Invariants of homology spheres and trivial cocycles on the Torelli group

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Let Σ_g be an oriented surface of genus g with a little disc D_2 on it, and $\mathcal{M}_g$ its mapping class group. i.e. $\mathcal{M}_g = \pi_0(\mathrm{Diff}(\Sigma_g; \mathrm{rel}.D_2))$. Fix a Heegaard splitting of the oriented sphere $\mathbf{S}^3$, that is a decomposition of $\mathbf{S}^3$ into two handlebodies of genus g, $S^3 = H_g \cup_{\iota_g} -H_g$, where ι_g is a diffeomorphism of Σ_g along which the handlebodies are glued together. Then we get a map from $\mathcal{M}_g$ to the set $\mathcal{V}(3)$ of diffeomorphism classes of closed oriented 3-manifolds, by twisting the Heegaard splitting:

$$\begin{array}{rcl} \mathcal{M}_g : & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & \mathbf{S}^3_\phi = H_g \cup_{\iota_g \circ \phi} -H_g \end{array}$$

The little disc allows to define a stabilization map $\mathcal{M}_g \hookrightarrow \mathcal{M}_g$ which is compatible with the above construction. Furthermore, restricting the diffeomorphisms of either handlebodies to their boundary we get two subgroups $\mathcal{A}_g, \mathcal{B}_g \subset \mathcal{M}_g$, namely those classes which contain diffeomorphisms that extend to the exterior handlebody $-H_g$ and to the interior handlebody H_g respectively. The fundamental theorem on Heegaard splittings, due to Singer, now says:

Theorem 0.1 *The map*

$$\begin{array}{rcl} \lim_{g\to\infty} \mathcal{B}_g M_g/\mathcal{A}_g : & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & \mathbf{S}^3_\phi = H_g \cup_{\iota_g \circ \phi} -H_g \end{array}$$

is well defined and is a bijection.

Let $\mathcal{T}_g$ be the Torelli group, that is the kernel of the canonical map $\mathcal{M}_g \to \mathrm{Aut}(H_1(\Sigma_g, \mathbf{Z}))$. Twisting the Heegaard splitting by maps in the Torelli group produces homology 3-spheres, and all are obtained in this way. So, if $S(3)$ denotes the subset of $\mathcal{V}(3)$of homology spheres, we get an induced bijection

$$\begin{array}{rcl} \lim_{g\to\infty} T_g/\sim: & \longrightarrow & S(3) \\ \phi & \longmapsto & \mathbf{S}^3_\phi = H_g \cup_{\iota_g \circ \phi} -H_g \end{array}$$

where $\sim$ is the equivalence relation on $\mathcal{T}_g$ induced by the double cosets in $\mathcal{M}_g$. Any invariant of homology spheres with values in the integers, say F , can be viewed via this bijection as a sequence of compatible maps F_g on the Torelli groups $\mathcal{T}_g$ that are constant on equivalence classes. There is now an associated family of trivial 2-cocycles defined by

$$C_{F_g}(\phi, \psi) = F_g(\phi\psi) - F_g(\phi) - F_g(\psi)$$

In this talk we characterize those trivial cocycles that come from invariants and we explain how to recover the invariant out of the 2-cocycles. For instance we show that only the trivial map $F = 0$ is associated to the 0-cocycle, or equivalently that no non-trivial invariant is a group homomorphism. The characterization turns out to be purely cohomological. In particular we show how to construct the Casson invariant in a purely algebraic and elementary way, and characterize it as being the unique invariant whose cocycle C_g is a pull-back of a cocycle defined on an abelian quotient of the Torelli group. We will also show how from this constructon one can prove that some invariants that are induced from cocycles defined on automorphisms of free nipotent groups never vanish on the Johnson filtration of the set $S(3)$.