On the complexity of maximizing the minimum Shannon capacity in wireless networks by joint channel assignment and power allocation

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Abstract

We consider wireless telecommunications systems with orthogonal frequency bands, where each band is referred to as a channel, e.g., orthogonal frequency-division multiple access (OFDMA). For a snap-shot in time, a joint channel assignment and power allocation optimization problem is presented, both in downlink and in uplink, where the objective is to maximize the minimum total Shannon capacity of any mobile user in the system. The corresponding decision problem is proved to be NP-hard. Another proof gives that for any constant $\rho > 0$, a sufficiently large number of channels ensures that the optimization problem is not ρ-approximable, unless P is equal to NP. If assuming that the channel allocation is known, the remaining power allocation optimization problem also has this inapproximability property. This power allocation optimization problem formulation is not convex in general. However, if each transmitter only is allowed to use one single channel, then it is shown that every KKT point is a global optimum.

1. Introduction

This paper concerns wireless telecommunications systems consisting of base stations and mobile users, either in downlink or in uplink. A constant size of the frequency band is available for each base station. This bandwidth of each base station is divided into orthogonal parts of the same size, referred to as channels. It is assumed that the communication takes place over these orthogonal channels, e.g., orthogonal frequency-division multiple access (OFDMA). A static multi-user problem is considered, where a snap-shot in time has been taken. Each mobile user is to be connected with a certain base station and to be given one or more of its channels, where each channel of a base station only can be used by at most one mobile user. In this setting, we define a cell to be a base station and the mobile users using its channels. The orthogonality between the channels implies that the key property to analyse is the inter-cell interference, since the interference within a cell does not need to be considered. Users sharing the same channel will be connected to different

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base stations, and depending on their respective powers used to transmit, the interference will vary. In this setting we aim to maximize the minimum total Shannon capacity of any mobile user.

Given a set of base stations B, a set of mobile users M, and a set of channels C, we aim to locate a cell configuration, distribute the channels within each cell, and decide the amount of power to transmit on each channel, such that it is advantageous for the overall system performance.

One can increase the complexity by also considering a controller which, e.g., makes decision on which channel each base station can use, see, e.g., [13]. No such radio network controllers are considered within this paper.

An outline of this paper is as follows, from section 2 to section 4 a joint channel assignment and power allocation optimization problem is studied, while from section 5 to section 7 the power allocation optimization problem is studied. Finally, summary and conclusion are given in section 8.

In section 2 a system model is described, and the joint channel assignment and power allocation optimization problem and also the corresponding joint channel assignment and power allocation decision problem is formulated. As fare as we are aware, the joint channel and power allocation optimization problem has not been studied before. In section 3 it is proved that the joint channel assignment and power allocation decision problem is NP-hard, both in downlink and in uplink. A similar problem formulation, where the objective is to minimize the total power consumption [1], is proved to be NP-hard in [2]. In section 4 we prove that for any positive constant ρ, there is a sufficiently large number of channels $|C|$, such that the joint channel and power allocation optimization problem is not ρ-approximable, unless P is equal to NP.

In section 5 the power allocation optimization problem is formulated, under the assumption that a feasible channel assignment is given. In, e.g., [12], a channel allocation approach in downlink for the multi-cell system is given. To our knowledge, the entire power allocation optimization problem has not been formulated before, though a restricted version of it has been studied in, e.g., [15]. Section 6 contains some general properties of the power allocation optimization problem. One important result here is that for any positive constant ρ, there exists a sufficiently large number of channels $|C|$, such that the power allocation optimization problem is not ρ-approximable, unless P is equal to NP. Another property concerns when every mobile user must have the same total Shannon capacity in a local maximizer. It is also shown that at least one source will transmit with full power in a local maximizer, which also is proved to be applicable when the objective is to maximize the total Shannon capacity. This objective, i.e., to maximize the total rate, is another often used objective, see, e.g., [6] and [11]. In section 7 some convexity properties of the power allocation optimization problem are investigated. From Yates, [15], it is known that the setting where each transmitter only can use one channel has a unique optimal value, which each mobile user can obtain, and there also are algorithms on how to obtain the optimal value. Our contribution is to show that any first-order KKT point of this optimization problem is a global maximizer. This is despite that the power allocation optimization problem, where each transmitter only is allowed to use one channel, is not convex in general.

2. System model and problem formulation

In this section the system model is described, and from this model a joint channel assignment and power allocation optimization problem is formulated. To abbreviate, the joint channel assignment and power allocation optimization problem is also referred to as the *overall optimization problem*.

Given a set of sources S and a set of destinations D, we model that sources use power to transmit information to destinations in a network. In downlink the source nodes are the base stations B and the destination nodes are the mobile users M, while in uplink it is the other way around. To be able to represent both downlink and uplink simultaneously, the sets S and D are sometimes used instead of the sets B and M.

Introduce a real non-negative variable p_{ik}, which denotes the power that source $i \in S$ use to transmit information on channel $k \in C$. Also introduce the binary variable x_{ijk}, which indicates if information can be sent by source $i \in S$ to destination $j \in D$, on channel k, where

$$x_{ijk} = \begin{cases} 1, & \text{if } i \in S \text{ can send information to } j \in D \text{ on channel } k \in C, \\ 0, & \text{otherwise.} \end{cases}$$

Let $y_{ij} \in [0, 1]$, $i \in S$, $j \in D$, which due to constraints will behave as a binary variable, where

$$y_{ij} = \begin{cases} 1, & \text{if } x_{ijk} = 1 \text{ on any channel } k \in C, \\ 0, & \text{otherwise.} \end{cases}$$

To measure the quality of the signal, some well known concepts are introduced. The effective *path gain*, g_{ij}, is the proportion of the power that source $i \in S$ uses that will reach destination $j \in D$. Within this paper it is assumed that the path gain vector g, of the entire system is known by each source $i \in S$ and each destination $j \in D$. It holds that

$$g_{ij} \geq 0, \quad i \in S, j \in D.$$

The *signal to interference and noise ratio* (SINR) is a ratio of the desired signal over the undesired disturbance, which for source i, destination j and channel k, is given by

$$\mathrm{SINR}_{ijk} = \frac{g_{ij} p_{ik}}{\sigma_j^2 + \sum_{m \in S \setminus \{i\}} g_{mj} p_{mk}}, \quad i \in S, j \in D, k \in C, \tag{2.1}$$

where $\sigma_j^2 > 0$ is the interior noise at destination j, which is assumed to be known. The *Shannon capacity* of the mobile user on arc (i, j), i.e., source i and destination j, on channel k, is given by

$$\eta_{ijk} = W \log(1 + \mathrm{SINR}_{ijk}), \quad i \in S, j \in D, k \in C, \tag{2.2}$$

where W is a positive scalar given by the size of the frequency band, i.e., in our setting W is a known constant. Note that the Shannon capacity η_{ijk} is related to the mobile user on arc (i, j), whether it is receiving in downlink or transmitting in uplink. The total Shannon capacity of the mobile user on arc (i, j) is given by

$$\eta_{ij} = \sum_{k \in C} \eta_{ijk}, \quad i \in S,\, j \in D, \tag{2.3}$$

where η_{ijk} is given in (2.2).

Let us introduce a utility function which aims for good quality of the signals, for all mobile users in the system. In this paper, our objective is to maximize the minimum total Shannon capacity η_{ij}, given in (2.3).

Before formulating the joint channel assignment and power allocation optimization problem, some constraints ensuring that the system behaves as expected is needed. One constraint ensures that each mobile user communicates with exactly one base station, while each base station in general can transmit to many mobile users simultaneously. Another constraint ensures that each base station uses each channel $k \in C$ at most once. Finally there is a power restriction on each source $i \in S$, ensuring that the total power used is at most the upper bound power consumption constant $P_{S,i}^{\max}$.

From the introduced setting above, the joint channel assignment and power allocation optimization problem, in downlink, is given by

$$\begin{aligned}
&\underset{\eta, p_{ik}, x_{ijk}, y_{ij}}{\text{maximize}}\ \eta && \text{(2.4a)}\\
&\text{subject to } y_{ij}\eta \le \sum_{k \in C} x_{ijk}\eta_{ijk}, \quad i \in S,\, j \in D, && \text{(2.4b)}\\
&\sum_{i \in B} y_{ij} = 1, \quad j \in M, && \text{(2.4c)}\\
&\sum_{j \in M} x_{ijk} \le 1, \quad i \in B,\, k \in C, && \text{(2.4d)}\\
&\sum_{k \in C} p_{ik} \le P_{S,i}^{\max}, \quad i \in S, && \text{(2.4e)}\\
&x_{ijk} \le y_{ij}, \quad i \in S,\, j \in D,\, k \in C, && \text{(2.4f)}\\
&0 \le y_{ij} \le 1, \quad i \in S,\, j \in D, && \text{(2.4g)}\\
&x_{ijk} \in \{0, 1\}, \quad i \in S,\, j \in D,\, k \in C, && \text{(2.4h)}\\
&p_{ik} \ge 0, \quad i \in S,\, k \in C, && \text{(2.4i)}
\end{aligned}$$

where $S = B$, $D = M$, and η_{ijk} is given in (2.2). If we let $S = M$, $D = B$, and replace equations (2.4c) and (2.4d) with

$$\sum_{j \in B} y_{ij} = 1, \quad i \in M, \tag{2.5a}$$

$$\sum_{i \in M} x_{ijk} \le 1, \quad j \in B,\, k \in C, \tag{2.5b}$$

the corresponding uplink formulation is obtained.

For each of the two optimization problems, (2.4), in downlink and in uplink, a corresponding joint channel assignment and power allocation decision problem is formulated in Problem 2.1.

Problem 2.1. (Joint channel assignment and power allocation) *The joint channel assignment and power allocation problem is defined by the following instance and question.*
Instance: *Parameters P, W, σ, g and the scalar $\bar{\eta}$.*
Question: *Is there a choice of p, x, y and η which gives a feasible solution to the optimization problem* (2.4)*, such that $\eta \geq \bar{\eta}$?*

The complexity of these joint channel assignment and power allocation decision problems, Problem 2.1, are studied in section 3, while the complexity of the joint channel assignment and power allocation optimization problems (2.4) are studied in section 4.

3. The overall decision problems are NP-hard

In this section the complexity of the joint channel assignment and power allocation decision problems, given in Problem 2.1, are proved to be NP-hard.

In order to prove that these decision problems, Problem 2.1, are NP-hard, a reduction is made from the graph K-colorability problem.

Problem 3.1. (Graph K-colorability) *The K-colorability problem is defined by the following instance and question.*
Instance: *Graph $G = (V, E)$, positive integer $K \leq |V|$.*
Question: *Is G a K-colorable graph, i.e., does there exist a function $f : V \rightarrow \{1, 2, ..., K\}$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$?*

The graph K-colorability problem is one of the classical problems known to be NP-complete, see, e.g., [7]. This result was originally given by Karp [9], and is summarized in the following theorem.

Theorem 3.1. (Karp [9]) *The graph K-colorability problem,* Problem 3.1*, is NP-complete for $K \geq 3$.*

To prove that the decision problems, Problem 2.1, are NP-hard, we take an arbitrary instance of the graph K-colorability problem and create an instance for Problem 2.1, such that solving this created instance also implies solving the graph K-colorability problem.

Given an arbitrary graph G, our construction of the problem instance is divided into five stages. For clarity the construction is also illustrated graphically, originating from a given graph G, illustrated in Figure 1, while the stages are presented in Figure 2, 3 and 4.

1. For each vertex $i \in V$ create a base station $\bar{B}_i$ and a mobile user $\bar{M}_i$. Let the path gain between base station $\bar{B}_i$ and mobile user $\bar{M}_i$ be denoted by $\bar{g}$. Define the set $C = \{1, ..., K\}$, and let $p_{ik}^{\bar{S}}$ denote the power variable of source $\bar{S}_i$, on channel $k \in C$. See Figure 2.

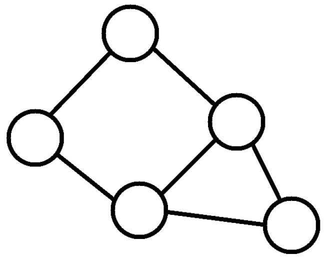

Figure 1: A given graph G.

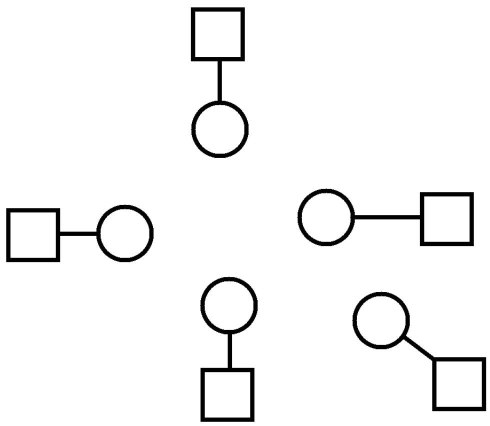

Figure 2: First stage of creating the problem instance. For each node $i \in V$ a source $\bar{S}_i$ is represented by a square, destination $\bar{D}_i$ is represented by a circle, and the path gain between them is illustrated as a link $(\bar{S}_i, \bar{D}_i)$.

2. For each edge $(i, j) \in E$ of the original graph G, let the path gain between base station $\bar{B}_i$ and mobile user $\bar{M}_j$ be δ, and also let the path gain between base station $\bar{B}_j$ and mobile user $\bar{M}_i$ be δ. Define $B = \cup_i \bar{B}_i$ and $M = \cup_i \bar{M}_i$. This stage is illustrated in Figure 3.

3. Once again create a base station $\tilde{B}_i$ and a mobile user $\tilde{M}_i$ for each node $i \in V$. Denote the path gain between base station $\tilde{B}_i$ and mobile user $\tilde{M}_i$ by $\tilde{g}$. Let the path gain between source $\tilde{S}_i$ and destination $\bar{D}_i$ be denoted by $\tilde{G}$. Let variable $p_{ik}^{\tilde{S}}$ denote the transmission power of source $\tilde{S}_i$ on channel $k \in C$. Also define $B = \cup_i \left(\bar{B}_i \cup \tilde{B}_i\right)$ and $M = \cup_i \left(\bar{M}_i \cup \tilde{M}_i\right)$, i.e., we have $S = \cup_i \left(\bar{S}_i \cup \tilde{S}_i\right)$ and $D = \cup_i \left(\bar{D}_i \cup \tilde{D}_i\right)$. See Figure 4, where this stage is illustrated.

4. Let all path gains that have not been defined in stages 1 to 3 be equal to zero. These links are not illustrated graphically as they only correspond to zero links.

5. Introduce optimization problem (2.4), in downlink or in uplink.

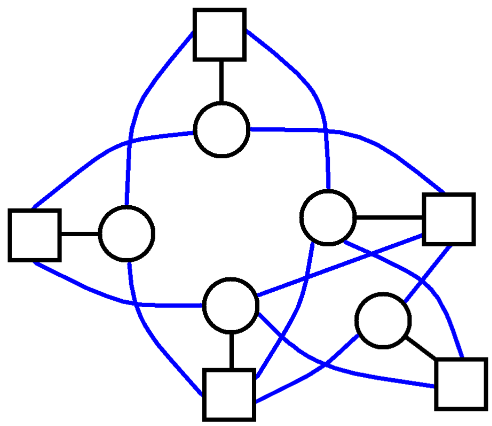

Figure 3: Second stage of creating the problem instance. Introduce the two blue links $(\bar{S}_i, \bar{D}_j)$ and $(\bar{S}_j, \bar{D}_i)$ for each edge $(i, j) \in E$ of the graph G, to illustrate that the path gain is δ between these sources and destinations.

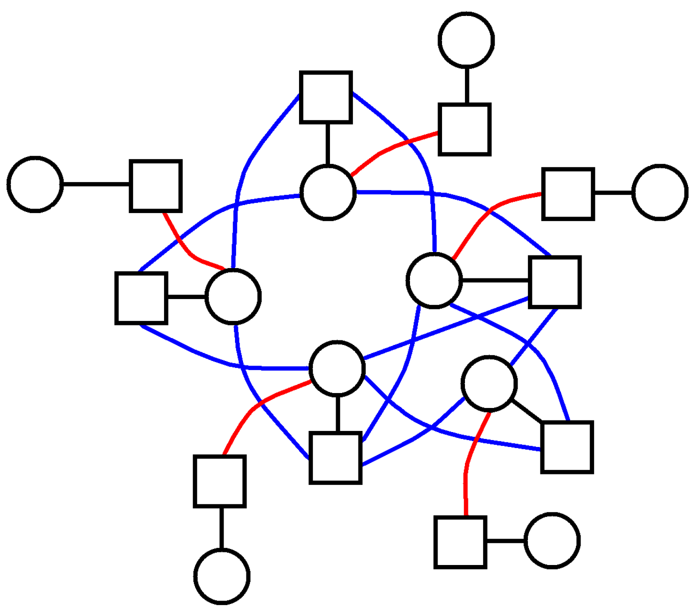

Figure 4: Third stage of creating the problem instance. Source $\tilde{S}_i$ is represented by a square, destination $\tilde{D}_i$ by a circle. The path gain between base station $\tilde{B}_i$ and mobile user $\tilde{M}_i$ is illustrated by a black link $(\tilde{S}_i, \tilde{D}_i)$ and the path gain between source $\tilde{S}_i$ and destination $\bar{D}_i$ is illustrated by a red link $(\tilde{S}_i, \bar{D}_i)$. This is for each node $i \in V$ of the graph G.

In stage 1 the number of colors in the K-coloring problem is made equal to the number of channels in the constructed problem instance, i.e., $|C| = K$. The reason for defining B and M in both stage 2 and 3 is that a similar construction is to be used in Section 4, where stage 3 is omitted. In stage 5 the introduced power variables p_{ik}, $i \in S$, $k \in C$, either correspond to $p_{ik}^{\bar{S}}$ or $p_{ik}^{\tilde{S}}$ depending on if $i \in \bar{S}$ or $i \in \tilde{S}$, where $\bar{S} = \cup_i \bar{S}_i$ and $\tilde{S} = \cup_i \tilde{S}_i$.

Some parameters and constants are needed to complete the construction of the decision problem instance of Problem 2.1. The path gains g are composed by $\bar{g}$, δ, $\tilde{g}$, $\tilde{G}$ and zero, according to the stages 1 to 4. Let the parameters and constants of the problem instances be chosen as follows:

$$P = P_{S,i}^{\max} = 1, \quad i \in S, \tag{3.1a}$$
$$W = 1, \tag{3.1b}$$
$$\sigma^2 = \sigma_j^2 < 1, \quad j \in D, \tag{3.1c}$$
$$\bar{g} > 1, \tag{3.1d}$$
$$\bar{\eta} = \log\left(1 + \frac{\bar{g}}{\sigma^2}\right), \tag{3.1e}$$
$$\delta = 1, \tag{3.1f}$$
$$\tilde{g} = (K-1)\sigma^2 \left(\left(1 + \frac{\bar{g}}{\sigma^2}\right)^{\frac{1}{K-1}} - 1 \right), \tag{3.1g}$$
$$\tilde{G} > \frac{1}{\tilde{p}} \left(\frac{\bar{g}}{\exp\left(\frac{1}{K-1}\left(\bar{\eta} - \log\left(1 + \max\{\frac{\bar{g}}{\sigma^2+1}, \frac{1}{\sigma^2}\}\right)\right)\right) - 1} - \sigma^2 \right), \tag{3.1h}$$

where $\tilde{p}$ denotes the optimal value of the optimization problem

$$\begin{array}{ll} \underset{p \in \mathbb{R}^K}{\text{minimize}} & p_2 \\ \text{subject to} & \sum_{i=1}^K \log\left(1 + p_i \frac{\tilde{g}}{\sigma^2}\right) \geq (K-1)\log\left(1 + \frac{1}{K-1}\frac{\tilde{g}}{\sigma^2}\right), \\ & 0 \leq p_1 \leq p_2 \leq \ldots \leq p_K, \\ & \sum_i p_i \leq 1. \end{array} \tag{3.2}$$

If $p_1 = 0$ in (3.2), then $p_i = 1/(K-1)$, $i = 2, \ldots, K$, according to Lemma A.1 in the appendix. Hence, $p_2 = 0$ is infeasible, and it follows that $\tilde{p} > 0$. The value of $\tilde{g}$ in (3.1g) is chosen such that

$$\bar{\eta} = (K-1)\log\left(1 + \frac{1}{K-1}\frac{\tilde{g}}{\sigma^2}\right), \tag{3.3}$$

which means that a mobile communicating over a link with path gain $\tilde{g}$ must use at least $K-1$ channels to obtain $\bar{\eta}$ as Shannon capacity, and if only $K-1$ channels are used then all available power of the source is needed together with no interference taking place, according to Lemma A.1 in the appendix. The value of $\tilde{G}$ has to be "sufficiently large". Specifically, the value of $\tilde{G}$ in (3.1h) equivalently gives

$$\bar{\eta} > \log\left(1 + \max\left\{\frac{\bar{g}}{\sigma^2+1}, \frac{1}{\sigma^2}\right\}\right) + (K-1)\log\left(1 + \frac{\bar{g}}{\sigma^2 + \tilde{G}\tilde{p}}\right). \tag{3.4}$$

Note that the lower bound on $\tilde{G}$ in (3.1h) is well defined, as $\tilde{p} > 0$. Also note that the path gains of g are non-negative, as $\tilde{g} > 0$ in (3.1g) and $\tilde{G}$ is chosen positive.

The decision problem instances have been constructed. These constructed instances are used to prove Lemma 3.1, which then will be used to verify that the decision problems, Problem 2.1, are NP-hard.

Lemma 3.1. *Let $G = (V, E)$ be a given graph, $K \geq 3$. Then G is K-colorable if and only if the associated instances of* Problem 2.1*, created in the stages* 1 *to* 5*, together with the parameters chosen according to* (3.1)-(3.2)*, have feasible p, x, y and η such that $\eta \geq \bar{\eta}$.*

Proof. We first prove that graph K-colorability implies that the optimal value of the corresponding instance of (2.4), is at least $\bar{\eta}$. Assume that we have a K-colorable graph G, and then construct the corresponding instance of problem (2.4), with parameters and constants chosen according to (3.1)-(3.2). As G is K-colorable, we can associate one channel k_i for each $i \in V$ such that $k_i \neq k_j$, $(i, j) \in E$. Let each source $\bar{S}_i$ transmit with full power, i.e., 1, over the link with path gain $\bar{g}$ on the selected channel k_i. Let each source $\tilde{S}_i$ transmit with power $1/(K-1)$ on each of the $K-1$ channels that remain when k_i is omitted. This channel assignment introduces no interference. The mobiles $\bar{M}_i$ obtain Shannon capacity $\bar{\eta}$, given by (3.1e), as their information is on a channel without interference, with power 1 and path gain $\bar{g}$. Due to the choice of $\tilde{g}$ in (3.1g), it follows from (3.3) and Lemma A.1 that mobiles $\tilde{M}_i$ also obtain Shannon capacity $\bar{\eta}$, as their information is on $K-1$ channels without interference, path gain $\tilde{g}$ and power $1/(K-1)$ on each of these channels. It follows that there is a feasible solution to the created instance of (2.4), with $\eta = \bar{\eta}$, as required.

We now prove that no graph K-colorability implies that the objective value of the corresponding instance of (2.4) is less than $\bar{\eta}$. We do so by showing that an objective function value of $\bar{\eta}$ eventually leads to a contradiction. Let us begin to prove that no transmission can take place over any link with path gain $\tilde{G}$. In this part of the proof the approach differs between downlink and uplink.

Let us first prove this statement in downlink. As discussed after (3.3), the choice of $\tilde{g}$ in (3.1g) implies that at least $K-1$ channels are needed on a link with path gain $\tilde{g}$ if the mobile user of that link is to obtain Shannon capacity $\bar{\eta}$. Hence, the only situation when a link with path gain $\tilde{G}$ can be used for transmission is when exactly $K-1$ channels are used on the corresponding link with path gain $\tilde{g}$. But in this situation, all transmission power of source $\tilde{S}_i$ is used on the $K-1$ channels of the link with path gain $\tilde{g}$, such that the mobile user of that link obtains Shannon capacity $\bar{\eta}$. Therefore no transmission power is left for the link with path gain $\tilde{G}$.

Let us now prove the same statement in uplink. If $\tilde{M}_i$ transmit to $\bar{B}_i$ for any $i \in V$, then there will exist a mobile user $\bar{M}_j$, $j \in V$, which do not transmit to any base station. As this situation is infeasible, no transmission is made over a link with path gain $\tilde{G}$. Therefore it can be concluded that power transmissions only take place over the links with path gain δ, $\bar{g}$ or $\tilde{g}$.

However, links with path gain δ can also be excluded as a candidate. First recall that a link with path gain $\tilde{g}$ has a positive lower bound $\tilde{p}$ on its second smallest power, if the mobile user of that link is to obtain Shannon capacity $\bar{\eta}$. Therefore transmission on a link with path gain $\tilde{g}$ causes interference along the corresponding link with path gain $\tilde{G}$, with at least $\tilde{p}$ on $K-1$ channels. Hence, on a link with path gain δ the total Shannon capacity

$$\eta < \log\left(1+\frac{1}{\sigma^2}\right)+(K-1)\log\left(1+\frac{1}{\sigma^2+\tilde{G}\tilde{p}}\right). \quad (3.5)$$

By the choice of $\tilde{G}$, a combination of (3.4) and (3.5) gives $\eta < \bar{\eta}$, showing that the Shannon capacity is not high enough if transmitting power over a link with path gain δ. Hence, if the Shannon capacity is to be at least $\bar{\eta}$ for every mobile user, then every power transmission is over a link with path gain either $\bar{g}$ or $\tilde{g}$.

For each node $i \in V$ of the original graph, the associated base station $\tilde{B}_i$ communicates with mobile user $\tilde{M}_i$ on at least $K-1$ channels with path gain $\tilde{g}$. For each i, let $\tilde{k}_i$ be the channel on which least power is used, breaking ties arbitrarily. As the graph is not K-colorable, there will be two neighboring nodes $i \in V$ and $j \in V$, where $\tilde{k}_i = \tilde{k}_j$. To simplify notation, denote that channel as $\tilde{k}$, i.e., $\tilde{k} = \tilde{k}_i = \tilde{k}_j$. An upper bound on the Shannon capacity of mobile user $\bar{M}_i$ is then given by

$$\log\left(1+\frac{\bar{g}p^{\bar{S}}_{i\tilde{k}}}{\sigma^2+p^{\bar{S}}_{j\tilde{k}}}\right)+(K-1)\log\left(1+\frac{\tilde{g}}{\sigma^2+\tilde{G}\tilde{p}}\right),$$

and analogously for mobile user $\bar{M}_j$. This upper bound is obtained by assuming that the link with path gain $\tilde{G}$ cause no interference on the channel $\tilde{k}$, and interfere with power $\tilde{p}$ on the remaining $K-1$ channels. An upper bound on the least Shannon capacity of these two mobile users $\bar{M}_i$ and $\bar{M}_j$ on the channel $\tilde{k}$, leads to

$$\underset{\substack{0\le p^{\bar{S}}_{i\tilde{k}}\le 1\\ 0\le p^{\bar{S}}_{j\tilde{k}}\le 1}}{\text{maximize}}\left\{\min\left\{\log\left(1+\frac{\bar{g}p^{\bar{S}}_{i\tilde{k}}}{\sigma^2+p^{\bar{S}}_{j\tilde{k}}}\right),\log\left(1+\frac{\bar{g}p^{\bar{S}}_{j\tilde{k}}}{\sigma^2+p^{\bar{S}}_{i\tilde{k}}}\right)\right\}\right\}. \quad (3.6)$$

The two expressions are equal, since otherwise the objective value of (3.6) could be improved by increasing one power and decreasing the other. As the logarithmic function is strictly increasing, the maximizer is $p^{\bar{S}}_{i\tilde{k}} = p^{\bar{S}}_{j\tilde{k}} = 1$. Hence an upper bound on the total Shannon capacity is given by

$$\log\left(1+\frac{\bar{g}}{\sigma^2+1}\right)+(K-1)\log\left(1+\frac{\bar{g}}{\sigma^2+\tilde{G}\tilde{p}}\right),$$

By (3.4), this upper bound is less than $\bar{\eta}$, which is a contradiction. Hence, by our construction it is not possible to obtain an optimal value greater than or equal to $\bar{\eta}$ when the graph is not K-colorable. ∎

From Lemma 3.1 it is now straightforward to prove that the decision problems, Problem 2.1, in both downlink and in uplink are NP-hard, which is stated in Theorem 3.2.

Theorem 3.2. *The joint channel assignment and power allocation decision problems, given in* Problem 2.1*, are NP-hard.*

Proof. Lemma 3.1 shows that if we can solve the constructed problem instance, given in Problem 2.1, then we also can solve the graph K-colorability problem. As the graph K-colorability problem, given in Problem 3.1, is NP-complete, according to Theorem 3.1, the theorem follows. ∎

4. On the approximability of the overall problem

In this section it is proved that for any positive constant ρ, there exists a sufficiently large number of channels $|C|$, such that the joint channel assignment and power allocation optimization problems (2.4), in downlink and in uplink, are not ρ-approximable, unless P is equal to NP. To prove this statement, a known NP-hard K-colorable graph result by Khot, see [10, Theorem 1.6], is to be used, and is therefore restated in the following theorem.

Theorem 4.1. (Khot [10]) *For all sufficiently large constants K, it is NP-hard to color a K-colorable graph with $K^{\frac{1}{25}(\log K)}$ colors. Moreover this hardness result holds for graphs with bounded degree, in fact graphs with degree at most $2^{K^{O(\log K)}}$.*

The definition of an ρ-approximable algorithm is given in the following definition.

Definition 4.1. *Let A be an optimization maximization problem with positive cost function c, and let $\mathcal{A}$ be an algorithm which, given an instance I of A, returns a feasible solution $f_{\mathcal{A}}(I)$; denote the optimal solution of I by $\hat{f}(I)$. Then $\mathcal{A}$ is called an ρ-approximate algorithm for A for some constant $\rho > 0$ if and only if*

$$c(f_{\mathcal{A}}(I)) \geq \rho c(\hat{f}(I))$$

for all instances I.

Let us now consider an arbitrary K-colorable graph G, $K \geq 3$, and construct a problem instance by using stages 1, 2, 4 and 5 in Section 3. The path gain g is composed by $\bar{g}$, δ and zero, according to the construction in stages 1, 2 and 4 in Section 3. Let the constants within this problem instance be chosen as follows:

$$P_{S,i}^{\max} = 1, \quad i \in S, \tag{4.1a}$$
$$W = 1, \tag{4.1b}$$
$$\sigma_j^2 = 1, \quad j \in D, \tag{4.1c}$$
$$\bar{g} = 1, \tag{4.1d}$$
$$\delta = \frac{1}{K^2}. \tag{4.1e}$$

Given this constructed problem instance, one can prove the following theorem.

Theorem 4.2. *For any constant $\rho > 0$, there exists a positive $N \in \mathbb{R}$ such that there is no ρ-approximate polynomial algorithm of the optimization problem (2.4), in downlink or in uplink, for $K \geq N$, unless P is equal to NP.*

Proof. Let $G = (V, E)$ be a K-colorable graph. Let us construct the problem instance of optimization problem (2.4), either considering downlink or uplink, using stages 1, 2, 4 and 5 in section 3, together with the constants given in (4.1).

With this constructed problem instance, the proof consists of two parts. In the first part it is proved that there exists a solution, independent of K, with positive total Shannon capacity, i.e., $c(\hat{f}(I)) > 0$. So, due to the K-colorability of the graph, a positive objective value, independent of K, can be obtained. The second part of the proof is to verify that for a sufficiently large K, there exists an instance I for which $c(f_{\mathcal{A}}(I))$ can be made arbitrarily small. The theorem follows by combining these two parts of the proof.

Let us begin to prove the first part. As the graph is K-colorable one can associate a channel $k_i \in C$ for each node $i \in V$ such that $k_i \neq k_j$, $(i, j) \in E$. Now if each source $\bar{S}_i$ transmit information by using full power, i.e., 1, over the link with path gain $\bar{g} = 1$ on channel k_i, and no power on its other channels, then the total Shannon capacity is

$$\log(1 + \bar{g}) > 0,$$

for each mobile user, independently of K. Hence, $c(\hat{f}(I)) > 0$, independently of K, for an arbitrary problem instance I, which concludes the first part of the proof.

The second part of the proof is to verify that for some instance I, $c(f_{\mathcal{A}}(I)) \to 0$ as $K \to \infty$. Our first step is to verify that no transmission can take place over a link with path gain δ, if the algorithm is to be ρ-approximable for some constant $\rho > 0$. If a link with path gain δ is used, then the maximum Shannon capacity of the mobile user using the link is $\log(1 + \frac{1}{K^2})$. Hence, the maximum total Shannon capacity is at most

$$K \log\left(1 + \frac{1}{K^2}\right) = \frac{1}{K} + o\left(\frac{1}{K}\right), \tag{4.2}$$

for the mobile user using the link with path gain δ. As the objective function is to maximize the minimum total Shannon capacity of all mobile users, it follows that if a mobile user transmit over a link with path gain δ, then the objective value can be at most the total Shannon capacity of that mobile user. As (4.2) converges to zero as $K \to \infty$, this implies that no transmission can take place over a link with path gain δ, if the algorithm is to be ρ-approximable for some constant $\rho > 0$. Hence, all communication must take place over the links with path gain $\bar{g} = 1$, i.e., by base station-mobile user pairs, where each pair originates from the same node $i \in V$.

As the communication only takes place between base station-mobile user pairs originating from the same node $i \in V$, the communication can be represented as possible colors of the original graph G. The remaining part of the proof is now to construct a coloring of the graph, which for an approximation algorithm gives a positive objective value, but uses to few colors, and by that contradicting the possibility of existence of an ρ-approximable algorithm, for any constant $\rho > 0$, unless P is equal to NP.

To construct a cloring of the graph G, we first introduce a positive scalar

$$\gamma = \sqrt{K^3/T}, \tag{4.3}$$

where $T = K^{\frac{1}{25}(\log K)}$. The choice of γ in (4.3) will turn out to be suitable. Let us also introduce a set Γ_i for each communicating base station-mobile user pair $i \in V$, where Γ_i contains the channels on which transmission takes place with power of at least γ, i.e.,

$$\Gamma_i = \{k \in C : p_{ik} \geq \gamma\}, \quad i \in V.$$

We refer to the members of the set Γ_i as the possible colors of node $i \in V$.

To simplify notation, we let η_{ik}, $i \in V$, $k \in C$, denote the Shannon capacity of the mobile user of the communicating base station-mobile user pair $i \in V$, on channel $k \in C$.

For $\hat{i} \in V$, $k \notin \Gamma_{\hat{i}}$, the Shannon capacity $\eta_{\hat{i}k}$ is bounded, according to

$$\eta_{\hat{i}k} \leq \log(1+\gamma), \quad k \notin \Gamma_{\hat{i}}. \tag{4.4}$$

Hence, if $\Gamma_{\tilde{i}} = \emptyset$ for some $\tilde{i}$, then (4.4) gives an upper bound on the Shannon capacity for each channel. We will return to this setting, but for now, let us instead consider the situation where $\Gamma_i \neq \emptyset$ for all $i \in V$.

As $\Gamma_i \neq \emptyset$, $i \in V$, we can introduce F_{ik} as the number of neighboring nodes of $i \in V$ for which color k is possible, i.e.,

$$F_{ik} = |\{j \in V : k \in \Gamma_j,\, (i,j) \in E\}|, \quad i \in V,\, k \in \Gamma_i. \tag{4.5}$$

Let k_i be the favorite color of node i, defined as the possible color with the least number of neighbors of $i \in V$, breaking ties arbitrarily, i.e.,

$$k_i = \operatorname*{argmin}_{k \in \Gamma_i} F_{ik}, \quad i \in V. \tag{4.6}$$

We can now create a coloring of the given graph G based on the colors k_i by creating copies of the colors when needed, to avoid the conflicts. If F is defined as

$$F = \max_{i \in V} F_{ik_i}, \tag{4.7}$$

then it gives the largest number of conflicts for any node $i \in V$ on its favorite color k_i. Hence, at most $F+1$ copies of any color is needed, and therefore this construction gives a coloring of the graph G that in total contains at most $K(F+1)$ colors.

Let us now consider a node $\hat{i} \in V$ for which $F_{\hat{i}k_{\hat{i}}} = F$. Then $F_{\hat{i}k} \geq F$ for each color $k \in \Gamma_{\hat{i}}$ according to (4.5)-(4.7). Hence, for each $k \in \Gamma_{\hat{i}}$, the corresponding base station-mobile user pair $\hat{i}$ receives interference from at least F other pairs over links with path gain δ, where each of these pairs use power γ or more. Hence, the Shannon capacity $\eta_{\hat{i}k}$, of mobile user $\hat{i}$ on channel k, is bounded as

$$\eta_{\hat{i}k} \leq \log\left(1 + \frac{1}{1 + F\delta\gamma}\right), \quad k \in \Gamma_{\hat{i}}. \tag{4.8}$$

According to Theorem 4.1 there exists, for all sufficiently large constants K, an instance I for which $K(F+1) > T$. For this instance I it follows that $F \geq T/K$, which in conjunction with (4.8) gives

$$\eta_{\hat{i}k} \leq \log\left(1 + \frac{1}{1 + \frac{T}{K}\delta\gamma}\right) \leq \log\left(1 + \frac{1}{\frac{T}{K}\delta\gamma}\right), \quad k \in \Gamma_{\hat{i}}. \tag{4.9}$$

Let us now consider (4.4) and (4.9). The choice of γ in (4.3) gives that their upper bounds become identical. The joint bound, of the Shannon capacity of mobile user $\hat{\imath}$ on a channel $k \in C$, is then given by

$$\eta_{\hat{\imath}k} \leq \log\left(1 + \sqrt{\frac{K^3}{T}}\right), \quad k \in C. \tag{4.10}$$

By summing (4.10) over all channels, an upper bound of the total Shannon capacity of mobile user $\hat{\imath}$ is obtained as

$$\eta \leq K \log\left(1 + \sqrt{\frac{K^3}{T}}\right). \tag{4.11}$$

The choice of $T = K^{\frac{1}{25}(\log K)}$, which was obtained from Theorem 4.1, gives that the upper bound in (4.11) is approximately $K^{\frac{5}{2}-\frac{1}{50}\log(K)}$ for large K. As there for all sufficiently large K exists an instance I where the total Shannon capacity is bounded by (4.11), it follows that $c(f_{\mathcal{A}}(I))$ converges to zero as $K \to \infty$, and the theorem follows. ∎

The result of Theorem 4.2 is for a rather large value of K. However, there should be difficulties when only considering three channels, i.e., $|C| = K = 3$. This is due to the known difficulty of coloring an arbitrary 3-colorable graph, for which it is unknown whether it is a NP-hard problem. The best known algorithm gives a $|V|^{0.2072}$ coloring of the graph [5]. Hence, a configuration with only three channels might not be approximable in general.

5. The power allocation optimization problem

From section 3 and section 4 we know that the joint channel assignment and power allocation problems are difficult. However, we are still aiming for "a good solution" to these problems. Given a feasible channel assignment, it is of interest to be able to solve the remaining power allocation optimization problem efficiently. One possibility is to solve the channel assignment and the power allocation problems interchangeably, which motivates our study of the power allocation optimization problems that will be defined here, and take place in section 6 and section 7.

As mentioned, the power allocation setting is obtained by assuming that a feasible channel assignment is known, i.e., both x_{ijk}, $i \in S$, $j \in D$, $k \in C$, and y_{ij}, $i \in S$, $j \in D$, are known. Hence for each source i we know which destination j it communicates with. This motivates a natural extension of notation by introducing the concept of links (i, j), where a link contains the source $i \in S$ and destination $j \in D$ that are communicating, i.e., $i \in S$ and $j \in D$ such that $y_{ij} = 1$. Let us introduce some sets, which will be used when the power allocation optimization problems are considered. These sets are given in the following definition.

Definition 5.1. *Define the following sets:*

$$
\begin{array}{rcl}
L & = & \{(i,j) : y_{ij} = 1,\, i \in S,\, j \in D\}, \\
L_k & = & \{(i,j) \in L : x_{ijk} = 1\}, \quad k \in C, \\
C_{ij} & = & \{k \in C : x_{ijk} = 1\}, \quad (i,j) \in L, \\
C_i & = & \cup_{j \in D:\,(i,j)\in L} C_{ij}, \quad i \in S.
\end{array}
$$

Note that for a given channel assignment, the sets in Definition 5.1 are known. If $L_{\hat{k}} = \emptyset$ for some channel $\hat{k}$, then we can simply exclude that channel by letting $C \leftarrow C\backslash\{\hat{k}\}$, since no user transmit information on that channel anyway. If $C_{\hat{i}\hat{j}} = \emptyset$, for some link $(\hat{i},\hat{j}) \in L$, then the objective value is trivially zero, since we maximize the minimum Shannon capacity. Hence, for the power allocation optimization problem to be *well-posed*, we will assume that the two conditions, given in Definition 5.2, are fulfilled.

Definition 5.2. *For the power allocation optimization problem to be well-posed, the given channel assignment must imply that $L_k \neq \emptyset$, $k \in C$, and $C_{ij} \neq \emptyset$, $(i,j) \in L$.*

Before formulating the power allocation optimization problem, let us first redefine the concepts of signal to interference and noise ratio and Shannon capacity, for a given feasible channel assignment. Let the SINR be given by (2.1) for the existing links, i.e., for $(i,j) \in L$. Let the Shannon capacity of a mobile user on link $(i,j) \in L$, at channel k, be given as

$$\eta_{ijk} = W \log\left(1 + \mathrm{SINR}_{ijk}\right), \quad (i,j) \in L,\, k \in C_{ij}, \tag{5.1}$$

and let the total Shannon capacity of a mobile user j be given by

$$\eta_{ij} = \sum_{k \in C_{ij}} \eta_{ijk}, \quad (i,j) \in L, \tag{5.2}$$

where η_{ijk} is given in (5.1). Note that the remaining links $(i,j) \notin L$ are not be used, so their SINR and Shannon capacity need not be defined.

Given a feasible channel assignment, the power allocation optimization problem is given by

$$
\begin{array}{ll}
\underset{\eta,\, \eta_{ij},\, p_{ik}}{\text{maximize}} & \eta \\
\text{subject to} & \eta \leq \eta_{ij}, \quad (i,j) \in L, \\
& \sum_{k \in C_i} p_{ik} \leq P^{\max}_{S,i}, \quad i \in S, \\
& p_{ik} \geq 0, \quad i \in S,\, k \in C_i,
\end{array} \tag{5.3}
$$

where η_{ij}, $(i,j) \in L$ is given in (5.2), and $P^{\max}_{S,i}$ is the same upper bound power consumption constant as in (2.4e). In (5.3), $S = B$ gives the downlink formulation, while $S = M$ gives the uplink formulation.

We can obtain a simplified version of the power allocation optimization problem (5.3) by introducing an additional constraint on our channel assignment. The constraint is to only allow each link $(i,j) \in L$ to use one single channel, i.e., $|C_{ij}| = 1$,

$(i,j) \in L$. Let us refer to this formulation as a *single channel on each link power allocation optimization problem*, which is given by

$$\begin{array}{ll} \underset{\eta,\eta_{ij},p_{ik}}{\text{maximize}} & \eta \\ \text{subject to} & \eta \le \eta_{ijk}, \quad (i,j) \in L,\, k \in C_{ij} \\ & \sum_{k\in C_i} p_{ik} \le P_{S,i}^{\max}, \quad i \in S, \\ & p_{ik} \ge 0, \quad i \in S,\, k \in C_i, \end{array} \tag{5.4}$$

where $|C_{ij}| = 1$, $(i,j) \in L$, η_{ijk}, $(i,j) \in L$, $k \in C_{ij}$ is given in (5.1), and $P_{S,i}^{\max}$ is the same upper bound power consumption constant as in (2.4e) and (5.3). As one channel only occupies each link, the optimization problem (5.4) in uplink can be decomposed into a separate optimization problem for each channel $k \in C$, since $|C_i| = |C_{ij}|$, $(i,j) \in L$, in uplink. However, in downlink this is not necessarily the case, since base stations can have links to several mobile users.

The power allocation optimization problems (5.3) and (5.4) have been defined above, and some mathematical properties are now to be presented in section 6 and section 7.

6. On properties of the power allocation optimization problem

Given a feasible channel assignment, the arising power allocation optimization problem is given in (5.3). Some general mathematical properties of this problem is to be given here, while some convexity properties are given in section 7.

If we use a similar construction as for the proof of Theorem 4.2, a complexity result for the power allocation optimization problem (5.3) can be obtained. The result is given in the following corollary.

Corollary 6.1. *For any constant $\rho > 0$, there exists a positive $N \in I\!R$ such that there is no ρ-approximate polynomial algorithm of the optimization problem* (5.3)*, in downlink or in uplink, for $K \ge N$, unless P is equal to NP.*

Proof. Consider the proof of Theorem 4.2, for the joint channel assignment and power allocation optimization problem (2.4). Let $x_{\bar{S}_i\bar{D}_ik} = 1$, $i \in V$, $k \in C$, $x_{\bar{S}_i\bar{D}_jk} = 0$, $i \in V$, $j \in V$, $i \ne j$, $k \in C$, $y_{\bar{S}_i\bar{D}_i} = 1$, $i \in V$, and $y_{\bar{S}_i\bar{D}_j} = 0$, $i \in V$, $j \in V$, $i \ne j$. As the entire complexity remains for this instance, the same proof as for Theorem 4.2 gives the corollary. ∎

Let us now for the remaining part of this section assume that each path gain

$$g_{ij} > 0, \quad i \in S,\ j \in D.$$

If so, then a local maximizer to a well-posed power allocation optimization problem (5.3), where each power $p_{ik} > 0$, $i \in S$, $k \in C$, fulfils that every Shannon capacity η_{ij}, $(i,j) \in L$, given in (5.2), are equal in downlink. This result also holds in uplink, but with the additional assumption of the sets L and C being *connected*, see the following definition.

Definition 6.1. *The sets L and C are connected if there for every non-empty $\tilde{L} \subset L$, $\tilde{L} \neq L$, and $\tilde{C} = \{k \in C : x_{ijk} = 1$ for some $(i,j) \in \tilde{L}\}$, exists a channel $\tilde{k} \in \tilde{C}$ and a link $(\hat{i},\hat{j}) \in L \backslash \tilde{L}$ such that $(\hat{i},\hat{j}) \in L_{\tilde{k}}$.*

Let us use a straightforward argument to prove our claim, which is that a local maximizer of (5.3) with only positive powers, will have an objective value $\eta^* = \eta_{ij}$, $(i,j) \in L$, given that the sets L and C are connected. First we note that a feasible channel assignment in downlink always is connected. Now assume that there exists a link $(i,j) \in L$ where $\eta_{ij} > \eta^*$. Then we can decrease all its powers p_{ik}, $k \in C_i$, which will improve the Shannon capacity for all users using a channel in C_i. Make sure that the decrease of the powers p_{ik}, $k \in C_i$ are sufficiently small, i.e., such that η_{ij} remain strictly greater than η^*. Let $\tilde{L} = \{(i,j) \in L : \eta_{ij} > \eta^*\}$, if $\tilde{L} = L$ we are done, so assume that $\tilde{L} \neq L$. Due to the assumption of connectedness, see Definition 6.1, it follows that with the set of links $\tilde{L}$ there exists a channel $\tilde{k} \in \tilde{C}$ and a link $(\hat{i},\hat{j})$ which both belongs to $L \backslash \tilde{L}$ and to $L_{\tilde{k}}$. Now, we can decrease the power of the sources belonging to links $(i,j) \in \tilde{L}$, which are using the channel $\tilde{k} \in \tilde{C}$. Make sure that the decrease is sufficiently small, i.e., such that η_{ij} remain strictly greater than η^* for all $(i,j) \in \tilde{L}$, and also becomes strictly greater than η^* for all $(i,j) \in L_{\tilde{k}}$. As the sets L and C are connected, this procedure can be repeated until all links $(i,j) \in L$ eventually have an improved Shannon capacity, which contradicts that the suggested solution is a local maximizer, and our claim follows.

To motivate our assumptions, used to prove our claim above, we now illustrate some examples in downlink and in uplink where it does not hold that the Shannon capacity of each mobile user has to be equal in a local maximizer.

In a local maximizer of the power allocation optimization problem (5.3) in downlink, where some base station does not transmit on each channel $k \in C$, there might exists $(\hat{i},\hat{j}) \in L$ such that $\eta_{\hat{i}\hat{j}} > \eta^*$, where η^* is the objective value. Let $B = \{1,2\}$, $M = \{1,2,3\}$, $C = \{1,2\}$ and $L = \{(1,1),(2,2),(2,3)\}$. Let $L_1 = \{(1,1),(2,2)\}$, $L_2 = \{(2,3)\}$, $C_{11} = \{1\}$, $C_{22} = \{1\}$, and $C_{23} = \{2\}$. It follows that the sets L_k, $k \in C$, and C_{ij}, $(i,j) \in L$, are well-posed. Let $W = 1$, $\sigma_j = 1$, $j \in M$, and $P_{B,i}^{\max} = 1$, $i \in B$, $g = (g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{23}) = (1/2, 1/2, -, 1, 1, 5)$, where the value of g_{13} does not affect the solution. A local maximizer, with objective value η^*, is for example $(p_{11}, p_{21}, p_{22}) = (1, 1/2, 1/2)$, as $\eta^* = \eta_{11} = \eta_{22} = \log(1 + 1/3) < \eta_{23}$, where not both η_{11} and η_{22} can be improved.

In a local maximizer of the power allocation optimization problem (5.3) in uplink, where some base station does not receive information from sources using non-zero power on each channel $k \in C$, there might exists $(\hat{i},\hat{j}) \in L$ such that $\eta_{\hat{i}\hat{j}} > \eta^*$, where η^* is the objective value. Let $M = \{1,2,3\}$, $B = \{1,2\}$, $C = \{1,2\}$ and $L = \{(1,1),(2,2),(3,2)\}$. Let $L_1 = \{(1,1),(2,2)\}$, $L_2 = \{(1,1),(3,2)\}$, $C_{11} = \{1,2\}$, $C_{22} = \{1\}$, and $C_{32} = \{2\}$. It follows that the sets L_k, $k \in C$, and C_{ij}, $(i,j) \in L$, are well-posed. Let $W = 1$, $\sigma_j = 1$, $j \in B$, and $P_{M,i}^{\max} = 1$, $i \in M$, $g = (g_{11}, g_{12}, g_{21}, g_{22}, g_{31}, g_{32}) = (5, 1, -, 1, 1, 5)$, where the value of g_{21} does not affect the solution. One local maximizer, with objective value η^*, is for example $(p_{11}, p_{12}, p_{21}, p_{32}) = (0, 1, 1, 1)$, as $\eta^* = \eta_{22} = \log(1 + 1) < \eta_{11} = \eta_{32} = \log(1 + 5/2)$, where η_{22} cannot be improved. Note that if channel 1 on link $(1,1)$ is removed, i.e.,

$L_1 = \{(2,2)\}$ and $C_{(1,1)} = \{2\}$ instead, then the sets L_k, $k \in C$, and C_{ij}, $(i,j) \in L$ remain well-posed, and this new problem can be decomposed into two connected subsets, one for each channel $k \in C$, where these connected subsets have different objective values.

Another property of the power allocation optimization problem (5.3) is that at least one source will use full power at any local maximizer, given that the objective value is positive. To prove this, let us assume that p_{ik}, $(i,j) \in L$, $k \in C_{ij}$, is a local maximizer of optimization problem (5.3), with objective value $\eta^* > 0$, where $\sum_{k \in C_i} p_{ik} < P_{S,i}^{\max}$, $i \in S$. Then there exists some $\alpha > 1$ such that $\sum_{k \in C_i} \alpha p_{ik} \leq P_{S,i}^{\max}$, $i \in S$. Let η_{ij} denote the Shannon capacity of the mobile user of link $(i,j) \in L$, when the power p_{ik}, $i \in S$, $k \in C_i$ is used, and let η_{ij}^{α} denote the Shannon capacity of the mobile user of link $(i,j) \in L$, when the power αp_{ik}, $i \in S$, $k \in C_i$ is used. As $\sigma_j^2/\alpha < \sigma_j^2$, $j \in D$, it follows that

$$\min_{(i,j) \in L} \eta_{ij}^{\alpha} > \min_{(i,j) \in L} \eta_{ij},$$

which contradicts that the suggested p_{ik}, $(i,j) \in L$, $k \in C_i$, is a local maximizer, and our claim follows. For the single channel on each link power allocation optimization problem (5.4), our claim implies that at least one source $i \in S$ will use power equal to $P_{S,i}^{\max}$ at a local maximizer.

A similar result is given in [8], though with the objective to maximize the total Shannon capacity over one single channel with two base stations, communicating with one mobile user each. This setting is proved to have at least one source that uses full power. This result can now be extended, using the same proof as above. If considering a local maximizer of a modified power allocation optimization problem (5.3), where the modification is to replace the objective function into being to maximize the total Shannon capacity, then the same proof, gives that

$$\sum_{(i,j) \in L} \eta_{ij}^{\alpha} > \sum_{(i,j) \in L} \eta_{ij},$$

which contradicts that the suggested p_{ik}, $(i,j) \in L$, $k \in C_i$, is a local maximizer. Hence, this last result is also applicable when the objective function is to maximize the total Shannon capacity.

7. On convexity of the restricted problem

This section contains some convexity results for the power allocation optimization problems given in (5.3) and (5.4). A first observation, due to Corollary 6.1, is that it is probably unlikely to find a general convex optimization problem that corresponds to our power allocation optimization problem (5.3), since a convex optimization problem tend to be solvable in polynomial time in general.

We can also observe that neither problem (5.3) nor (5.4) is convex in general, which follows from an example with two base stations, two mobile users and one channel. Let $S = \{1,2\}$, $D = \{1,2\}$, $C = \{1\}$ and $L = \{(1,1), (2,2)\}$, together with the constants $W = 1$, $\sigma_1 = \sigma_2 = 0.1$, $P_{S,1}^{\max} = P_{S,2}^{\max} = 20$ and $(g_{11}, g_{12}, g_{21}, g_{22}) =$

(0.011, 0.089, 0.002, 0.004). This gives a non-concave objective function, see Figure 5. Hence, the optimization problems (5.3) and (5.4) are not convex in general.

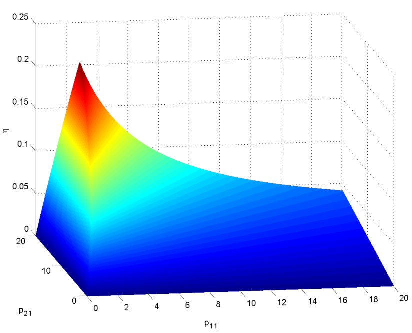

Figure 5: The Shannon capacity $\eta = \min\{\eta_{11}, \eta_{22}\}$ is a function of the power variables p_{11} and p_{21}, which both uses the single channel $|C| = 1$.

Despite not being convex in general, the single channel on each link power allocation optimization problem (5.4), with a non-zero objective value $\eta^* = \eta_{ij}$, $(i,j) \in L$, given in (5.2), and with each transmitter only using a single channel, has a convexity property in terms of each local maximizer being also a global maximizer. This is to be proved, by formulating a corresponding convex single channel optimization problem, obtained from (5.4), where $\gamma = \log\{\eta\}$ and $\exp(\bar{p}_{ik}) = p_{ik}$, $(i,j) \in L$, $k \in C_{ij}$, and then verify that the relation between their KKT-points are one-to-one. The convex single channel optimization problem is given by

$$\begin{array}{ll} \underset{\gamma, \bar{p}_{ik}}{\text{maximize}} & \gamma \\ \text{subject to} & \gamma \leq \log\Big(W \log(1 + \overline{\text{SINR}}_{ijk})\Big), \quad (i,j) \in L, k \in C_{ij}, \\ & \log\Big(\sum_{k \in C_i} \exp(\bar{p}_{ik})\Big) \leq \log\Big(P_{S,i}^{\max}\Big), \quad i \in S, \end{array} \tag{7.1}$$

where

$$\overline{\text{SINR}}_{ijk} = \frac{g_{ij} \exp(\bar{p}_{ik})}{\sigma_j^2 + \sum_{(m,n) \in L_k \setminus \{(i,j)\}} g_{mj} \exp(\bar{p}_{mk})}, \quad (i,j) \in L, k \in C_{ij}.$$

Note that if (7.1) is to be well defined, then the corresponding η and p_{ik}, $(i,j) \in L$, $k \in C_{ij}$, of (5.4) must be non-negative. From [14] it is known that $W \log(1 + \overline{\text{SINR}}_{ijk})$, $(i,j) \in L$, $k \in C_{ij}$ is a quasiconcave function, i.e., not closed under addition, and that $\log\Big(W \log(1 + \overline{\text{SINR}}_{ijk})\Big)$, $(i,j) \in L$, $k \in C_{ij}$ is a concave function. Hence, it follows that (7.1) is a convex optimization problem, see, e.g., [4].

Lemma 7.1. *In uplink, the two optimization problems* (5.4) *and* (7.1) *are equivalent in the sense that their KKT points are one to one, given that the powers* $p_{ik} = \exp(\bar{p}_{ik})$, $(i,j) \in L$, $k \in C_{ij}$, *the objective value* η^* *of* (5.4) *is strictly positive, and* $\eta^* = \eta_{ij}$, $(i,j) \in L$, *given in* (5.2).

This result also holds in downlink, under the additional assumption $p_{ik} = p_{i\hat{k}}$, $i \in B$, $k \in C_{ij}$, $\hat{k} \in C_{i\hat{j}}$.

Proof. The first-order necessary optimality conditions are necessary and sufficient for global optimality of the corresponding convex single channel optimization problem (7.1), since it satisfies the Slater constraint qualification. Let us begin by evaluating the first-order necessary optimality conditions of (7.1) and then compare it with the first-order necessary optimality conditions of (5.4). The optimality conditions of (7.1) take the form

$$\sum_{m=1}^{|M|} \bar{\lambda}_m = 1, \tag{7.2a}$$

$$\begin{aligned} \frac{\bar{\mu}_i}{\exp(\bar{p}_{\hat{i}k})} = &- \sum_{(\hat{i},\hat{j}) \in L_k \setminus \{(i,j)\}} \frac{\bar{\lambda}_{\hat{m}}}{\eta_{\hat{m}}} \frac{W g_{\hat{i}\hat{j}} \exp(\bar{p}_{\hat{i}k})}{\left(1 + \overline{\mathrm{SINR}}_{\hat{i}\hat{j}k}\right)} \frac{g_{i\hat{j}}}{\left(\sigma_{\hat{j}}^2 + \sum_{(\tilde{i},\tilde{j}) \in L_k \setminus \{(\hat{i},\hat{j})\}} g_{\tilde{i}\hat{j}} \exp(\bar{p}_{\tilde{i}k})\right)^2} \\ &+ \frac{\bar{\lambda}_m}{\eta_m} \frac{W}{\left(1 + \overline{\mathrm{SINR}}_{ijk}\right)} \frac{g_{ij}}{\left(\sigma_j^2 + \sum_{(\hat{i},\hat{j}) \in L_k \setminus \{(i,j)\}} g_{\hat{i}j} \exp(\bar{p}_{\hat{i}k})\right)}, \\ &(i,j) \in L, k \in C_{ij}, \end{aligned} \tag{7.2b}$$

$$0 = \bar{\lambda}_m \left(\log\left(W \log(1 + \overline{\mathrm{SINR}}_{ijk})\right) - \gamma\right), \quad (i,j) \in L, \tag{7.2c}$$

$$0 = \bar{\mu}_i \left(\log\left(P_{S,i}^{\max}\right) - \log\left(\sum_{k \in C_i} \exp(\bar{p}_{ik})\right)\right), \quad i \in S, \tag{7.2d}$$

$$\begin{aligned} \bar{\lambda}_m &\geq 0, \quad m \in M, \\ \bar{\mu}_i &\geq 0, \quad i \in S, \end{aligned} \tag{7.2e}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are the Lagrange multipliers. Note that index $m \in M$, while index $i \in S$ and index $j \in D$.

Now, if we let $p_{ik} = \exp(\bar{p}_{ik})$, $(i,j) \in L$, $k \in C_{ij}$, $\lambda_m = \bar{\lambda}_m$, $m \in M$, and

$$\mu_i = \frac{\eta^*}{\exp(\bar{p}_{ik})} \bar{\mu}_i, \quad i \in S, k \in C_i, \tag{7.3}$$

we obtain the KKT points of (5.4). Note that equation (7.3) gives no additional constraint in uplink, since $|C_i| = |C_{ij}| = 1$, $(i,j) \in L$. However, in downlink equation (7.3) states that

$$\bar{p}_{ik} = \bar{p}_{i\hat{k}}, \quad i \in B, k \in C_{ij}, \hat{k} \in C_{i\hat{j}}.$$

which gives the additional assumption needed in downlink. ∎

In downlink of (5.4), there is no motivation for each base station $i \in B$ using equal power on every link, more than that it is needed in the proof of Lemma 7.1. If we in addition assume that each base station only is allowed to use one single channel, then this problem is equivalent with the optimization problem (5.4) in uplink, which can be decomposed into $|C|$ optimization problems, one for each channel. For this problem it holds that any KKT point is a global maximizer, where the objective value $\eta^* = \eta_{ij}$, $(i,j) \in L$. This result is given in the following corollary.

Corollary 7.1. *Any first-order KKT point of a well-posed power allocation optimization problem* (5.4)*, with objective value $\eta^* > 0$, and where every source $i \in S$ has each of its powers equal, is also a global maximizer, where $\eta^* = \eta_{ij}$, $(i, j) \in L$, given in* (5.2).

Proof. Consider a first-order KKT point of a well-posed power allocation optimization problem (5.4), with objective value $\eta^* > 0$, where each source $i \in S$ has every of its powers equal. Given this setting, we know from Section 6, that $\eta^* = \eta_{ij}$, $(i, j) \in \bar{L}$ both in downlink and in uplink. Then Lemma 7.1 gives that the KKT point is a global maximizer, and the corollary follows. ∎

It is already known that the unique global maximizer of (5.4) in uplink has the objective value $\eta^* = \eta_{ij}$, $(i, j) \in L$. There also exists algorithms which obtain this global maximizer, see, e.g., Yates [15]. However, Corollary 7.1 gives that it is sufficient to obtain a first-order KKT point, to ensure global optimality of (5.4). One setting for Corollary 7.1 is, e.g., when $|C_i| = 1$, $i \in S$.

This convexification approach, used to prove Lemma 7.1 and Corollary 7.1, is not applicable for the more general power allocation optimization problem (5.3), for several reasons. First, it is still necessary for each source $i \in S$ to have equal power on every link. If reformulating (7.1) for the corresponding (5.3) situation, it changes the first constraints into

$$\gamma \leq \log\left(\sum_{k \in C_i} W \log(1 + \overline{\text{SINR}}_{ijk})\right), \quad (i, j) \in L, \tag{7.4}$$

instead. In the proof of Lemma 7.1 and Corollary 7.1, this modification introduces an extra factor $1/\gamma_{ij}$, $(i, j) \in L$, on each term of the right-hand-side of equation (7.2c). So, if those results are still to hold, then we need the additional assumption of

$$\gamma^* = \gamma_{ij}, \quad (i, j) \in L, \quad \text{as well as} \quad \eta^* = \eta_{ij}, \, (i, j) \in L.$$

Hence, such a setting only covers the situation where the number of channels on each link is the same, together with that the Shannon capacity of each channel is the same, and that each source $i \in S$ use equal power on every link. Another aspect is that optimization problem (7.1) might no longer be convex, due to (7.4), which follows from the previously mentioned characteristic of quasiconcave functions not being closed under addition, see, e.g., [3].

In the slightly more general setting where each base station has two channels, there exist examples both in downlink and in uplink with more than one local maximizer. Consider the following setting where $B = \{1, 2, 3\}$, $M = \{1, 2, 3, 4\}$ and $C = \{1, 2\}$ in Figure 6. The constructed examples evolve from a setting of two base station-mobile user pairs and two channels, where the sources of each pair uses all power on one channel each. This setting gives rise to two different local maximizers, which both are equally good in terms of optimal value. However, introducing a third base station, with two mobile users, two different local maximizers, with different objective value, can be obtained.

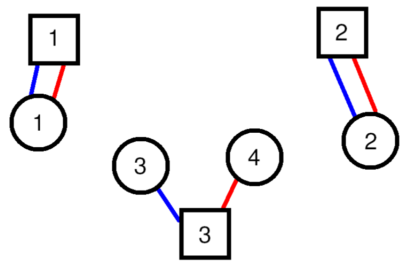

Figure 6: A setting consisting of three base stations illustrated as squares, four mobile users illustrated as circles and two channels illustrated as links.

In downlink this setting, of Figure 6, gives $L = \{(1,1), (2,2), (3,3), (3,4)\}$, where channel 1 can be used on the links $(1,1)$, $(2,2)$ and $(3,3)$, and where channel 2 can be used on the links $(1,1)$, $(2,2)$ and $(3,4)$. Let $W = 1$, $\sigma_j^2 = 0.1$, $j \in M$, $P_{B,i}^{\max} = 10$, $i \in B$, and

$$g = \begin{pmatrix} 1 & 10 & 10 & 1 \\ 10 & 1 & 1 & 1 \\ 0.01 & 0.01 & 2 & 2 \end{pmatrix}.$$

If solving the optimization problem (5.3) in downlink with these sets and constants, one can obtain the two KKT points

$$\begin{aligned} &\eta \approx 1.4066 \\ &p \approx \begin{pmatrix} 0.5961 & 0 \\ 0 & 0.3286 \\ 9.3396 & 0.6604 \end{pmatrix} \end{aligned} \quad \text{and} \quad \begin{aligned} &\eta \approx 2.1791 \\ &p \approx \begin{pmatrix} 0 & 1.1758 \\ 1.1758 & 0 \\ 5.0000 & 5.0000 \end{pmatrix} \end{aligned}$$

which both are local maximizers.

In uplink this setting, of Figure 6, gives $L = \{(1,1), (2,2), (3,3), (4,3)\}$, where channel 1 can be used on the links $(1,1)$, $(2,2)$ and $(3,3)$, and where channel 2 can be used on the links $(1,1)$, $(2,2)$ and $(4,3)$. Let $W = 1$, $\sigma_j^2 = 0.1$, $j \in B$, $P_{M,i}^{\max} = 10$, $i \in M$, and

$$g = \begin{pmatrix} 1 & 10 & 10 \\ 10 & 1 & 1 \\ 0.01 & 0.01 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

If solving the optimization problem (5.3) in uplink with these sets and constants, one can obtain the two KKT points

$$\begin{aligned} &\eta \approx 0.3785 \\ &p \approx \begin{pmatrix} 2.1560 & 4.3366 \\ 10.0000 & 0 \\ 7.2839 & - \\ - & 10.0000 \end{pmatrix} \end{aligned} \quad \text{and} \quad \begin{aligned} &\eta \approx 0.8771 \\ &p \approx \begin{pmatrix} 1.4065 & 0 \\ 0.0802 & 9.9198 \\ 10.0000 & - \\ - & 7.0338 \end{pmatrix} \end{aligned}$$

which both are local maximizers. Note that a "$-$" in p indicates that the corresponding variable p_{ik} does not exist, i.e., the transmitting mobile user $i \in M$ does not have access to channel $k \in C$.

8. Summary and conclusions

In both downlink and uplink a joint channel assignment and power allocation optimization problem has been formulated, with objective to maximize the minimum total Shannon capacity of any mobile user. The corresponding decision problems were shown to be NP-hard, and for any constant $\rho > 0$, there exists a sufficiently large number of channels, such that the joint channel assignment and power allocation optimization problem is not ρ-approximable, unless P is equal to NP.

As the joint channel and power allocation optimization problem is difficult, one approach is to divide the problem into two separate parts, where one part concerns the channel allocation and the other part solves the power allocation optimization problem. These two parts can then be solved interchangeably.

The second part of this paper is focused on properties of the power allocation optimization problem. This optimization problem is not convex in general, though if each transmitter only is allowed to use one channel, then any KKT point ensures global optimality. It was shown that for any constant $\rho > 0$, there exists a sufficiently large number of channels, such that the power allocation optimization problem is not ρ-approximable, unless P is equal to NP. Hence, the difficulties remain in the power allocation optimization problem, even though the binary combinatorics, in terms of the channel assignment, is assumed to be known.

A. A mathematical result

Lemma A.1. *Let $\tilde{g} > 0$, $\sigma > 0$ and N a positive integer. Then the optimization problem*

$$\begin{array}{ll} \underset{p \in I\!R^N}{\text{maximize}} & \sum_{k=1}^N \log\left(1 + \frac{\tilde{g} p_k}{\sigma^2}\right) \\ \text{subject to} & \sum_{k=1}^N p_k \leq 1, \\ & p_k \geq 0, \quad k = 1, ..., N. \end{array} \tag{A.1}$$

has a unique global optimal solution where $p_k = 1/N$, $k = 1, ..., N$, and the optimal value is $N \log(1 + \frac{\tilde{g}}{N\sigma^2})$.

Proof. The Langrange function of this problem is given by

$$\mathcal{L}(p, \mu, \lambda) = -\sum_{k=1}^N \log\left(1 + \frac{\tilde{g} p_k}{\sigma^2}\right) - \mu\left(1 - \sum_{k=1}^N p_k\right) - \sum_{k=1}^N p_k \lambda_k,$$

where $p \in I\!R^N$, $\mu \in I\!R$ and $\lambda \in I\!R^N$. The derivative of the Lagrange function, with respect to each p_k, is given by

$$\frac{\partial \mathcal{L}}{\partial p_k} = -\frac{\tilde{g}}{\sigma^2 + p_k \tilde{g}} + \mu - \lambda_k = 0, \quad k = 1, ..., N.$$

Let $p_k = 1/N$ and $\lambda_k = 0$, $k = 1, ..., N$, then

$$\mu = \frac{\tilde{g}}{\sigma^2 + \frac{1}{N}\tilde{g}} > 0,$$

and as $(1 - \sum_k p_k) = 0$, this point fulfils the first-order necessary optimality conditions.

Linear constraints together with that the logarithmic function is strictly increasing gives that it is a strict convex optimization problem (A.1), which gives the lemma. ∎

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