

Definition 3.1. Let $(C, x_1, \ldots, x_n)$ be a nodal curve, with points on it.

(1) The curve C is *prestable* if the points x_i are distinct and in the smooth locus of C, the *genus* of the prestable curve is its arithmetic genus $h^1(C, \mathcal{O}_C)$,
(2) A *special point* on $(C, x_1, \ldots, x_n)$ is one of the x_i or a node. We make the convention that a node of an irreducible component of C accounts for 2 special points,
(3) The prestable curve $(C, x_1, \ldots, x_n)$ is *stable* if every geometric genus 0 irreducible component has at least 3 special points on it, and every genus 1 component has at least 1 special point on it. The genus of the prestable curve is again its arithmetic genus.

Exercise 3.2. *Note that the stability condition is a topological condition on a curve C. Observe that a prestable curve is stable if and only if its automorphism group is finite.*

Exercise 3.3. *You can associate to each prestable curve a weighted multigraph, where the vertices are the irreducible components. Each node of the curve corresponds to an edge that connects the two vertices associated to the irreducible components that intersect in that node. Each such graph is then decorated with a genus function, which assigns to each vertex the geometric genus of the corresponding irreducible component.*

(1) prove that the curve is stable if and only if the associated graph has only finitely many automorphisms,
(2) if g, n is fixed, there are only finitely many graphs associated to genus g, n-pointed stable curves,

Definition 3.4. A stable map is the datum of (C, x_i, f), where:

(1) $(C, x_1, \dots, x_n)$ is a prestable curve,
(2) $f : C \to X$ is a morphism,
(3) If the map f contracts an irreducible component C_k to a point, then
  - If C_k has geometric genus 0, it must have at least 3 special points on it,
  - If C_k has geometric genus 1, it must have at least 1 special point on it. a stable map represents a class $\beta \in H_2(X)$ if $f_*([C]) = \beta$.

We describe now the moduli problem of stable, n-pointed maps.

Definition 3.5. We define the groupoid $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$, for S any algebraic scheme of finite type over $\mathbb{C}$. The *objects* of $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ are:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, shift left, "\pi"] & X \\ S \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

where:

(1) The morphism π is flat and projective,
(2) The morphisms s_i are n sections of π,
(3) For any $s \in S$ geometric point, $(C_s, x_1, \dots, x_n)$ is a prestable curve of genus g
(4) For any $s \in S$ geometric point, $f_s : C_s \to X$ is a stable map that represents the class β.

Exercise 3.6. *Define the morphisms between families of stable pointed maps and realize that families of stable pointed maps over a base S form indeed a groupoid.*

Definition 3.7. The associated set $\overline{M}_{g,n}(X,\beta)(S)$. The association $\overline{M}_{g,n}(X,\beta)$ is a contravariant functor from the category of schemes of finite type over $\mathbb{C}$ to the category of sets.

Theorem 3.8. *$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a Deligne–Mumford stack. There is a coarse moduli space for the functor $\overline{M}_{g,n}(X,\beta)$. The first is a proper stack, the latter is a projective scheme.*

Proof. See [BM96, Theorem 3.14] for the first stacky theorem, and [FP97] for the projectivity of the coarse moduli space. The generality adopted by Behrend-Manin is slightly bigger, the case stated here corresponds to the case when their graph τ is the graph with one vertex. The stability condition gives the fact that each automorphism group of the objects is finite (so the stack id Deligne–Mumford).

The notes of Fulton-Pandharipande [FP97, Sections 2,3,4,5] construct the coarse moduli space, assuming the projectivity of the moduli space of curves $\overline{M}_{g,n}$ (a result estabilished by Deligne–Mumford–Knudsen). A nice overview of their construction can be found in [A96, Section 1.6].

Assuming that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is an algebraic scheme, an alternative proof of projectivity can be found in [Co95]. $\square$

Example 3.9. (1) $\overline{\mathcal{M}}_{g,n}(X,0) \cong \overline{\mathcal{M}}_{g,n} \times X$,
(2) $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^m,1)$ is the Grassmannian of lines in $\mathbb{P}^m$.

Exercise 3.10. *Study the moduli stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,2)$. Its coarse moduli space can be identified with the classical space of complete conics (*[KV07, Section 2.9]*,* [HKKPTVVZ03, Section 24.2]*).*

Exercise 3.11. *Study the moduli stack $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)$. It gives an example of moduli stack reducible, non reduced, of impure dimension, where the open part is not dense. Observe that when $d=1$ the open part is empty, but the compactification is not!*

Exercise 3.12. *Show that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,1)$ is a locally trivial fibration with fiber the Fulton–Macpherson compactification $\mathbb{P}^1[m]$ over $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^m,1)$.*

From the example, we see that in general $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is possibly not smooth (even as a Deligne–Mumford stack), reducible, non reduced, of impure dimension. Moreover it is not the closure of the open part, as we are adding lots of new "extraneous" components

Definition 3.13. (some properties) (see [HKKPTVVZ03, Section 24.3])

(1) Evaluation maps: there are n evaluation maps $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$, defined by $ev_i(C, x_i, f) := f(x_i)$. The proof that these are well-defined maps of stacks can be found in [BM96, Proposition 5.5].
(2) Forgetful map, $\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ ([BM96, Proposition 4.5])
(3) If $g : X \to Y$, there is an induced map: $f_g : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X, g_*\beta)$, as long as the space on the right exists. If Y is a point, this gives a map $stab : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$

Exercise 3.14. *Universal curve. Let $\overline{\mathcal{C}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ be the universal curve. Prove that there is an isomorphism of $\overline{\mathcal{C}}_{g,n}(X,\beta)$ and $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$ over $\overline{\mathcal{M}}_{g,n}(X,\beta)$, where the last map is the forgetful map defined above.*

4. Genus 0 invariants

In this section, we define the genus 0 Gromov–Witten invariants. We thus focus on the moduli stack $\overline{\mathcal{M}}_{0,n}(X,\beta)$. We introduce some further hypothesis on X in order to have a better-behaved moduli space:

Definition 4.1. A projective variety X is convex when, for every $f : \mathbb{P}^1 \to X$, the vector space $H^1(\mathbb{P}^1, f^*(T_X))$ vanishes.

Definition 4.2. ([H75, Section 12.1, Section 21.3]) A projective variety is said to be *homogeneous*, if it is isomorphic to the quotient G/P where G is a linear algebraic group, and P is a parabolic subgroup.

Exercise 4.3. *Prove by induction the following statement: if X is convex, then $H^1(C, f^*(T_C)) = 0$ for any $f : C \to X$, where C is a genus 0 prestable curve.*

Exercise 4.4. *Prove that an homogeneous variety is convex, using the fact that the tangent bundle to an homogeneous variety is generated by its global sections.*

Example 4.5.
1. The projective space $\mathbb{P}^k$ is homogeneous,
2. The grassmannians are homogeneous,
3. Flag varieties are homogeneous,
4. Abelian varieties are convex, but not homogeneous,
5. Projective bundles over curves are convex (if the genus of the curve is bigger than 0), but not homogeneous.

Theorem 4.6. *Let X be a convex variety. Then $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is a smooth Deligne–Mumford stack with normal crossing divisors. If X is furthermore homogeneous, then $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is irreducible. In both cases, its dimension is* $\dim(X) + n - 3 + \int_\beta c_1(T_X)$.

Proof. See [FP97, Theorem 2, Theorem 3] and [BM96, Proposition 7.4]. The original proof that if X is homogeneous, then the moduli space of maps from genus 0 curve is irreducible was given in [KP01, Theorem 1].

A nice intuitive argument that motivates the dimension by computing the tangent space at points where f is a closed immersion is in [HKKPTVVZ03, Section 24.4] (??? David: is there one such point for each moduli space so that this becomes a proof???). $\square$

From now on, in this section and in the following one, we shall assume that X is homogeneous. Benefits of this choice, besides Theorem 4.6:

1. The cycle map $cyc : A^* \to H^*$ is an isomorphism ([Fu84, Chapter 19]),
2. The evaluation maps are flat.
3. The monoid of effective classes $N^+(X) \subset H_2(X,\mathbb{Z})$ is generated by finitely many classes $\langle \beta_1, \dots, \beta_p \rangle$, where $\beta_i = \mu_{i*}(\mathbb{P}^1)$ for certain μ_i closed embedding.

Exercise 4.7. *Using the torus action (see* [H75]*) on an homogeneous variety X, prove this last assertion (stated in the proof of* [FP97, Lemma 15]*).*

Exercise 4.8. *Using generic flatness, and the fact that there is a transitive group action on X homogeneous, prove that the evaluation maps are flat. Observe that the total evaluation map $ev : \overline{\mathcal{M}}_{0,n}(X,\beta) \to X^n$ are not flat!*

Definition 4.9. (genus 0 Gromov-Witten invariants for homogeneous projective varieties) Let X be an homogeneous projective variety. We define the following linear maps:

$$\langle\rangle_{0,n,\beta}^{X} : H^*(X)^{\otimes n} \to \mathbb{Q}$$

as follows:

$$\langle \alpha_1, \ldots, \alpha_n \rangle := \prod_{i=1}^{n} ev_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{0,n}(X,\beta)] = \int_{\overline{\mathcal{M}}_{0,n}(X,\beta)} ev_1(\alpha_1) \cup \ldots \cup ev_n(\alpha_n)$$

Example 4.10. If P is the class of a point in $\mathbb{P}^2$, the number of points N_d should morally be equal to $\langle P, \ldots, P\rangle_{0,3d-1,d}$. We shall prove this fact rigourously as a consequence of Proposition 4.12.

We now focus on the enumerativity property of Definition 4.9. To prove it, we shall need the following intuitive lemma, due to Kleiman, and usually referred to as Kleiman–Bertini:

Lemma 4.11. [K74] *Let G be a connected linear algebraic group, and X an homogeneous $G-$variety, and $f : Y \to X$, $g : Z \to X$ be two morphisms. For $\sigma \in G$ we denote Y^σ the scheme Y as a scheme over X via the map $\sigma \circ f$. Then:*

(1) There exists an open, dense subset $G^0 \subset G$ such that $Y^\sigma \times_X Z$ is either empty or of pure dimension $\dim(Y) + \dim(Z) - \dim(X)$ for all $\sigma \in G^0$,

(2) If Y and Z are nonsingular, G^0 can be chosen in such a way that $Y^\sigma \times_X Z$ is nonsingular too.

With this Lemma, one can prove the enumerativity property of genus 0 Gromov-Witten invariants:

Proposition 4.12. *Let $\Gamma_1, \ldots, \Gamma_n$ be subvarieties of an homogeneous projective variety $X = G/P$, and γ_i be the corresponding cohomology classes. Now suppose that the codimensions of Γ_i add up to the dimension of the moduli space $\overline{\mathcal{M}}_{0,n}(X,\beta)$ for a certain class β. Then for generic $g = (g_1, \ldots, g_n) \in G^n$, we have that the scheme-theoretic intersection:*

$$ev_1^{-1}(g_1\Gamma_1) \cup \ldots \cup ev_n^{-1}(g_n\Gamma n)$$

is a finite number of reduced points supported on the open part $\overline{\mathcal{M}}_{0,n}(X,\beta)$. Moreover, one has the equality:

$$\langle \gamma_1, \ldots, \gamma_n\rangle_{0,n,\beta} = \#\{ev_1^{-1}(g_1\Gamma_1) \cup \ldots \cup ev_n^{-1}(g_n\Gamma_n)\}$$

Proof. The proof of this is given in [FP97, Lemma 14]. A more friendly and detailed exposition is in [A96, Section 1.3]. □

Let us now show that the linear maps $\langle\rangle_{0,n,\beta}^{X}$ satisfy some convenient and geometrically intuitive properties (which we used in Section 1 to prove Kontsevich's formula):

- Splitting property

- Mapping to a point
- Fundamental class insertion
- Divisor insertion

The following properties are stated and then proved in [FP97, p.35].

Proposition 4.13. *(Mapping to a point) Suppose $\beta = 0$. Then $\langle\rangle_{0,n,0}$ is identically zero, unless n equals 3, in this case $\langle \alpha_1, \alpha_2, \alpha_3\rangle_{0,3,0} = \alpha_1 \cup \alpha_2 \cup \alpha_3$*

Proposition 4.14. *(Fundamental class insertion) $\langle \alpha_1, \dots, \alpha_n, 1\rangle_{0,n+1,\beta}$ is always zero, unless $\beta = 0, n = 3$, in this case it is $\alpha_1 \cup \alpha_2$.*

Proposition 4.15. *(Divisor insertion) Let $D \in H^2(X)$, then $\langle D, \alpha_1, \dots, \alpha_n\rangle_{0,n+1,\beta} = \langle \alpha_1, \dots, \alpha_n\rangle_{0,n,\beta} \int_\beta D$*

For the Splitting property, we need a bit of notation and of understanding of the boundary of $\overline{\mathcal{M}}_{0,n}(X,\beta)$, analogous to the one we carried out for $X = \mathbb{P}^2$ in Section 1. Let $A \sqcup B = [n]$ be a partition $[n]$, and $\beta = \beta_1 + \beta_2$ such that $\beta_i \in H_2(X,\mathbb{Z})_+$.

Definition 4.16. We define $D(A,\beta_1|B,\beta_2)$ as the closure of the substack of $\overline{\mathcal{M}}_{0,n}(X,\beta)$, whose generic element is a stable map f from a curve C made of two irreducible components C_1 and C_2, where C_1 is an $A-$pointed and C_2 is a $B-$pointed rational curve. Thus f can be written as (f_1, f_2), and we require $f_{i*}([C_i]) = \beta_i$.

Exercise 4.17. *Let Y be the fiber product in the following diagram:*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,A\sqcup *}(X,\beta_2) \arrow[d, "ev_*"] \\ \overline{\mathcal{M}}_{0,B\sqcup \bullet}(X,\beta_1) \arrow[r, "ev_\bullet"] & X \end{tikzcd}$$

Then gluing the two curves in Y produces a map $Y \to D(A,\beta_1|B,\beta_2)$, which is an isomorphism.

From the previous exercise, we have a natural map $i : D(A,\beta_1|B,\beta_2) \to \overline{\mathcal{M}}_{0,A\sqcup\bullet}(X,\beta_1)\times \overline{\mathcal{M}}_{0,B\sqcup\star}(X,\beta_2)$. We also have an inclusion map $\alpha : D(A,\beta_1|B,\beta_2) \to \overline{\mathcal{M}}_{0,n}(X,\beta)$.

It is thus a natural (and technically important) question, to relate the invariants on the two sides via the divisor D. For this we introduce a basis for $H^*(X)$: $T_0, \dots, T_m$, and we fix $g^{ij} := \int T_i \cup T_j$.

Lemma 4.18. *(Splitting property) The following equality holds:*

$$i_*\alpha^*(\prod ev_i^*(\alpha_i)) = \sum_{e,f} g^{ef} \left(ev_\bullet^*(T_e) \prod_{i\in A} ev_i^*(\alpha_i)\right)\left(ev_*^*(T_f) \prod_{j\in B} ev_j^*(\alpha_j)\right)$$

Proof. This is proved in [FP97, Lemma 16]. □

We will see in the following sections a nicer and more compact way to express this property. It will be the constituency part for the formal properties of a Gromov–Witten theory. In particular we will see that this property can be recasted as an associativity property of a certain product, or even as the flatness property on a certain connection defined on the manifold $H^*(X)$.

5. Quantum Cohomology

Let X be a smooth, projective, homogeneous variety. In this section we introduce two quantum cohomology products, a small and a big one. From our view point so far, they are a nice way to organize the information obtained from the Gromov–Witten invariants.

We start by observing that on $H^*(X)$ there is an associative cup product and a bilinear nondegenerate pairing $\langle , \rangle$:

$$\langle \alpha, \beta \rangle := \int \alpha \cup \beta$$

The cup product and $\langle , \rangle$, together give $H^*(X)$ the structure of a Frobenius algebra, with unit the fundamental class $1_X \in H^0(X)$.

Now we generalize this structure. We define:

$$\alpha_1 *_\beta \alpha_2 := ev_{3*} \left(ev_1^*(\alpha_1) \cup ev_2^*(\alpha^2) \right)$$

where the moduli space considered is $\overline{\mathcal{M}}_{0,3}(X, \beta)$. We introduce a formal parameter q^β for each element $\beta \in H_2(X, \mathbb{Z})_+$ with the rule $q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2}$. Thus we define:

$$\alpha_1 * \alpha_2 := \sum_\beta \alpha_1 *_\beta \alpha_2 q^\beta$$

and extend this product $\mathbb{Q}[[H_2(X, \mathbb{Z})_+]]$-linearly to $H^*(X) \otimes \mathbb{Q}[[H_2(X, \mathbb{Z})_+]]$. We call the resulting structure on this vector space $QH_s^*(X)$.

Exercise 5.1. *Show that, if α_1 and α_2 are fixed, there are only finitely many β for which $*_\beta$ is nonzero.*

Proposition 5.2. *The small quantum cohomology $QH_s^*(X)$ is a Frobenius algebra with the same unit of $H^*(X)$.*

Proof. The assertion on the unity boils down to the fundamental class insertion property of the previous section. The most complicated thing to prove as usual is associativity. To do this we consider the following big diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,4}(X,\beta_1+\beta_2) \arrow[drrrr, dotted, bend left, "ev_l"] \arrow[ddrrrr, dotted, "ev_k"] \arrow[dddrr, dotted, "ev_j"'] \arrow[ddd, dotted, bend right, "ev_i"'] & & & & \\
& & D(12\beta_1|34\beta_2) \arrow[ull] \arrow[r] \arrow[d] & \mathcal{M}_{0,3}(X,\beta_2) \arrow[r] \arrow[d] \arrow[dr] & X \\
& & \mathcal{M}_{0,3}(X,\beta_1) \arrow[r] \arrow[dll] \arrow[d] & X & X \\
X & & X & &
\end{tikzcd}$$

We call the four marked points on $\overline{\mathcal{M}}_{0,4}(X,\beta_1+\beta_2)$ i,j,k,l. With some entertaining diagram-chaising, the associativity is reduced to (using also that the product is clearly commutative):

$$\sum_{\beta_1+\beta_2=\beta} ev_i^*(\alpha_1)ev_j^*(\alpha_2)ev_k^*(\alpha_3)\cap[D(12\beta_1|34\beta_2)] = \sum_{\beta_1+\beta_2=\beta} ev_i^*(\alpha_2)ev_j^*(\alpha_3)ev_k^*(\alpha_1)\cap[D(12\beta_1|34\beta_2)]$$

reordering the terms on the right hand side, one obtains:

$$ev_i^*(\alpha_1)ev_j^*(\alpha_2)ev_k^*(\alpha_3)\cap[D(23\beta_1|14\beta_2)] = ev_i^*(\alpha_2)ev_j^*(\alpha_3)ev_k^*(\alpha_1)\cap[D(12\beta_1|34\beta_2)]$$

so the associativity of $*$ follows from the fact that:

$$\sum_{\beta_1+\beta_2=\beta}[D(12\beta_1|34\beta_2)] = \sum_{\beta_1+\beta_2=\beta}[D(23\beta_1|14\beta_2)]$$

This in cohomology is a consequence of the fact that points in $\overline{\mathcal{M}}_{0,4}$ are cohomologically equivalent, and if $F : \overline{\mathcal{M}}_{0,4}(X,\beta) \to \overline{\mathcal{M}}_{0,4}$ is the forgetful map, the two sides of the last equation are just two ways of computing $F^*(Q)$ for Q a point. $\square$

Observe that, after fixing a basis $T_0,\dots,T_m$ for $H^*(X)$, we have that the product is given by:

$$\alpha_1 * \alpha_2 = \sum_{\beta,j}\langle\alpha_1,\alpha_2,T_i\rangle_{0,3,\beta}\, g^{ij}T_jq^\beta$$

Exercise 5.3. *(see* [CK99, Example 8.1.2.1]*) The small quantum cohomology ring of* $\mathbb{P}^r$ *is isomorphic to* $\mathbb{Q}[H][[q]]/(H^{r+1}-q)$ *where* H *is the class of an hyperplane in* $H^2(\mathbb{P}^r)$*.*

If on one hand this result satisfactorily shows that the small quantum cohomology is a deformation of the usual cohomology ring, on the other hand we see (by solving the exercise) that the only enumerative information contained is the 3-points information. In other words, the numbers N_d do not appear in the small quantum cohomology of $\mathbb{P}^2$.

To really collect all the information from $n-$point functions, we need the *big quantum cohomology*. Let us define the *Gromov–Witten potential*:

$$\Phi(\gamma) := \sum_{n\geq 3}\sum_{\beta}\frac{\langle\gamma^n\rangle_{0,n,\beta}}{n!}q^\beta$$

Here we use the notation that $\gamma^n = \gamma,\ldots,\gamma$, $n-$times. Now putting $\gamma = \sum y_i T_i$, we obtain a formal power series in y_i:

$$\Phi(y_0,\ldots,y_m) = \sum_{n_0+\ldots+n_m=n,\beta\in H_2(X,\mathbb{Z})}\langle T_0^{n_0},\ldots,T_m^{n_m}\rangle_{0,n,\beta}\frac{y_0^{n_0}}{n_0!}\cdots\frac{y_m^{n_m}}{n_m!}$$

Exercise 5.4. *(*[FP97, Lemma 15]*) This is indeed a power series. In particular, for any n, there are only finitely many β such that the number $\langle\gamma^n\rangle_{0,n,\beta}$ is nonzero.*

We denote with $\Phi_{ijk} := \partial_i\partial_j\partial_k\Phi$. Observe that:

$$\Phi_{ijk}(\gamma) = \sum_{n\geq 0,\beta}\frac{\langle T_i,T_j,T_k,\gamma^n\rangle_{0,n,\beta}}{n!}$$

Definition 5.5. The big quantum product is defined on the basis $T_0,\ldots,T_m$ as:

$$T_i *_b T_j := \sum_{e,f}\Phi_{ije}g^{ef}T_f$$

and then extended $\mathbb{Q}[[y_0,\ldots,y_m]]$-linearly to a product on the whole $H^*(X)\otimes\mathbb{Q}[[y_0,\ldots,y_m]]$.

One can easily check that the product $*$ is well defined, in other words if one changes the basis $T_0,\ldots,T_m$ in $T_0',\ldots,T_m'$, the linear change of coordinates from $H^*(X)\otimes\mathbb{Q}[[y_0,\ldots,y_m]]$ to $H^*(X)\otimes\mathbb{Q}[[y_0',\ldots,y_m']]$ identifies the two product structures.

Theorem 5.6. *The big quantum cohomology ring $H^*(X)\otimes\mathbb{Q}[[y_0,\ldots,y_m]]$ is a Frobenius algebra.*

Proof. The detailed proof is in [FP97, Theorem 4]. Commutativity follows from the fact that the product is defined in terms of partial derivatives of Φ. The fundamental class insertion property show that the usual 1 in cohomology is a unity for this new algebra. Associativity follows from the splitting property and the usual equivalence of points on $\overline{\mathcal{M}}_{0,4}\cong\mathbb{P}^1$. □

Exercise 5.7. [FP97]*[Section 9]The Gromov–Witten potential for $\mathbb{P}^2$ is:*

$$\Phi^{\mathbb{P}^2}(y_0 1 + y_1 H + y_2 H^2) = \frac{1}{2}(xy^2+x^2z)+\sum_d N_d\frac{y_2^{3d-1}}{(3d-1)!}e^{dy_1}$$

where N_d are the numbers defined in Section 1. The big Quantum Cohomology ring of $\mathbb{P}^2$ is isomorphic to:

$$QH^*(\mathbb{P}^2) = \mathbb{Q}[[y_0, y_1, y_2]][Z]/(Z^3 - \Phi_{111}Z^2 - 2\Phi_{112}Z - \Phi_{122})$$

6. Formal properties of Quantum Cohomology

In this section we give a shadow of the equivalence of three structures on (H, g), a finite dimensional complex vector space H endowed with a symmetric bilinear nondegenerate form g, and a fixed basis of it $T_0, \ldots, T_m$. We borrow this presentation from the book of Manin [M99], where the equivalence is fully proved. We address the reader who seeks for more details to [M99], and in particular to Chapter 0.

Let $\Phi \in \mathbb{Q}[[y_0, \ldots, y_m]]$, we say that Φ is *a formal solution of the associativity equation* (or that it is a potential), if the following is satisfied for all a, b, c, d:

$$\sum_{e,f} \Phi_{abe} g^{ef} \Phi_{fcd} = \sum_{e,f} \Phi_{bce} g^{ef} \Phi_{fad} \tag{6.1}$$

where $g^{ij} = g(T_i, T_j)$.

The name comes from the fact that, if one defines a product as in the section above:

$$T_i * T_j := \sum \Phi_{ije} g^{ef} T_f$$

this product is associative as a consequence of 6.1. Moreover this product is invariant under adding terms of degree < 3 to the formal power series Φ.

Now let I_n be a family of linear maps:

$$I_n : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{0,n}), \quad n \geq 3$$

For any splitting σ of $[n]$ into $A \sqcup B$, we have an induced gluing map:

$$\phi_\sigma : \overline{\mathcal{M}}_{0,A+1} \times \overline{\mathcal{M}}_{0,B+1} \to \overline{\mathcal{M}}_{0,n}$$

The family of linear maps I_n is said to be a *cohomological Field Theory* if they satisfy:

$$\phi_\sigma^* \circ I_n(\gamma_1, \ldots, \gamma_n) = I_{A+1} \otimes I_{B+1} \left(\bigotimes_{i \in A} \gamma_i \otimes \Delta \otimes \bigotimes_{j \in B} \gamma_j \right) \tag{6.2}$$

where $\Delta \in H^{\otimes 2}$ is the diagonal class:

$$\Delta := \sum_{i,j} g^{ij} T_i \otimes T_j$$

each linear map I_n is then called a *correlator*.

Now the theorem Manin proves is:

Theorem 6.3. *(*[M99, Theorem 0.5]*) There is a natural bijection between the set of formal solution of the associativity relation, modulo terms of degree ≤ 2, and structures of a Cohomological Field Theory on (H, g).*

Exercise 6.4. *(for those who now operads) Recast the notion of Cohomological field theory using Poincaré duality on $H^*(\overline{\mathcal{M}}_{0,n})$, as a map:*

$$H^*(\overline{\mathcal{M}}_{0,n}) \to \mathrm{Hom}(H^{\otimes n}, H)$$

thus giving H the structure of an algebra over the operad $H^*(\overline{\mathcal{M}}_{0,n})$. (suggested reading: [A96, Section 3.6] [M99, Chapter 4]*)*

We now see a third way of giving a Cohomological Field Theory. Let (H, g) be as before. If a potential Φ is given, we can define the following connection on the (linear) manifold H. In coordinates, we define the structure constants of the connection as:

$$A_{ij}^k := \sum_e \Phi_{ije} g^{ek}$$

and then the connection ∇ is defined by:

$$\nabla_{\partial_i} \partial_j := \sum_k A_{ij}^k \partial_k$$

The curvature $(3,1)$-tensor R writes, in terms of A (with Einstein convention):

$$R_{ijk}^m := A_{ik}^f A_{fj}^m - A_{jk}^f A_{fi}^m + \partial_j A_{ik}^m - \partial_i A_{jk}^m$$

Now the curvature tensor is clearly torsion-less, as the last two terms in the sum vanish as a consequence of the commutativity of the partial derivatives $\partial_i \partial_j \Phi = \partial_j \partial_i \Phi$.

Exercise 6.5. *Check that the associativity property of Φ is equivalent to the vanishing of the first two terms. Thus an associative potential Φ gives rise to a flat connection on the manifold H. (*[M99, Definition 0.4, Theorem 0.5]*,* formal Frobenius manifold*).*

The equivalent structures of Cohomological Field Theory, associative potential, formal Frobenious manifold, are a first glimpse of the three different avatars of the formal Quantum Cohomology that appear in enumerative geometry, theoretical phyisics, and mathematical physics (integrable systems, in the Dubrovin formalism).

Now we set $H := H^*(X)$ where X is an homogeneous projective manifold, and $T_0, \ldots, T_m$ a basis of it, with $g^{ij} := \int T_i \cup T_j$ as in the former section. The linear maps $\langle\rangle_{0,n,\beta}$ factor through the cohomology of $\overline{\mathcal{M}}_{0,n}$:

$$\begin{tikzcd} H^*(X)^{\otimes n} \arrow[rr, "\langle\rangle_{0,n,\beta}"] \arrow[dr, "I_{0,n,\beta}"'] & & \mathbb{Q} \\ & H^*(\overline{\mathcal{M}}_{0,n}) \arrow[ur, "\int"'] & \end{tikzcd}$$

where $I_{0,n,\beta}$ is defined by:

$$I_{0,n,\beta}(\alpha_1 \otimes \ldots \otimes \alpha_n) := stab_*(\prod ev_i^*(\alpha_i))$$

(the map *stab* is defined in 3.13).

The correlators of this sections I_n are then obtained by taking the sum over all $\beta \in H_2(X,\mathbb{Z})$ of $I_{0,n,\beta}$. Arguing as in 5.4, one can see that given $\alpha_1, \ldots, \alpha_n$ this sum is always finite, so that:

$$I_n := \sum_{\beta \in H_2(X,\mathbb{Z})} I_{0,n,\beta}$$

is a good definition.

A last observation. This section motivates us to use the correlators $I_{0,n,\beta}$ instead of the functions $\langle\rangle_{0,n,\beta}$. Indeed, what we have done is to abstract the result of Lemma 4.18. The formula in Lemma 4.18 is then expressed by the (equivalent) equations 6.1 and 6.2.

In genus 0 though, the information contained in the correlators and in the functions $\langle\rangle_{0,n,\beta}$ is equivalent, (this is a consequence of Theorem 6.3). Two more important reasons to prefer the correlators instead of the functions $\langle\rangle_{0,n,\beta}$ are:

(1) The theory can be extended to include the so-called gravitational descendents (see [HKKPTVVZ03, 26.9]). In this note we will not discuss the importance of this.
(2) In higher genus the correlator functions $I_{g,n,\beta}$ carry strictly more informations than $\langle\rangle_{g,n,\beta}$. In the next section we will construct Gromov–Witten theory using the approach of correlators.

7. Axioms for Gromov–Witten theory

We now present the axiomatic theory of Gromov–Witten classes. Historically, Kontsevich discovered his formula (that we treated in Section 1) around 1993. This motivated him to write the paper [KM94], where they build the theory that allows the formulation of *reconstruction theorems*, results that permit the reconstruction of all datas of the theory out of fewer datas. The first reconstruction theorem for $\mathbb{P}^2$ becomes the result that we saw in Section 1, allowing one to reconstruct all the N_d's from the knowledge of N_1: the number of lines passing through two points in the plane. One important point to recall is that, although the theory is stated in full generality, the results are described only for the restriction to the genus 0 case (that we called Cohomological Field Theory, and they call tree level system).

The result reconstruction theorems of Kontsevich and Manin are formal, because they count the the number of Gromov–Witten invariants, so one needs a result like 4.12 to obtain enumerative datas. As for the whole theory in arbitrary genus, the existence of sensible Gromov–Witten theories was proved a few years later the paper [KM94] was published. The key technical point for these theories is the existence of a virtual fundamental class for the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The presentation of this section borrows heavily from [A96, Section 3.7]. We fix X a smooth projective variety (from now on, the more restrictive homogeneity assumption is dropped).

Definition 7.1. ([KM94, Definition 2.2]) A system of Gromov–Witten classes on X is the datum, for all g, n such that $2g + n \geq 3$ and $\beta \in H_2(X, \mathbb{Z})$, of linear maps:

$$I^X_{g,n,\beta} : H^*(X)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

satisfying a series of axioms (GW0), ... (GW9).

Remark 7.2. The axiom (GW9) is taken from [CK99, Chapter 7.3], and it motivates the adjective *invariants*, in that the Gromov–Witten classes are deformation invariants. Even if this axiom is not explicit in the requests of Kontsevich and Manin, it is a desirable property that is then obtained for instance with the construction of Virtual Fundamental class that we shall see in the following section.

Before stating the axioms, we want to make a couple of considerations to motivate some of them. We have the following diagram of maps in mind:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[r, "ev"] \arrow[d, "forg"] & X^n \\ \mathfrak{M}_{g,n} & \end{tikzcd}$$

Where $\mathfrak{M}_{g,n}$ is the stack of $n-$pointed prestable curves of genus g (it is an Artin stack, nonseparated, not of finite type, see [B97], [BM96]). This last stack is unobstructed (hence smooth) of dimension $3g-3+n$, and contains $\overline{\mathcal{M}}_{g,n}$ as an open (but still proper!) substack. We can compute the expected dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$: it will be $3g - 3 + n$ plus the dimension of the fibers of $forg$. If (C, x_i, f) is a point of $\overline{\mathcal{M}}_{g,n}(X, \beta)$, the dimension of the fiber $forg^{-1}(C, x_i)$ is determined by deformation theory, as the tangent space to deforming the map $f : C \to X$ by leaving C, x_i fixed. Deformation theory shows us that this space is obstructed. Its tangent space is $H^0(C, f^*(T_X))$, and a natural choice for an obstruction space is $H^1(C, f^*(T_X))$. Thus our expected dimension will be:

$$expdim := 3g - 3 + n + \chi(f^*(T_X)) = 3g - 3 + n + \dim(X)(1 - g) + \int_\beta c_1(T_X) =$$

$$(\dim(X) - 3)(1 - g) + \int_\beta c_1(T_X) + n$$

now we state the axioms. We start with the group of axioms that we consider more formal: (GW1), (GW6), (GW7):

- (GW1): S_n-equivariance. If $\sigma \in S_n$, then

$$I_{g,n,\beta}(\alpha_1 \otimes \ldots, \alpha_n) = I_{g,n,\beta}(\alpha_{\sigma(1)} \otimes \ldots, \alpha_{\sigma(n)})$$

- (GW6): Splitting axiom. For all σ partition of g and $[n]$ in $g_1 + g_2 = g$, $A \sqcup B = n$, we have a (gluing) map:
$$\phi_\sigma : \overline{\mathcal{M}}_{g_1,A+1} \times \overline{\mathcal{M}}_{g_2,B+1} \to \overline{\mathcal{M}}_{g,n}$$
and the axiom asks for any partition ϕ_σ, the following to hold:
$$\phi_\sigma^* \circ I_{g,n,\beta}\left(\bigotimes_{i=1}^{n} \alpha_i\right) = \sum_{\beta_1+\beta_2=\beta} I_{g_1,A+1,\beta_1} \otimes I_{g_2,B+1,\beta}\left(\bigotimes_{i\in A} \alpha_i \otimes \Delta \otimes \bigotimes_{j\in B} \alpha_j\right)$$
where $\Delta \in H^*(X)^{\otimes 2}$ is the diagonal class under the Künneth decomposition.
- (GW7): Genus reduction. There are maps $\psi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, the axiom asks for each such map the following to hold:
$$\psi^* \circ I_{g,n\beta}(\ldots) = I_{g-1,n+2,\beta}(\ldots \otimes \Delta)$$
Where Δ is as in the previous axiom.

These three axioms give $H^*(X)$ the structure of an *algebra over the modular operad* $H^*(\overline{\mathcal{M}}_{g,n})$ (see [GK98] for the definition of modular operad and algebra over it, it is a generalization of the structure of algebra over the operad $H^*(\overline{\mathcal{M}}_{0,n})$ that we have observed in 6.4). These three axioms for the genus 0 case become the axioms for a Cohomological Field theory or a tree-level system, which we have studied in the previous section.

Now we see the remaining, "geometric" axioms:

- (GW0): Effectivity. If β is not an effective class, then $I_{g,n,\beta}$ is identically zero.
- (GW2): Degree. The degree of $I_{g,n,\beta}$ is equal to:
$$2((g-1)\dim(X) - \int_\beta c_1(T_X))$$
This is motivated by the fact that the fibers of $forg$ have (complex) dimension $-(g-1)\dim(X) + \int_\beta c_1(T_X)$.
- (GW3): Fundamental class insertion. If $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgets the last point, we have:
$$I_{g,n+1,\beta}(\ldots \otimes 1_X) = \pi^* I_{g,n,\beta}$$
moreover, in the special case of $g = 0, n = 3$:
$$I_{0,3,\beta}(\alpha_1, \alpha_2, 1_X) = \begin{cases} \int_X \alpha_1 \cup \alpha_2 & \text{if } \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$
- (GW4): Divisor insertion. With the same notation of the previous axiom, if $\gamma \in H^2(X)$ is the class of a divisor, then:
$$\pi_* I_{g,n+1,\beta}(\ldots \otimes \gamma) = I_{g,n,\beta} \int_\beta \gamma$$

- (GW5): Mapping to a point. This says what happens when $\beta = 0$. Let $\mathbb{E} := \pi_*(\omega_\pi)$ be the Hodge bundle. Then, if we let p_1 and p_2 be the two projection maps:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\ \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

and $E := p_1^*(\mathbb{E}^\vee) \otimes p_2^*(T_X)$, the axiom fixes the value of $I_{g,n,0}$ to be:

$$I_{g,n,0}(\alpha_1 \otimes \ldots \otimes \alpha_n) = p_{1*}\left(p_2^*\left(\prod \alpha_i\right) \cup c_{top}(E)\right)$$

Exercise 7.3. *We can make sense of the previous axiom in the following way. The dimension of $\overline{\mathcal{M}}_{g,n}(X,0)$ is simply $3g-3+n+\dim(X)$, so we are off from the expected dimension by $\dim(X)g$. Indeed the obstruction at each point (C, x_i, f) equals $H^1(C, f^*(T_X)) = H^1(C, \mathcal{O}_C) \otimes T_{f(C)}C$ since f is a constant map as $\beta = 0$. Moreover, by elementary Serre-Duality, $H^1(C, \mathcal{O}_C) = H^0(C, \omega_C)^\vee$, where ω_C is the dualizing sheaf on C. One can easily see that $E = p_1^*(\mathbb{E}^\vee) \otimes p_2^*(T_X)$ is the sheaf of obstruction patched together on $\overline{\mathcal{M}}_{g,n} \times X$ (for instance the fiber of it at a point C, x_i, f is $H^0(C, \omega_C)^\vee \otimes T_{f(C)}C$).*

And now the last two axioms:

(1) (GW8): Motivic axiom. The correspondences $I_{g,n,\beta}$ can be described in terms of elements in $A_*(\overline{\mathcal{M}}_{g,n} \times X^n)$, the Chow group. Indeed, we consider the diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r, "ev"] \arrow[d, "stab"] & X^n \\ \overline{\mathcal{M}}_{g,n} & \overline{\mathcal{M}}_{g,n} \times X^n \arrow[l, "q"] \arrow[u, "p"'] \end{tikzcd}$$

The axiom requires the existence of classes $c_{g,n,\beta} \in A_*(\overline{\mathcal{M}}_{g,n} \times X^n)$ and the following equalities:

$$I_{g,n,\beta} = q_*(p^*() \cap c_{g,n,\beta})$$

(2) (GW9): Deformation axiom. Suppose $\mathcal{X} \to T$ is a smooth projective map over a connected base T, and we call $\mathcal{X}_t$ the fibers of the map over $t \in T$ a geometric point. Then we have a family of maps:

$$I^{\mathcal{X}_t}_{g,n,\beta_t} : H^*(\mathcal{X}_t) \to H^*(\overline{\mathcal{M}}_{g,n})$$

If α_i are locally constant sections of $H^*(\mathcal{X}_t)$ and β_t is a locally constant section of $H_2(\mathcal{X}_t, \mathbb{Z})$, then the axiom imposes:

$$I^{\mathcal{X}_t}_{g,n,\beta_t}(\alpha_1 \otimes \ldots \otimes \alpha_n)$$

to be constant.

Example 7.4. (Ruan's example) In [R94], the Deformation invariance is exploited to show an example of two diffeomorphic manifolds that are not deformation invariant. One argues as follows. Take two algebraic surfaces V and W that are homeomorphic but such that V is minimal and W is not. These are nondiffeomorphic, but $V \times S^2$ and $W \times S^2$ are diffeomorphic, and Ruan gives lots of examples (starting with V equal to the Barlow surface and W equal to the 8-point blowup of $\mathbb{C}P^2$) where the diffeomorphism can be arranged to intertwine the first Chern classes, whence by a theorem of Wall the almost complex structures are isotopic. However, the distinction between the GW invariants between V and W (which holds because V is minimal and W is not) survives to $V \times S^2$ and $W \times S^2$, so that they are not deformation equivalent. (source: Mathoverflow).

Now the program to pursue enumerative geometry on X, following this approach, becomes:

(A) To construct a system of Gromov–Witten classes on X. We will see an example of system of Gromov–Witten classes constructed in three papers by Behrend [B97], Behrend–Fantechi [BF97] and Behrend–Manin [BM96].
(B) To compute the Gromov–Witten classes (and therefore, the Gromov–Witten invariants $\langle\rangle_{g,n,\beta}$);
(C) To prove (or to study) enumerativity, like in 4.12.

Example 7.5. The restriction of the axioms to the case of genus 0 curves: in this case we have a system of Gromov–Witten classes on X a smooth projective homogeneous variety constructed as in the previous sections.

Example 7.6. Again if $g = 0$, a trivial example of a system of Gromov–Witten classes is given by the zero map for all $I_{0,n,\beta}$, unless $\beta = 0$. In this case the mapping to a point axiom fixes:

$$I_{0,n,0}(\alpha_1 \otimes \ldots \otimes \alpha_n) = 1_{\overline{\mathcal{M}}_{0,n}} \int \alpha_1 \cup \ldots \cup \alpha_n$$

This shows in particular that the system of Gromov–Witten classes is not unique. Intuitively, given a system of Gromov–Witten classes, one can also rescale them by using appropriate constants to obtain another system of Gromov–Witten classes.

Let us now see the first reconstruction theorem due to Kontsevich-Manin. Its motivation is the reconstruction theorem we have studied in Section 1. We strongly suggest the reader to study its proof.

Theorem 7.7. *(*[KM94, Theorem 3.1]*) Let X be a smooth projective variety, for which there is a system of genus* 0 *Gromov–Witten classes $\{I_{0,n,\beta}\}$. Suppose furthermore that $H^*(X)$ is generated by $H^2(X)$. Then the system of Gromov–Witten classes $I^X_{0,n,\beta}$ can be* *uniquely reconstructed from the set of datas:*

$$\left\{ I_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3)| \; - \int_\beta c_1(T_X) \leq 2\dim(X)+1,\ \sum |\gamma_i| = 2(\int_\beta c_1(T_X) + \dim(X),\ |\gamma_3| = 2 \right\}$$

8. An introduction to the Virtual fundamental class

In this section we construct the system of Gromov–Witten classes defined in three papers by Behrend [B97], Behrend–Fantechi [BF97] and Behrend–Manin [BM96].

The idea is to construct the system of Gromov–Witten classes in the following way:

$$I^X_{g,n,\beta}(\alpha_1 \otimes \dots \alpha_n) = stab_* \left(ev^* \left(\bigotimes \alpha_i \right) \cap [\overline{\mathcal{M}}_{g,n}]^{virt} \right)$$

where $[\overline{\mathcal{M}}_{g,n}]^{virt}$ is a virtual fundamental class and the maps are, as usual:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r, "ev"] \arrow[d, "stab"] & X^n \\ \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

The idea is: the usual fundamental class does not do the job (we do not have Poincaré duality on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ as it is not smooth), but there is a valid "virtual" substitute for it, which makes the moduli space behave af if it were smooth. (see [CK99, 7.1.2] for an intuitive but more precise and detailed explanation of this).

The techniques needed for this construction are: deformation theory, intersection theory, algebraic stacks, the cotangent complex. We give just a short presentation, based on [B99]. An other good reference is a series of online lectures given by Fantechi at SISSA [F10].

So we give now an overview of the construction. It is based on two steps (see [B99, Section 3]):

(A) The construction of the Intrinsic normal Cone ([BF97]) for any "space" X. This is a construction intrinsic to the space (scheme, Deligne–Mumford stack ...);

(B) The *choice* of an obstruction theory for X. In particular, since our space is $\overline{\mathcal{M}}_{g,n}(X,\beta)$ there is a natural obstruction theory given by the moduli problem the space itself solves. Note that this is not intrinsic on the space, and therefore the virtual fundamental class we will construct depends upon the choice of this obstruction theory.

As for the point (A), if we are given a space M that can be locally embedded in a smooth space Y via a map i, then we have that $i^*(T_Y)$ acts on the normal cone of M in Y, $C_M Y$. We can take the stack quotient:

$$[C_M Y / i^*(T_Y)]$$

this is a cone (Artin) stack of pure dimension 0, and Behrend-Fantechi show that this stack is independent of the choice of i. So the local construction glue to a global cone stack $\mathcal{C}_M$, again an Artin stack of dimension 0. There is also a relative version of this construction over an algebraic stack S, where one starts with:

$$\begin{tikzcd} M \arrow[rr, "i"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

and Y is smooth over S, and then the outcome is an Artin stack $\mathcal{C}_{M|S}$, of dimension equal to $\dim(S)$.

In our special case, we take $M := \overline{\mathcal{M}}_{g,n}(X, \beta)$ and $S := \mathfrak{M}_{g,n}$, and thus get an intrinsic normal cone $\mathcal{C}_{\overline{\mathcal{M}}_{g,n}(X,\beta)|\mathfrak{M}_{g,n}}$.

As for point (B), let us consider the diagram:

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,n}(X, \beta) & \xrightarrow{f} & X \\ & & \\ \overline{\mathcal{M}}_{g,n}(X, \beta) & & \end{array}$$

where $\overline{\mathcal{C}}_{g,n}(X, \beta)$ is the universal curve, and consider:

$$R\pi_* f^* T_X \in D^{[0,1]}(\overline{\mathcal{M}}_{g,n}(X, \beta))$$

(an element of the bounded derived category of coherent sheaves on the DM stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$). Behrend–Fantechi show that it is possible to find a complex of vector bundles on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ that is quasiisomorphic to $R\pi_* f^* T_X$:

$$R^0\pi_* f^* T_X \to E_0 \to E_1 \to R^1\pi_* f^* T_X$$

One can then construct the quotient as a vector bundle (Artin) stack: $[E_1/E_0]$, and this depends only on $R\pi_* f^* T_X$ as an element in the derived category. The fact that $R\pi_* f^* T_X$ is an *obstruction theory* for $\overline{\mathcal{M}}_{g,n}(X, \beta)$, says precisely that there is a closed embedding:

$$\mathcal{C}_{\overline{\mathcal{M}}_{g,n}(X,\beta)|\mathfrak{M}_{g,n}} \subset [E_1/E_0]$$

We have thus the following diagram:

$$\begin{tikzcd} \mathcal{C}_{\overline{\mathcal{M}}_{g,n}(X,\beta)|\mathfrak{M}_{g,n}} \arrow[r] & {[E_1/E_0]} \arrow[d, shift right] \\ & \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[u, bend right, "0"'] \end{tikzcd}$$

Where 0 is the zero section of the vector bundle (Artin) stack. So one defines:

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{virt} := 0^* \left([\mathcal{C}_{\overline{\mathcal{M}}_{g,n}(X,\beta)|\mathfrak{M}_{g,n}}] \right)$$

With this construction, one has then the Theorem:

Theorem 8.1. *([B97]) The definition:*

$$I_{g,n,\beta}(\alpha_1 \otimes \ldots \otimes \alpha_n) := stab_* \left(ev^*(\alpha) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt} \right)$$

defines a system of Gromov–Witten classes as defined in the previous section.

Remark 8.2. The original definition of a virtual fundamental class by Behrend, Behrend–Fantechi was slightly more complicated, as a pull–back map from an Artin stack ($[E_1/E_0]$) was not defined. This is nowadays available as a consequence of Kresch' thesis [K99].

9. Orbifold cohomology

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