

# Robust Formation Control in $\mathbb{SE}(3)$ for Tree-Graph Structures with Prescribed Transient and Steady State Performance

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## Abstract

This paper presents a novel control protocol for distance and orientation formation control of rigid bodies, whose sensing graph is a static and undirected tree, in the special Euclidean group  $\mathbb{SE}(3)$ . The proposed control laws are decentralized, in the sense that each agent uses only local relative information from its neighbors to calculate its control signal, as well as robust with respect to modeling (parametric and structural) uncertainties and external disturbances. The proposed methodology guarantees the satisfaction of inter-agent distance constraints that resemble collision avoidance and connectivity maintenance properties. Moreover, certain predefined functions characterize the transient and steady state performance of the closed loop system. Finally, simulation results verify the validity and efficiency of the proposed approach.

*Key words:* Multi-Agent Systems, Cooperative Control, Formation Control, Connectivity Maintenance, Robust Control, Prescribed Performance Control.

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## 1 Introduction

During the last decades, decentralized control of multi-agent systems has gained a significant amount of attention due to the great variety of its applications, including multi-robot systems, transportation, multi-point surveillance as well as biological systems. Among the various research topics in multi-agent systems, the most popular ones can be considered to be (i) multi-agent navigation [1], where the agents need to navigate to predefined positions of the state space, and (ii) consensus [2], where the agents aim to converge to a common state. At the same time, the agents might need to fulfill certain transient properties, such as network connectivity [3] and/or collision avoidance [4]. Another important problem considered in multi-agent systems is formation control [5], where the agents aim to form a predefined shape in the state space, and which can be seen as a combination of the navigation and consensus problems.

Formation control is categorized in ([5]) position-based, distance-based and orientation-based formation control, as well as a combination of the two, which is also the focus of this work.

Distance-based formation control has been well-studied in the related literature (see, indicatively, [6–17]). In these works, however, the authors consider simplified single-integrator models for the agent dynamics. Double integrator schemes have been studied in [18–20]. Orientation-based formation control has been investigated in [21–24], whereas the authors in [24–26] have considered the combination of distance- and orientation-based formation, also employing single integrator or 2D unicycle dynamics.

The use of simplified dynamics however, like in the aforementioned works, does not apply to realistic engineering applications, where the systems may have complicated and uncertain dynamics. Moreover, such systems are inherently under the presence of exogenous disturbances. Two more characteristics not taken into account in most of the aforementioned works is (i) connectivity preservation among the agents, and (ii) inter-agent collision avoidance. Both of these properties are important, inherent from the limited sensing capabilities of multi-agent systems, and dimensionless agents/robots in potential real-time applications, respectively. In this paper,

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we aim to handle collision avoidance and connectivity maintenance properties among the initially connected agents of a static sensing graph. We note that these properties are taken into account in works that utilize potential functions [27], or aim at achieving position consensus or velocity alignment (flocking) [28–31], some of which consider collision avoidance between any pair of agents, and not only the initially connected ones. Potential-based formation control schemes suffer from inherent local minima issues, which is not exhibited in the velocity alignment problems, since the agents converge to the same velocity regardless of their positions. In addition, simplified dynamics (single or double-integrators) are considered in the aforementioned works, whereas the more general dynamics considered here correspond to more realistic scenarios. Finally, as we indicate below, prescribed performance guarantees are given for the agents’ transient and steady-state behavior.

More specifically, this paper presents a novel control protocol for the formation control of multiple rigid bodies forming a tree sensing graph in  $\mathbb{SE}(3)$ . We employ the Prescribed Performance Control methodology, initially proposed in [32], to achieve predefined transient and steady-state performance. Prescribed performance control has been considered in the framework of multi-agent systems in [33–36]. In [33, 34] the authors tackle the position-based formation control problem, by taking into account position-based connectivity maintenance in [34], and [35, 36] consider the consensus problem. The proposed methodology exhibits the following attributes: 1) It is decentralized, in the sense that each agent computes its own control signal based on its local sensing capabilities, without needing to communicate with the rest of the agents, or to know the pose of a global coordinate frame. 2) It is robust to bounded external disturbances and uncertainties of the dynamic model, since these are not employed in the control design. 3) It guarantees satisfaction of certain distance constraints among the initially connected agents, which resemble collision avoidance and connectivity maintenance specifications. 4) It guarantees convergence to a feasible formation configuration with predefined transient and steady-state performance from almost all initial conditions. Moreover, in contrast to standard continuous control methodologies on  $\mathbb{SO}(3)$  (where the closer the initial condition is to the unstable equilibrium, the more the stabilization time approaches infinity), it guarantees convergence to the formation configuration *arbitrarily fast*, regardless of the distance of the initial system configuration to the unstable equilibrium. This paper constitutes an extension of our previous works [37], [38]. In both of these works we addressed the same problem using Euler angles that suffer from representation singularities as well as knowledge of a common global inertial frame; [38] employs a potential function-based solution, inherently exhibiting local minima, and [37] also uses the idea of prescribed performance control.

## 2 Notation and Preliminaries

The set of positive integers is denoted as  $\mathbb{N}$ . The real  $n$ -coordinate space, with  $n \in \mathbb{N}$ , is denoted as  $\mathbb{R}^n$ ;  $\mathbb{R}_{\geq 0}^n$  and  $\mathbb{R}_{> 0}^n$  are the sets of real  $n$ -vectors with all elements nonnegative and positive, respectively. Given a set  $S$ , denote by  $|S|$  its cardinality, by  $S^n = S \times \dots \times S$  its  $n$ -fold Cartesian product, and by  $2^S$  the set of all its subsets. The notation  $\|x\|$  is used for the Euclidean norm of a vector  $x \in \mathbb{R}^n$ . Given a symmetric matrix  $A$ ,  $\lambda_{\min}(A) := \min\{|\lambda| : \lambda \in \text{eig}(A)\}$  denotes the minimum eigenvalue of  $A$ , respectively, where  $\text{eig}(A)$  is the set of all the eigenvalues of  $A$  and  $\text{rank}(A)$  is its rank;  $\|A\|_{\text{F}} := \text{tr}(A^{\text{T}}A)$  is the Frobenius norm of  $A$ , and  $\text{tr}[\cdot]$  is its trace;  $\det(A)$  denotes the determinant of a matrix  $A \in \mathbb{R}^{n \times n}$ . The notation  $\text{diag}\{A_1, \dots, A_n\}$  stands for the block diagonal matrix with the matrices  $A_1, \dots, A_n$  in the main block diagonal;  $A \otimes B$  denotes the Kronecker product of matrices  $A, B \in \mathbb{R}^{m \times n}$ , as was introduced in [39]. Define by  $I_n \in \mathbb{R}^{n \times n}$  and  $0_{m \times n} \in \mathbb{R}^{m \times n}$  the unitary matrix and the  $m \times n$  matrix with all entries zeros, respectively;  $\mathcal{B}(c, r) := \{x \in \mathbb{R}^3 : \|x - c\| \leq r\}$  is the vector-valued mapping representing the 3D ball of radius  $r \in \mathbb{R}_{> 0}$  and center  $c \in \mathbb{R}^3$ . Given  $x, y \in \mathbb{R}^3$ ,  $S : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$  is the skew-symmetric matrix defined according to  $S(x)y = x \times y$ , and  $S^{-1} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$  is its inverse, where  $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3 \times 3} : x^{\text{T}}S(\cdot)x = 0, \forall x \in \mathbb{R}^3\}$  is the space of skew-symmetric matrices. The special Euclidean group is denoted by  $\mathbb{SE}(3) := \{(c, R) \in \mathbb{R}^3 \times \mathbb{SO}(3)\}$ , where  $\mathbb{SO}(3) := \{R \in \mathbb{R}^{3 \times 3} : R^{\text{T}}R = I_3, \det(R) = 1\}$ . Moreover, the tangent space to  $\mathbb{SO}(3)$  at  $R$  is denoted by  $T_R\mathbb{SO}(3)$  and we also use  $\mathbb{T}_R := \mathbb{R}^3 \times T_R\mathbb{SO}(3)$ . We define the induced norm in  $\mathbb{SO}(3)^N$  as  $\|R\|_{\text{T}} := \sum_{i \in \{1, \dots, N\}} \|R_i\|_{\text{F}}$  for any  $R = (R_1, \dots, R_N) \in \mathbb{SO}(3)^N$ . Finally, all the differentiations are performed with respect to an inertial frame of reference unless otherwise stated. Some useful properties of skew symmetric matrices [40]:  $x^{\text{T}}S(y)x = 0$ ;  $S(Rx) = RS(x)R^{\text{T}}, -\frac{1}{2}\text{tr}[S(x)S(y)] = x^{\text{T}}y$ ,  $\text{tr}[AS(x)] = \frac{1}{2}\text{tr}[S(x)(A - A^{\text{T}})] = -x^{\text{T}}S^{-1}(A - A^{\text{T}})$ , for every  $x, y \in \mathbb{R}^3$ ,  $A \in \mathbb{R}^{3 \times 3}$  and  $R \in \mathbb{SO}(3)$ .

### 2.1 Prescribed Performance Control

Prescribed Performance Control (PPC), originally proposed in [32], describes the behavior where a tracking error  $e(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  evolves strictly within a predefined region that is bounded by certain functions of time, achieving prescribed transient and steady state performance. The mathematical expression of prescribed performance is given by the inequalities  $-\rho_L(t) < e(t) < \rho_U(t)$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ , where  $\rho_L(t), \rho_U(t)$  are smooth and bounded decaying functions of time, satisfying  $\lim_{t \rightarrow \infty} \rho_L(t) > 0$  and  $\lim_{t \rightarrow \infty} \rho_U(t) > 0$ , called performance functions. Specifically, for the exponential performance functions  $\rho_i(t) = (\rho_{i0} - \rho_{i\infty})e^{-l_i t} + \rho_{i\infty}$ , with  $\rho_{i0}, \rho_{i\infty}, l_i \in \mathbb{R}_{> 0}, i \in \{U, L\}$ , appropriately

chosen constants,  $\rho_{L0} = \rho_L(0)$ ,  $\rho_{U0} = \rho_U(0)$  are selected such that  $\rho_{U0} > e(0) > \rho_{L0}$  and the constants  $\rho_{L\infty} = \lim_{t \rightarrow \infty} \rho_L(t) < \rho_{L0}$ ,  $\rho_{U\infty} = \lim_{t \rightarrow \infty} \rho_U(t) < \rho_{U0}$  represent the maximum allowable size of the tracking error  $e(t)$  at steady state, which may be set arbitrarily small to a value reflecting the resolution of the measurement device, thus achieving practical convergence of  $e(t)$  to zero. Moreover, the decreasing rate of  $\rho_L(t)$ ,  $\rho_U(t)$ , which is affected by the constants  $l_L, l_U$  in this case, introduces a lower bound on the required speed of convergence of  $e(t)$ . Therefore, the appropriate selection of the performance functions  $\rho_L(t)$ ,  $\rho_U(t)$  imposes performance characteristics on the tracking error  $e(t)$ .

## 2.2 Dynamical Systems

**Theorem 1** [41, Theorem 2.1.1] *Let  $\Omega$  be an open set in  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ . Consider a function  $g : \Omega \rightarrow \mathbb{R}^n$  that satisfies the following conditions: 1) For every  $z \in \mathbb{R}^n$ , the function  $t \rightarrow g(z, t)$  defined on  $\Omega_z := \{t : (z, t) \in \Omega\}$  is measurable. For every  $t \in \mathbb{R}_{\geq 0}$ , the function  $z \rightarrow g(z, t)$  defined on  $\Omega_t := \{z : (z, t) \in \Omega\}$  is continuous; 2) For every compact  $S \subset \Omega$ , there exist constants  $C_S, L_S$  such that:  $\|g(z, t)\| \leq C_S$ ,  $\|g(z, t) - g(y, t)\| \leq L_S \|z - y\|$ ,  $\forall (z, t), (y, t) \in S$ . Then, the initial value problem  $\dot{z} = g(z, t)$ ,  $z_0 = z(t_0)$ , for some  $(z_0, t_0) \in \Omega$ , has a unique and maximal solution defined in  $[t_0, t_{\max})$ , with  $t_{\max} > t_0$  such that  $(z(t), t) \in \Omega, \forall t \in [t_0, t_{\max})$ .*

**Theorem 2** [41, Theorem 2.1.4] *Let the conditions of Theorem 1 hold in  $\Omega$  and let a maximal solution of the initial value problem  $\dot{z} = g(z, t)$ ,  $z_0 = z(t_0)$ , exists in  $[t_0, t_{\max})$  such that  $(z(t), t) \in \Omega, \forall t \in [t_0, t_{\max})$ . Then, either  $t_{\max} = \infty$  or  $\lim_{t \rightarrow t_{\max}^-} \left[ \|z(t)\| + \frac{1}{d_S((z(t), t), \partial\Omega)} \right] = \infty$ , where  $d_S : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is the distance of a point  $x \in \mathbb{R}^n$  to a set  $A$ , defined as  $d_S(x, A) := \inf_{y \in A} \{\|x - y\|\}$ .*

## 2.3 Graph Theory

An *undirected graph*  $\mathcal{G}$  is a pair  $(\mathcal{N}, \mathcal{E})$ , where  $\mathcal{N}$  is a finite set of  $N \in \mathbb{N}$  nodes, representing a team of agents, and  $\mathcal{E} \subseteq \{\{i, j\} : \forall i, j \in \mathcal{N}, i \neq j\}$ , with  $K := |\mathcal{E}|$ , is the set of edges that model the sensing capabilities between neighboring agents. For each agent, its neighboring set  $\mathcal{N}_i$  is defined as  $\mathcal{N}_i := \{j \in \mathcal{N} : \{i, j\} \in \mathcal{E}\}$ . If there is an edge  $\{i, j\} \in \mathcal{E}$ , then  $i, j$  are called *adjacent*. A *path* of length  $r$  from vertex  $i$  to vertex  $j$  is a sequence of  $r + 1$  distinct vertices, starting with  $i$  and ending with  $j$ , such that consecutive vertices are adjacent. For  $i = j$ , the path is called a *cycle*. If there is a path between any two vertices of the graph  $\mathcal{G}$ , then  $\mathcal{G}$  is called *connected*. A connected graph is called a *tree* if it contains no cycles. Consider an arbitrary orientation of  $\mathcal{G}$ , which assigns to each edge  $\{i, j\} \in \mathcal{E}$  precisely one of the ordered pairs  $(i, j)$  or  $(j, i)$ . When selecting the pair  $(i, j)$ , we say that  $i$  is the tail and  $j$  is the head of the edge  $\{i, j\}$ . By

considering a numbering  $k \in \mathcal{K} := \{1, \dots, K\}$  of the graph's edge set, we define the  $N \times K$  *incidence matrix*  $D(G) = [d_{ij}]$ , where:  $d_{ij} = 1$ , if  $i$  is the head of edge  $j$ ;  $d_{ij} = -1$ , if  $i$  is the tail of edge  $j$ ; and  $d_{ij} = 0$ , otherwise.

**Lemma 1** [17, Section III] *Assume that the graph  $\mathcal{G}$  is a tree. Then,  $D(\mathcal{G})^\top \Delta D(\mathcal{G})$  is positive definite for any positive definite matrix  $\Delta \in \mathbb{R}^{N \times N}$ .*

**Proposition 1** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ , with  $f(x) := \exp(x) [\exp(x) - 1] - x^2$ . Then it holds that  $f(x) \geq 0$ ,  $\forall x \in \mathbb{R}_{\geq 0}$ .*

**Proposition 2** [42] *Let  $R_1, R_2 \in \mathbb{SO}(3)$ , and  $e_R := S^{-1}(R_1^\top R_2 - R_2^\top R_1)$ . Then  $\|e_R\|^2 := \|R_1 - R_2\|_F^2 \left(1 - \frac{1}{8} \|R_1 - R_2\|_F^2\right)$ .*

**Proposition 3** [43] *Let  $R_1, R_2 \in \mathbb{SO}(3)$ . Then, for the rotation matrix  $R_2^\top R_1 \in \mathbb{SO}(3)$  it holds that  $-1 \leq \text{tr}[R_2^\top R_1] \leq 3$ ;  $\text{tr}[R_2^\top R_1] = 3$  if and only if  $R_2^\top R_1 = I_3 \Leftrightarrow R_1 = R_2$ ;  $\text{tr}[R_2^\top R_1] = -1$  when  $R_1 = R_2 \exp(\pm \pi S(x))$ , for every  $x$  in the unit sphere, where  $\exp(\cdot)$  here is the matrix exponential.*

## 3 Problem Formulation

Consider a set of  $N$  rigid bodies, with  $\mathcal{N} = \{1, 2, \dots, N\}$ ,  $N \geq 2$ , operating in a workspace  $W \subseteq \mathbb{R}^3$ . We consider that each agent occupies a ball  $\mathcal{B}(p_i, r_i)$ , where  $p_i \in \mathbb{R}^3$  is the position of the agent's center of mass with respect to an inertial frame  $\mathcal{F}_o$  and  $r_i \in \mathbb{R}_{>0}$  is the agent's radius. We also denote as  $R_i \in \mathbb{SO}(3)$  the rotation matrix associated with the orientation of the  $i$ th rigid body. Moreover, we denote by  $v_{i,L} \in \mathbb{R}^3$  and  $\omega_i \in \mathbb{R}^3$  the linear and angular velocity of agent  $i$  with respect to frame  $\mathcal{F}_o$ . The vectors  $p_i$  are expressed in  $\mathcal{F}_o$  coordinates, whereas  $v_{i,L}$  and  $\omega_i$  are expressed with respect to a local frame  $\mathcal{F}_i$  centered at each agent's center of mass. The position of  $\mathcal{F}_o$ , though, is not required to be known by the agents, as will be shown later. By defining  $x_i := (p_i, R_i) \in \mathbb{SE}(3)$  and  $v_i := [v_{i,L}^\top, \omega_i^\top]^\top \in \mathbb{R}^6$ , we model each agent's motion with the 2nd order Newton-Euler dynamics:

$$\dot{x}_i = (R_i v_{i,L}, R_i S(\omega_i)) \in \mathbb{T}_{R_i}, \quad (1a)$$

$$u_i = M_i \dot{v}_i + C_i(v_i) v_i + g_i(x_i) + w_i(x_i, v_i, t), \quad (1b)$$

where the matrix  $M_i \in \mathbb{R}^{6 \times 6}$  is the constant positive definite inertia matrix,  $C_i : \mathbb{R}^6 \rightarrow \mathbb{R}^{6 \times 6}$  is the Coriolis matrix,  $g_i : \mathbb{SE}(3) \rightarrow \mathbb{R}^6$  is the body-frame gravity vector,  $w_i : \mathbb{SE}(3) \times \mathbb{R}^6 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^6$  is a bounded vector representing model uncertainties and external disturbances, and  $\mathbb{T}_{R_i} = \mathbb{R}^3 \times T_R \mathbb{SO}(3)$ , as defined in Section 2. Finally,  $u_i \in \mathbb{R}^6$  is the control input vector representing the 6D body-frame generalized force acting on agent  $i$ . The following properties hold for the aforementioned terms:

- The terms  $M_i, C_i(\cdot), g_i(\cdot)$  are *unknown* to the agents,  $C_i(\cdot), g_i(\cdot)$  are continuous, and it holds that

$$0 < \underline{m}_i < \bar{m}_i < \infty \quad (2a)$$

$$\|g_i(x_i)\| \leq \bar{g}_i, \forall x_i \in \mathbb{SE}(3), \quad (2b)$$

where  $\bar{g}_i$  is a finite *unknown* positive constant and  $\underline{m}_i := \lambda_{\min}(M_i)$ , and  $\bar{m}_i := \lambda_{\max}(M_i)$ , which are also *unknown* to the agents,  $\forall i \in \mathcal{N}$ .

- The functions  $w_i(x_i, v_i, t)$  are assumed to be continuous in  $v_i \in \mathbb{R}^6$  and bounded in  $(x_i, t)$  by *unknown* positive finite constants  $\bar{w}_i$ .

The dynamics (1b) can be written in a vector form representation as:

$$\dot{x} = h_x(x, v), \quad (3a)$$

$$u = M\dot{v} + C(v)v + g(x) + w(x, v, t), \quad (3b)$$

where  $x := (x_1, \dots, x_N) \in \mathbb{SE}(3)^N$ ,  $v := [v_1^\top, \dots, v_N^\top]^\top \in \mathbb{R}^{6N}$ ,  $u := [u_1^\top, \dots, u_N^\top]^\top \in \mathbb{R}^{6N}$ , and  $h_x(x, v) := (h_{x_1}(x_1, v_1), \dots, h_{x_N}(x_N, v_N)) := ((R_1 v_{1,L}, R_1 S(\omega_1)), \dots, (R_N v_{N,L}, R_N S(\omega_N))) \in \mathbb{T}_{R_1} \times \dots \times \mathbb{T}_{R_N}$ ,  $M := \text{diag}\{[M_i]_{i \in \mathcal{N}}\} \in \mathbb{R}^{6N \times 6N}$ ,  $C(v) := \text{diag}\{[C_i(v_i)]_{i \in \mathcal{N}}\} \in \mathbb{R}^{6N \times 6N}$ ,  $g(x) := [g_1(x_1)^\top, \dots, g_N(x_N)^\top]^\top \in \mathbb{R}^{6N}$ ,  $w(x, v, t) := [w_1(x_1, v_1, t)^\top, \dots, w_N(x_N, v_N, t)^\top]^\top \in \mathbb{R}^{6N}$ .

It is also further assumed that each agent has a limited sensing range of  $s_i > \max_{i,j \in \mathcal{N}} \{r_i + r_j\}$ . Therefore, by defining the neighboring function  $\mathcal{N}_i(p) := \{j \in \mathcal{N} : p_j \in \mathcal{B}(p_i, s_i)\}$ , and  $p := [p_1^\top, \dots, p_N^\top]^\top \in \mathbb{R}^{3N}$ , agent  $i$  can measure the relative offset  $R_i^\top(p_i - p_j)$  (i.e., expressed in  $i$ 's local frame), the distance  $\|p_i - p_j\|$ , as well as the relative orientation  $R_j^\top R_i$  with respect to its neighbors  $j \in \mathcal{N}_i(p)$ . In addition, we consider that each agent can measure its own velocity subject to time- and state-varying bounded noise, i.e., agent  $i$  has continuous feedback of  $\tilde{v}_i := [\tilde{v}_{i,L}^\top, \tilde{\omega}_i^\top]^\top := v_i + n_i(x_i, t)$ ,  $\forall i \in \mathcal{N}$ ;  $n_i(x_i, t)$  are assumed to be bounded by *unknown* positive finite constants  $\bar{n}_i$  and  $n_{i,d}(x_i, \dot{x}_i, t) := \dot{n}_i(x_i, t)$  are assumed to be continuous in  $\dot{x}_i$  and bounded in  $(x_i, t)$  by *unknown* positive finite constants  $\bar{n}_{i,d}$ ,  $\forall i \in \mathcal{N}$ .

**Remark 1 [Local relative feedback]** *Note that the agents do not need to have information of any common global inertial frame. The feedback they obtain is relative with respect to their neighboring agents (expressed in their local frames) and they are not required to perform transformations in order to obtain absolute positions/orientations. In the same vein, note also that the velocities  $v_i$  are vectors expressed in the agents' local frames.*

The topology of the multi-agent network is modeled through the *undirected* graph  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ , with  $\mathcal{E} = \{(i, j) \in \mathcal{N}^2 : j \in \mathcal{N}_i(p(0)) \text{ and } i \in \mathcal{N}_j(p(0))\}$  (i.e., the

set of initially connected agents), which is assumed to be nonempty and *connected*. We further denote  $\mathcal{K} := \{1, \dots, K\}$  where  $K := |\mathcal{E}|$ . Given the  $k$ -th edge, we use the simplified notation  $(k_1, k_2)$  for the function that assigns to edge  $k$  the respective agents, with  $k_1, k_2 \in \mathcal{N}$ ,  $\forall k \in \mathcal{K}$ . Since the agents are heterogeneous with respect to their sensing capabilities (different sensing radii  $s_i$ ), the fact that the initial graph is nonempty, connected and undirected implies that

$$\|p_{k_2}(0) - p_{k_1}(0)\| < d_{k,\text{con}}, \quad (4)$$

with  $d_{k,\text{con}} := \min\{s_{k_1}, s_{k_2}\}$ ,  $\forall k \in \mathcal{K}$ . In other words, we consider that the position of the agents at  $t = 0$  is such that the agents for which (4) holds form a connected sensing graph. We also consider that  $\mathcal{G}$  is static in the sense that no edges are added to the graph. We do not exclude, however, edge removal through connectivity loss between initially neighboring agents, which we guarantee to avoid. That is, the proposed methodology guarantees that  $\|p_{k_2}(t) - p_{k_1}(t)\| < d_{k,\text{con}}$ ,  $\forall k \in \mathcal{K}$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ . It is also assumed that at  $t = 0$  the neighboring agents are at a collision-free configuration, i.e.,  $d_{k,\text{col}} < \|p_{k_2}(0) - p_{k_1}(0)\|$ ,  $\forall k \in \mathcal{K}$ , with  $d_{k,\text{col}} := r_{k_1} + r_{k_2}$ . Hence, we conclude that

$$d_{k,\text{col}} < \|p_{k_2}(0) - p_{k_1}(0)\| < d_{k,\text{con}}, \forall k \in \mathcal{K}. \quad (5)$$

The desired formation is specified by the constants  $d_{k,\text{des}} \in \mathbb{R}_{\geq 0}$ ,  $R_{k,\text{des}} \in \mathbb{SO}(3)$ ,  $\forall k \in \mathcal{K}$ , for which, the formation configuration is called *feasible* if the set  $\Phi := \{x \in \mathbb{SE}(3)^N : \|p_{k_2} - p_{k_1}\| = d_{k,\text{des}}, R_{k_2}^\top R_{k_1} = R_{k,\text{des}}, \forall k \in \mathcal{K}\}$  is nonempty. Apart from achieving a desired inter-agent formation while maintaining the initial edges, we aim at guaranteeing that the inter-agent distance of the edges  $k \in \mathcal{K}$  (initially connected agents) stays larger than  $r_{k_1} + r_{k_2}$ , complying with potential collision avoidance specifications. We also make the following required assumption:

**Assumption 1** *The sensing graph  $\mathcal{G}$  is a tree.*

The aforementioned assumption states the initially connected agents in  $\mathcal{E}$  must form a tree graph. In cases where the agents satisfying (4) form a graph that contains cycles, edges can be manually deleted according to certain criteria (e.g. neighboring priorities) in order to obtain a tree sensing graph.

**Problem 1** *Given  $N$  agents governed by the dynamics (1), under Assumption 1 and given the desired inter-agent configuration constants  $d_{k,\text{des}} \in \mathbb{R}_{\geq 0}$ ,  $R_{k,\text{des}} \in \mathbb{SO}(3)$ , with  $d_{k,\text{col}} < d_{k,\text{des}} < d_{k,\text{con}}$ ,  $\forall k \in \mathcal{K}$ , design decentralized control laws  $u_i \in \mathbb{R}^6$ ,  $i \in \mathcal{N}$  such that,  $\forall k \in \mathcal{K}$ , the following hold: 1)  $\lim_{t \rightarrow \infty} \|p_{k_2}(t) - p_{k_1}(t)\| = d_{k,\text{des}}$ ; 2)  $\lim_{t \rightarrow \infty} [R_{k_2}(t)]^\top R_{k_1}(t) = R_{k,\text{des}}$ ; 3)  $d_{k,\text{col}} < \|p_{k_2}(t) - p_{k_1}(t)\| < d_{k,\text{con}}$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ .*

The term ‘‘robust’’ here refers to robustness of the proposed methodology with respect to the unknown dynamics and external disturbances in (1) as well as the unknown noise  $n_i(\cdot)$  in the velocity feedback.

## 4 Main Results

Let us first introduce the distance and orientation errors:

$$e_k := \|p_{k_2} - p_{k_1}\|^2 - d_{k,\text{des}}^2 \in \mathbb{R}, \quad (6a)$$

$$\psi_k := \frac{1}{2} \text{tr} \left[ I_3 - R_{k,\text{des}}^\top R_{k_2}^\top R_{k_1} \right] \in [0, 2], \quad (6b)$$

$\forall k \in \mathcal{K}$ . The fact that  $\psi_k \in [0, 2]$  is derived by using Proposition 3. Regarding  $e_k$ , our goal is to guarantee  $\lim_{t \rightarrow \infty} e_k(t) \rightarrow 0$  from all initial conditions satisfying (5), while avoiding inter-agent collisions and connectivity losses among the initially connected agents specified by  $\mathcal{E}$ . Regarding  $\psi_k$ , we aim to guarantee the following: 1)  $\lim_{t \rightarrow \infty} \psi_k(t) \rightarrow 0$ , which according to Proposition 3 implies that  $\lim_{t \rightarrow \infty} R_{k_2}(t)^\top R_{k_1}(t) = R_{k,\text{des}}$ ; 2)  $\psi_k(t) < 2$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ , since the configuration  $\psi_k = 2$  is an undesired equilibrium, as will be clarified later.<sup>1</sup> By invoking the properties of skew symmetric matrices of Section 2, the errors (6) evolve according to the dynamics:

$$\dot{e}_k = 2(R_{k_1}^\top \tilde{p}_{k_2, k_1})^\top (R_{k_1}^\top R_{k_2} v_{k_2, L} - v_{k_1, L}), \quad (7a)$$

$$\dot{\psi}_k = \frac{1}{2} e_{R_k}^\top (R_{k_1}^\top R_{k_2} \omega_{k_2} - \omega_{k_1}), \quad (7b)$$

where  $\tilde{p}_{k_2, k_1} := p_{k_2} - p_{k_1}$  and  $e_{R_k} := S^{-1}(R_{k_1}^\top R_{k_2} R_{k,\text{des}} - R_{k,\text{des}}^\top R_{k_2}^\top R_{k_1})$ ,  $\forall k \in \mathcal{K}$ . By employing Proposition 2, we obtain  $\|e_{R_k}\|^2 = \|R_{k_2}^\top R_{k_1} - R_{k,\text{des}}\|_F^2 (1 - \frac{1}{8} \|R_{k_2}^\top R_{k_1} - R_{k,\text{des}}\|_F^2)$  as well as  $\|R_{k_2}^\top R_{k_1} - R_{k,\text{des}}\|_F^2 = \text{tr} \left[ (R_{k_2}^\top R_{k_1} - R_{k,\text{des}})^\top (R_{k_2}^\top R_{k_1} - R_{k,\text{des}}) \right] = \text{tr} \left[ 2I_3 - 2R_{k,\text{des}}^\top R_{k_2}^\top R_{k_1} \right] = 4\psi_k$ . Hence, it holds that:

$$\|e_{R_k}\|^2 = 2\psi_k(2 - \psi_k), \quad (8)$$

which implies that:  $\|e_{R_k}\| = 0 \Rightarrow \psi_k = 0$  or  $\psi_k = 2$ ,  $\forall k \in \mathcal{M}$ . The two configurations  $\psi_k = 0$  and  $\psi_k = 2$  correspond to the desired and undesired equilibrium, respectively.

The concepts and techniques of prescribed performance control (see Section 2.1) are adapted in this work in order to: a) achieve predefined transient and steady state response for the distance and orientation errors  $e_k$ ,  $\psi_k$ ,  $\forall k \in \mathcal{K}$ , as well as ii) avoid the violation of the distance

<sup>1</sup> It is well known that topological obstructions do not allow global stabilization on  $\mathbb{S}\mathbb{O}(3)$  with a continuous feedback control law (see [40, 42, 43])

and connectivity constraints between initially neighboring agents, as presented in Section 3. The mathematical expressions of prescribed performance are given by the inequality objectives:

$$-C_{k,\text{col}} \rho_{e_k}(t) < e_k(t) < C_{k,\text{con}} \rho_{e_k}(t), \quad (9a)$$

$$0 \leq \psi_k(t) < \rho_{\psi_k}(t) < 2, \quad (9b)$$

$\forall k \in \mathcal{K}$ , where  $\rho_{e_k} : \mathbb{R}_{\geq 0} \rightarrow \left[ \frac{\rho_{e_k, \infty}}{\max\{C_{k,\text{con}}, C_{k,\text{col}}\}}, 1 \right]$ ,  $\rho_{\psi_k} : \mathbb{R}_{\geq 0} \rightarrow [\rho_{\psi_k, \infty}, \rho_{\psi_k, 0}]$ , with  $\rho_{\psi_k}(t) := (\rho_{\psi_k, 0} - \rho_{\psi_k, \infty}) e^{-l_{\psi_k} t} + \rho_{\psi_k, \infty}$ ,  $\rho_{e_k}(t) := \left[ 1 - \frac{\rho_{e_k, \infty}}{\max\{C_{k,\text{con}}, C_{k,\text{col}}\}} \right] e^{-l_{e_k} t} + \frac{\rho_{e_k, \infty}}{\max\{C_{k,\text{con}}, C_{k,\text{col}}\}}$ , are designer-specified, smooth, bounded, and decreasing functions of time; the constants  $l_{e_k}$ ,  $l_{\psi_k} \in \mathbb{R}_{>0}$ , and  $\rho_{e_k, \infty} \in (0, \max\{C_{k,\text{con}}, C_{k,\text{col}}\})$ ,  $\rho_{\psi_k, \infty} \in (0, \rho_{\psi_k, 0})$ ,  $\forall k \in \mathcal{K}$ , incorporate the desired transient and steady state performance specifications respectively, as presented in Section 2.1, and  $C_{k,\text{col}}$ ,  $C_{k,\text{con}} \in \mathbb{R}_{>0}$ ,  $\forall k \in \mathcal{K}$ , are associated with the distance and connectivity constraints. In particular, we select

$$C_{k,\text{col}} := d_{k,\text{des}}^2 - d_{k,\text{col}}^2, C_{k,\text{con}} := d_{k,\text{con}}^2 - d_{k,\text{des}}^2, \quad (10)$$

$\forall k \in \mathcal{K}$ , which, since the desired formation is compatible with the constraints (i.e.,  $d_{k,\text{col}} < d_{k,\text{des}} < d_{k,\text{con}}$ ,  $\forall k \in \mathcal{K}$ ), ensures that  $C_{k,\text{col}}, C_{k,\text{con}} \in \mathbb{R}_{>0}$ ,  $\forall k \in \mathcal{K}$ , and consequently, in view of (5), that:  $-C_{k,\text{col}} \rho_{e_k}(0) < e_k(0) < \rho_{e_k}(0) C_{k,\text{con}}$ ,  $\forall k \in \mathcal{K}$ . Moreover, assuming that  $\psi_k(0) < 2$ ,  $\forall k \in \mathcal{K}$ , by choosing:

$$\rho_{\psi_k, 0} := \rho_{\psi_k}(0) \in (\psi_k(0), 2), \quad (11)$$

it is also guaranteed that:  $0 \leq \psi_k(0) < \rho_{\psi_k}(0)$ ,  $\forall k \in \mathcal{K}$ . Hence, if we guarantee prescribed performance via (9), by setting the steady state constants  $\rho_{e_k, \infty}$ ,  $\rho_{\psi_k, \infty}$  arbitrarily close to zero and by employing the decreasing property of  $\rho_{e_k}(t)$ ,  $\rho_{\psi_k}(t)$ ,  $\forall k \in \mathcal{K}$ , we guarantee practical convergence of the errors  $e_k(t)$ ,  $\psi_k(t)$  to zero and we further obtain:

$$-C_{k,\text{col}} < e_k(t) < C_{k,\text{con}}, 0 \leq \psi_k(t) < \rho_{\psi_k}(t), \quad (12)$$

$\forall t \in \mathbb{R}_{\geq 0}$ , which, owing to (10), implies:  $d_{k,\text{col}} < \|p_{k_2}(t) - p_{k_1}(t)\| < d_{k,\text{con}}$ ,  $\forall k \in \mathcal{K}$ ,  $t \in \mathbb{R}_{\geq 0}$ , providing, therefore, a solution to problem 1. Moreover, note that the choice of  $\rho_{\psi_k, 0}$  along with (12) guarantee that  $\psi_k(t) < 2$ ,  $\forall t \in \mathbb{R}_{\geq 0}$  and the avoidance of the unstable singularity equilibrium.

In the sequel, we propose a decentralized control protocol that does not incorporate any information on the agents’ dynamic model and guarantees (9) for all  $t \in \mathbb{R}_{\geq 0}$ . Given the errors  $e_k$ ,  $\psi_k$ , we perform the following steps:

**Step I-a:** Select the corresponding functions  $\rho_{e_k}(t)$ ,  $\rho_{\psi_k}(t)$  and positive parameters  $C_{k,\text{con}}$ ,  $C_{k,\text{col}}$ ,  $k \in \mathcal{K}$ ,

following (9), (11), and (10), respectively, in order to incorporate the desired transient and steady state performance specifications as well as the distance and connectivity constraints, and define the normalized errors,  $\forall k \in \mathcal{K}$ ,

$$\xi_{e_k} := \rho_{e_k}(t)^{-1}e_k, \quad \xi_{\psi_k} := \rho_{\psi_k}(t)^{-1}\psi_k. \quad (13)$$

**Step I-b:** Define the transformations  $T_{e_k} : (-C_{k,\text{col}}, C_{k,\text{con}}) \rightarrow \mathbb{R}$ ,  $k \in \mathcal{K}$ , and  $T_\psi : [0, 1) \rightarrow [0, \infty)$  by  $T_{e_k}(x) := \ln\left(\frac{1 + \frac{x}{C_{k,\text{col}}}}{1 - \frac{x}{C_{k,\text{con}}}}\right)$ ,  $T_\psi(x) := \ln\left(\frac{1}{1-x}\right)$ ,  $\forall k \in \mathcal{K}$ , and the transformed error states,  $\forall k \in \mathcal{K}$ ,

$$\varepsilon_{e_k} := T_{e_k}(\xi_{e_k}), \quad \varepsilon_{\psi_k} := T_\psi(\xi_{\psi_k}). \quad (14)$$

Next, we design the decentralized reference velocity vector for each agent  $v_{i,\text{des}} := [v_{i,L,\text{des}}^\top, \omega_{i,\text{des}}^\top]^\top$  as

$$v_{i,\text{des}} = \begin{bmatrix} v_{i,L,\text{des}} \\ \omega_{i,\text{des}} \end{bmatrix} = -\delta_i \begin{bmatrix} 2 \sum_{k \in \mathcal{M}} \alpha(i, k, R_{k_1}, R_{k_2}) \frac{r_{e_k}(\xi_{e_k})}{\rho_{e_k}(t)} \varepsilon_{e_k} R_{k_1}^\top \tilde{p}_{k_2, k_1} \\ \sum_{k \in \mathcal{K}} \alpha(i, k, R_{k_1}, R_{k_2}) \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} e_{R_k} \end{bmatrix}, \quad (15)$$

where  $\delta_i \in \mathbb{R}_{>0}$  are positive gains,  $\forall i \in \mathcal{N}$ ,  $r_{e_k} : (-C_{k,\text{col}}, C_{k,\text{con}}) \rightarrow [1, \infty)$ ,  $r_\psi : [0, 1) \rightarrow [1, \infty)$ , with  $r_{e_k}(x) := \frac{\partial T_{e_k}(x)}{\partial x}$ ,  $r_\psi(x) := \frac{\partial T_\psi(x)}{\partial x}$ , and  $\alpha$  is defined as  $\alpha(i, k, R_{k_1}, R_{k_2}) = -I_3$ , if  $i$  is the tail of the  $k$ th edge ( $i = k_1$ ),  $\alpha(i, k, R_{k_1}, R_{k_2}) = R_{k_2}^\top R_{k_1}$  if  $i$  is the head of the  $k$ th edge ( $i = k_2$ ), and 0 otherwise. The assignment of the head and tail in each edge can be done off-line according to the specified orientation of the graph, as mentioned in Section 2.3.

**Step II-a:** Define for each agent the velocity errors  $e_{v_i} := [e_{v_{i,1}}^\top, \dots, e_{v_{i,6}}^\top]^\top := \tilde{v}_i - v_{i,\text{des}}$ ,  $\forall i \in \mathcal{N}$ , and design the decreasing performance functions as  $\rho_{v_{i,\ell}} : \mathbb{R}_{\geq 0} \rightarrow [\rho_{v_{i,\ell}}^0, \rho_{v_{i,\ell}}^\infty]$ , with  $\rho_{v_{i,\ell}}(t) := (\rho_{v_{i,\ell}}^0 - \rho_{v_{i,\ell}}^\infty) \exp(-l_{v_{i,\ell}} t) + \rho_{v_{i,\ell}}^\infty$ , where the constants  $\rho_{v_{i,\ell}}^0$ ,  $\rho_{v_{i,\ell}}^\infty$ ,  $l_{v_{i,\ell}}$  incorporate the desired transient and steady state specifications, with the design constraints  $\rho_{v_{i,\ell}}^0 > |e_{v_{i,\ell}}(0)|$ ,  $\rho_{v_{i,\ell}}^\infty \in (0, \rho_{v_{i,\ell}}^0)$ ,  $\forall \ell \in \{1, \dots, 6\}$ ,  $i \in \mathcal{N}$ . The term  $e_{v_{i,\ell}}(0)$  can be measured by each agent at  $t = 0$  directly after the calculation of  $v_{i,\text{des}}(0)$ . Moreover, define the normalized velocity errors

$$\xi_{v_i} := [\xi_{v_{i,1}}, \dots, \xi_{v_{i,6}}]^\top := \rho_{v_i}(t)^{-1}e_{v_i}, \quad (16)$$

where  $\rho_{v_i}(\cdot) := \text{diag}\{[\rho_{v_{i,\ell}}(\cdot)]_{\ell \in \{1, \dots, 6\}}\}$ ,  $\forall i \in \mathcal{N}$ .

**Step II-b:** Define the transformation  $T_v : (-1, 1) \rightarrow \mathbb{R}$

as:  $T_v(x) := \ln\left(\frac{1+x}{1-x}\right)$ , and the transformed error states

$$\varepsilon_{v_i} := [\varepsilon_{v_{i,1}}, \dots, \varepsilon_{v_{i,6}}]^\top = [T_v(\xi_{v_{i,1}}), \dots, T_v(\xi_{v_{i,6}})]^\top, \quad (17)$$

Finally, design the decentralized control protocol for each agent  $i \in \mathcal{N}$  as

$$u_i := -\gamma_i [\rho_{v_i}(t)]^{-1} \bar{r}_v(\xi_{v_i}) \varepsilon_{v_i}, \quad (18)$$

where  $\bar{r}_v(\xi_{v_i}) := \text{diag}\{[r_v(\xi_{v_{i,\ell}})]_{\ell \in \{1, \dots, 6\}}\}$  with  $r_v : (-1, 1) \rightarrow [1, \infty)$ ,  $r_v(x) := \frac{\partial T_v(x)}{\partial x}$ , and  $\gamma_i \in \mathbb{R}_{>0}$  are positive gains,  $\forall i \in \mathcal{N}$ .

**Remark 2 [Control protocol intuition]** Note that the selection of  $C_{k,\text{col}}$ ,  $C_{k,\text{con}}$  and of  $\rho_{\psi_k}(t)$ ,  $\rho_{v_{i,\ell}}(t)$ , along with (5), guarantee that  $\xi_{e_k}(0) \in (C_{k,\text{col}}, C_{k,\text{con}})$ ,  $\psi_k(0) \in [0, 2)$ ,  $\xi_{v_{i,\ell}}(0) \in (-1, 1)$ ,  $\forall k \in \mathcal{K}$ ,  $\ell \in \{1, \dots, 6\}$ ,  $i \in \mathcal{N}$ . The prescribed performance control technique enforces  $\xi_{e_k}(t)$ ,  $\xi_{\psi_k}(t)$  and  $\xi_{v_{i,\ell}}(t)$  to remain strictly within the sets  $(-C_{k,\text{col}}, C_{k,\text{con}})$ ,  $[0, 2)$ , and  $(-1, 1)$ , respectively,  $\forall k \in \mathcal{K}$ ,  $\ell \in \{1, \dots, 6\}$ ,  $i \in \mathcal{N}$ ,  $t \geq 0$  (and providing thus a solution to Problem 1), by simply maintaining the boundedness of  $\varepsilon_{e_k}(t)$ ,  $\varepsilon_{\psi_k}(t)$  and  $\varepsilon_{v_i}(t)$  in a compact set,  $\forall t \geq 0$ .

**Remark 3 [Arbitrarily fast convergence to  $\psi_k = 0$ ]** The configurations where  $\|e_{R_k}\| = 0 \Leftrightarrow \psi_k = 0$  or  $\psi_k = 2$  are equilibrium configurations that result in  $\omega_{k_1,\text{des}} = \omega_{k_2,\text{des}} = 0$ ,  $\forall k \in \mathcal{K}$ . If  $\psi_k(0) = 2$ , the orientation formation specification for edge  $k$  cannot be met, since the system becomes uncontrollable. This is an inherent property of stabilization in  $\mathbb{SO}(3)$ , and cannot be resolved with a purely continuous controller [44]. Moreover, initial configurations  $\psi_k(0)$  starting arbitrarily close to 2 might take infinitely long to be stabilized at  $\psi_k = 0$  with common continuous methodologies [45]. Note however, that the proposed control law guarantees convergence to  $\psi_k = 0$  arbitrarily fast, given that  $\psi_k(0) < 2$ . More specifically, given the initial configuration  $\psi_k(0) < 2$ , we can always choose  $\rho_{\psi_k,0}$  such that  $\psi_k(0) < \rho_{\psi_k,0} < 2$ , regardless of how close  $\psi_k(0)$  is to 2. Then, as proved in the next section, the proposed control algorithm guarantees (9b) and the transient and steady state performance of the evolution of  $\psi_k(t)$  is determined solely by  $\rho_{\psi_k}(t)$  and more specifically, its convergence rate is determined solely by the term  $l_{\psi_k}$ . It can be observed from the desired angular velocities  $\omega_{i,\text{des}}$ , designed in (15), that close to the configuration  $\psi_k(0) = 2$ , the term  $e_{R_k}(0)$ , which is close to zero (since  $\psi_k(0) = 2 \Rightarrow \|e_{R_k}(0)\| = 0$ ), is compensated by the term  $r_\psi(\xi_{\psi_k}(0)) = \frac{1}{1-\xi_{\psi_k}(0)}$ , which attains large values (since  $\xi_{\psi_k}(0) = \frac{\psi_k(0)}{\rho_{\psi_k,0}}$  is close to 1). In previous related approaches, the term  $e_{R_k}(0)$  renders the control input arbitrarily small in configurations arbitrarily close to  $\psi_k(0) = 2$ , resulting thus in arbitrarily large stabilization time. Finally, note that potentially large values (but always bounded, as proved in the next section) for  $\omega_{i,\text{des}}$

and hence  $u_i$  due to the term  $r_\psi(\xi_{\psi_k}(0))$  can be compensated by tuning the control gains  $\delta_i$  and  $\gamma_i$ .

**Remark 4 [Decentralized manner, relative feedback, and robustness]** Notice by (15) and (18) that the proposed control protocols are decentralized in the sense that each agent uses only local relative information to calculate its own signal. In that respect, regarding every edge  $k$ , the parameters  $\rho_{e_k, \infty}, \rho_{\psi_k, \infty}, l_{e_k}, l_{\psi_k}$ , as well as the sensing radii  $s_j, \forall j \in \mathcal{N}_i(p(0))$ , which are needed for the calculation of the performance functions  $\rho_{e_k}(t), \rho_{\psi_k}(t)$ , can be transmitted off-line to the agents  $k_1, k_2 \in \mathcal{N}$ . In the same vein, regarding  $\rho_{v_{i,\ell}}(\cdot)$ , the constants  $\rho_{v_{i,\ell}}^\infty, l_{v_{i,\ell}}$  can be transmitted off-line to each agent  $i$ , which can also compute  $\rho_{v_i, 0}$ , given the initial velocity errors  $e_{v_i}(0)$ . Notice also from (15) that each agent  $i$  uses only relative feedback with respect to its neighbors. In particular, for the calculation of  $v_{i, L_{des}}$ , the tail of edge  $k$ , i.e., agent  $k_1$ , uses feedback of  $R_{k_1}^\top(p_{k_2} - p_{k_1})$ , and the head of edge  $k$ , i.e., agent  $k_2$ , uses feedback of  $R_{k_2}^\top R_{k_1} R_{k_1}^\top(p_{k_2} - p_{k_1}) = R_{k_2}^\top(p_{k_2} - p_{k_1})$ . Both of these terms are the relative inter-agent position difference expressed in the respective agent's local frames. For the calculation of  $\omega_{i, des}$ , agents  $k_1$  and  $k_2$  require feedback of the relative orientation  $R_{k_2}^\top R_{k_1}$ , as well as the signal  $S^{-1}(R_{k_1}^\top R_{k_2} R_{k, des} - R_{k, des}^\top R_{k_2}^\top R_{k_1})$ , which is a function of  $R_{k_2}^\top R_{k_1}$ . The aforementioned signals encode information related to the relative pose of each agent with respect to its neighbors, without the need for knowledge of a common global inertial frame. It should also be noted that the proposed control protocol (18) depends exclusively on the velocity of each agent (expressed in the agent's local frame) and not on the velocity of its neighbors. Finally, the proposed control law does not incorporate any prior knowledge of the model nonlinearities/disturbances, enhancing thus its robustness.

**Remark 5 [Inter-agent collision avoidance and tree graph topology]** Note that the proposed protocol does not guarantee collision avoidance among the agents that are not initially connected. Multi-agent collision avoidance under a higher level task (e.g., formation control) has been widely studied in the related literature ([1, 27, 46–51]), with the main issue being the possibility of local minima, due to conflicting objectives. Appropriately designed potential fields restrict these cases to attractor sets of measure zero (e.g., [1, 51]), without, however, eliminating them completely, and at the expense of simplified dynamics and possibly gain tuning. In this work, although we do not deal with collision avoidance among all agents, we derive important novelties with respect to the related literature by considering 2nd-order dynamics in  $\mathbb{SE}(3)$  with unknown dynamic terms and external disturbances, as well as achieving the desired task (inter-agent formation) with prescribed transient- and steady-state performance. The control scheme could be modified by adding appropriate terms that deal with

the collision avoidance among all agents, as well as discontinuous protocols that “pause” the performance functions, which drive the formation control objective, in cases of local minima. Such frameworks require however a significant amount of additional analysis and are thus beyond the scope of this paper. On the same note, the tree graph Assumption 1 is a technical assumption needed for the subsequent analysis, since it guarantees positive definiteness of the term  $D^\top D$ , where  $D$  is the incidence of the graph (see Lemma 1). The assumption is, in fact, a controllability assumption, since it can be proven that  $D^\top D$  is the term multiplying the stacked vector  $u$  of control inputs when considering the dynamics of the errors  $e_k, \psi_k$  (see also Assumption 1 of [32]). In addition, note that the tree topology may in fact be preferable in certain applications where communication costs are taken into account, since the minimum number of communicating edges corresponds to the tree topology.

**Remark 6 [Construction of performance functions and gain tuning]** The desired performance specifications concerning the transient and steady state response and the distance/connectivity constraints are introduced in the control schemes via  $\rho_{e_k}(t), \rho_{\psi_k}(t)$  and  $C_{k, col}, C_{k, con}, k \in \mathcal{K}$ . In addition, the velocity performance functions  $\rho_{v_{i,\ell}}(t)$  impose prescribed performance on the velocity errors  $e_{v_i} = v_i - v_{i, des}, i \in \mathcal{N}$ . The selection of these functions affects both the evolution of the errors within the corresponding performance envelopes as well as the control input characteristics (magnitude and rate). More specifically, relaxing the convergence rate and the steady state limit of the velocity performance functions leads to increased oscillatory behavior within the prescribed performance region, which is improved when considering tighter performance functions, enlarging, however, the control effort both in magnitude and rate. Nevertheless, the only hard constraint attached to their definition is related to their initial values. Specifically,  $\rho_{\psi_k, 0} = \rho_{\psi_k}(0) \in (\psi_k(0), 2), \rho_{v_{i,\ell}}^0 = \rho_{v_{i,\ell}}(0) > |e_{v_{i,\ell}}(0)|, \forall k \in \mathcal{K}, \ell \in \{1, \dots, 6\}, i \in \mathcal{N}$ . In the same vein, as will be verified by the proof of Theorem 3, the actual transient- and steady-state performance of the closed loop system is solely determined by the performance functions  $\rho_{e_k}(t), \rho_{\psi_k}(t), \rho_{v_{i,\ell}}(t)$ , and the constants  $C_{k, col}, C_{k, con}, k \in \mathcal{K}, \ell \in \{1, \dots, 6\}, i \in \mathcal{N}$ , without requiring any tuning of the gains  $\delta_i, \gamma_i, i \in \mathcal{N}$ . It should be noted, however, that their selection affects the control input characteristics and the state trajectory in the prescribed performance area. In particular, decreasing the gain values leads to increased oscillatory behavior within the prescribed performance area, which is improved when adopting higher values, enlarging, however, the magnitude and rate of the control input. Fine gain tuning is also needed in cases where the control input's magnitude and rate need to be bounded by pre-specified saturation values, since, although the proposed methodology yields bounded control inputs, it does not guarantee explicit bounds. In such cases, gain tuning might be needed to

guarantee that the magnitude and rate of the control input do not exceed these values. A detailed analysis regarding the acquirement of such bounds is found in [52].

**Remark 7 [Formation rigidity]** Note that the desired distance and orientation formation defined in this work is not “rigid”, in the sense that the agents can achieve it under more than one relative configurations. This contrasts with certain works in the related literature, where the desired formation can be visualized as a fixed geometric shape in the configuration space (see, e.g., [6, 8–10]).

#### 4.1 Stability Analysis

In this section we provide the main result of this paper, which is summarized in the following theorem.

**Theorem 3** Consider the multi-agent system described by the dynamics (3), under a static tree sensing graph  $\mathcal{G}$ , aiming at establishing a formation described by the desired offsets  $d_{k,des} \in (d_{k,col}, d_{k,con})$  and  $R_{k,des}, \forall k \in \mathcal{K}$ , while satisfying the distance and connectivity constraints between initially neighboring agents, represented by  $d_{k,col}$  and  $d_{k,con}, \forall k \in \mathcal{K}$ . Then, the control protocol (13)–(18) guarantees the prescribed transient and steady-state performance  $-C_{k,col}\rho_{e_k}(t) < e_k(t) < C_{k,con}\rho_{e_k}(t), 0 \leq \psi_k(t) < \rho_{\psi_k}(t), \forall k \in \mathcal{K}, t \in \mathbb{R}_{\geq 0}$ , under all initial conditions satisfying  $\psi_k(0) < 2, \forall k \in \mathcal{K}$  and (5), providing thus a solution to Problem 1.

**PROOF.** We start by defining some vector and matrix forms of the introduced signals and functions:  $e := [e_1, \dots, e_K]^T, \psi := [\psi_1, \dots, \psi_K]^T, e_R := [e_{R_1}^T, \dots, e_{R_K}^T]^T, e_v := [e_{v_1}^T, \dots, e_{v_N}^T]^T, \xi_a := [\xi_{a_1}, \dots, \xi_{a_K}]^T, \xi_v := [\xi_{v_1}^T, \dots, \xi_{v_N}^T]^T, \varepsilon_e := [\varepsilon_{e_1}, \dots, \varepsilon_{e_K}]^T, \varepsilon_\psi := [\varepsilon_{\psi_1}, \dots, \varepsilon_{\psi_K}]^T, \varepsilon_v := [\varepsilon_{v_1}^T, \dots, \varepsilon_{v_N}^T]^T, \tilde{p} := [\tilde{p}_{1_2,1_1}^T, \dots, \tilde{p}_{K_2,K_1}^T]^T, v_L := [v_{1,L}^T, \dots, v_{N,L}^T]^T, v_{Ldes} := [v_{1,Ldes}^T, \dots, v_{N,Ldes}^T]^T, \omega := [\omega_1^T, \dots, \omega_N^T]^T, \omega_{des} := [\omega_{1,des}^T, \dots, \omega_{N,des}^T]^T, v_{des} := [v_{1,des}^T, \dots, v_{N,des}^T]^T, \rho_a(t) := \text{diag}\{\rho_{a_k}(t)\}_{k \in \mathcal{K}}, \rho_v(t) := \text{diag}\{\rho_{v_i}(t)\}_{i \in \mathcal{N}}, r_e(\xi_e) := \text{diag}\{r_{e_k}(\xi_{e_k})\}_{k \in \mathcal{K}}, \Sigma_e(\xi_e, t) := r_e(\xi_e)\rho_e(t)^{-1}, \tilde{r}_\psi(\xi_\psi) := \text{diag}\{r_{\psi_k}(\xi_{\psi_k})\}_{k \in \mathcal{K}}, \Sigma_\psi(\xi_\psi, t) := \tilde{r}_\psi(\xi_\psi)\rho_\psi(t)^{-1}, \tilde{r}_v(\xi_v) := \text{diag}\{\tilde{r}_{v_i}(\xi_{v_i})\}_{i \in \mathcal{N}}, \Sigma_v(\xi_v, t) := \tilde{r}_v(\xi_v)\rho_v(t)^{-1}, where  $a \in \{e, \psi\}$ .$

With the introduced notation, (7) can be written in vector form as

$$\dot{e} = \mathbb{F}_p(\tilde{p})^T \hat{R} D_R(R, \mathcal{G})^T v_L, \quad (19a)$$

$$\dot{\psi} = \mathbb{F}_R(e_R)^T D_R(R, \mathcal{G})^T \omega, \quad (19b)$$

where  $\hat{R} := \text{diag}\{[R_{k_1}]_{k \in \mathcal{K}}\} \in \mathbb{R}^{3K \times 3K}, \mathbb{F}_p(\tilde{p}) :=$

$$2 \begin{bmatrix} \tilde{p}_{1_2,1_1} & \dots & 0_{3 \times 1} \\ \vdots & \ddots & \vdots \\ 0_{3 \times 1} & \dots & \tilde{p}_{K_2,K_1} \end{bmatrix} \in \mathbb{R}^{3K \times K}, \mathbb{F}_R(e_R) := \frac{1}{2} \begin{bmatrix} e_{R_1} & \dots & 0_{3 \times 1} \\ \vdots & \ddots & \vdots \\ 0_{3 \times 1} & \dots & e_{R_K} \end{bmatrix} \in \mathbb{R}^{3K \times K}, \text{ and } D_R \in \mathbb{R}^{3N} \times \mathbb{R}^{3K} \text{ is the orientation incidence matrix of the graph:}$$

$$D_R(R, \mathcal{G}) := \bar{R}^T [D(\mathcal{G}) \otimes I_3] \hat{R}, \quad (20)$$

with  $\bar{R} := \text{diag}\{[R_i]_{i \in \mathcal{N}}\} \in \mathbb{R}^{3N \times 3N}$ , and  $D(\mathcal{G})$  is the incidence matrix of the graph. The terms  $\bar{R}$  and  $\hat{R}$  in  $D_R(R, \mathcal{G})$  correspond to the block diagonal matrix with the agents’ rotation matrices along the main block diagonal, and the block diagonal matrix with the rotation matrix of each edge’s tail along the main block diagonal, respectively. These two terms have motivated the incorporation of the terms  $\alpha(\cdot)$  in the desired velocities  $v_{i,des}$  designed in (15), since, as shown next, the vector form  $v_{des}$  yields the orientation incidence matrix  $D_R(R, \mathcal{G})$ .

The desired velocities (15) and control inputs (18) can be written in vector form as

$$v_{Ldes} = -\Delta D_R(R, \mathcal{G}) \hat{R}^T \mathbb{F}_p(\tilde{p}) \Sigma_e(\xi_e, t) \varepsilon_e, \quad (21a)$$

$$\omega_{des} = -\Delta D_R(R, \mathcal{G}) [\Sigma_\psi(\xi_\psi, t) \otimes I_3] e_R, \quad (21b)$$

$$u = -\Gamma \Sigma_v(\xi_v, t) \varepsilon_v, \quad (21c)$$

where  $\Delta := \text{diag}\{[\delta_i I_3]_{i \in \mathcal{N}}\} \in \mathbb{R}^{3N \times 3N}$  and  $\Gamma := \text{diag}\{[\gamma_i I_6]_{i \in \mathcal{N}}\} \in \mathbb{R}^{6N \times 6N}$ . Note from (21c) and (13), (16), (14), (17) that  $u$  can be expressed as a function of the states  $u(x, v, t)$ . Hence, the closed loop system can be written as  $\dot{x} = h_x(x, v), \dot{v} = h_v(x, v, t) := -M^{-1}\{C(v)v + g(x) + w(x, v, t) - u(x, v, t)\}$ , and by defining  $z := (x, v) \in \mathbb{SE}(3)^N \times \mathbb{R}^{6N}$ :

$$\dot{z} = h(z, t) := (h_x(z), h_v(z, t)). \quad (22)$$

Next, define the set  $\Omega := \{(x, v, t) \in \mathbb{SE}(3)^N \times \mathbb{R}^{6N} \times \mathbb{R}_{\geq 0} : \xi_{e_k}(p_{k_1}, p_{k_2}, t) \in (-C_{k,col}, C_{k,con}), \xi_{\psi_k}(R_{k_1}, R_{k_2}, t) < 1, \xi_{v_i}(x, v_i, t) \in (-1, 1)^6, \forall k \in \mathcal{K}\}$ , where we have expressed  $\xi_{e_k}, \xi_{\psi_k}, \xi_{v_i}$  from (13), (16) as a function of the states. It can be verified that the set  $\Omega$  is open due to the continuity of the operators  $\xi_{e_k}(\cdot), \xi_{\psi_k}(\cdot), \xi_{v_i}(\cdot)$  and nonempty, due to (10). Our goal here is to prove firstly that (22) has a unique and maximal solution  $(z(t), t)$  in  $\Omega$  and then that this solution stays in a compact subset of  $\Omega$ .

It can be verified that the function  $h : \Omega \rightarrow \mathbb{T}_{R_1} \times \dots \times \mathbb{T}_{R_N} \times \mathbb{R}^{6N}$  is (a) continuous in  $t$  for each fixed  $(x, v) \in \{(x, v) \in \mathbb{SE}(3)^N \times \mathbb{R}^{6N} : (x, v, t) \in \Omega\}$ , and (b) continuous and locally lipschitz in  $(x, v)$  for each



fixed  $t \in \mathbb{R}_{\geq 0}$ . Therefore, the conditions of Theorem 1 are satisfied and hence, we conclude the existence of a unique and maximal solution of (22) for a timed interval  $[0, t_{\max})$ , with  $t_{\max} > 0$  such that  $(z(t), t) \in \Omega$ ,  $\forall t \in [0, t_{\max})$ . This implies that

$$\xi_{e_k}(t) = \rho_{e_k}(t)^{-1} e_k(t) \in (-1, 1), \quad (23a)$$

$$\xi_{\psi_k}(t) = \rho_{\psi_k}(t)^{-1} \psi_k(t) < 1, \quad (23b)$$

$$\xi_{v_i}(t) = \rho_{v_i}(t)^{-1} e_{v_i}(t) \in (-1, 1)^6, \quad (23c)$$

$\forall k \in \mathcal{K}$ ,  $i \in \mathcal{N}$ ,  $t \in [0, t_{\max})$ . Therefore, the signals  $e_k(t)$ ,  $\psi_k(t)$ ,  $e_{v_i}(t)$  are bounded for all  $t \in [0, t_{\max})$ . In the following, we aim to show that the solution  $(z(t), t)$  is bounded in a compact subset of  $\Omega$  and hence, by employing Theorem 2, that  $t_{\max} = \infty$ .

Consider the positive definite function  $V_e := \frac{1}{2} \|\varepsilon_e\|^2$ , which is well defined for  $t \in [0, t_{\max})$ , due to (23a). By differentiating  $V_e$ , we obtain  $\dot{V}_e = \varepsilon_e^\top \Sigma_e(\xi_e, t) \{-\dot{\rho}_e(t) \xi_e + \mathbb{F}_p(\tilde{p})^\top \hat{R} D_R(R, \mathcal{G})^\top v_L\}$ , which, by substituting  $v_L = \tilde{v}_L - n_p(x, t) = e_{v_p} + v_{L\text{des}} - n_p(x, t)$  and (19), becomes:

$$\begin{aligned} \dot{V}_e &= -\varepsilon_e^\top \Sigma_e(\xi_e, t) \mathbb{F}_p(\tilde{p})^\top \tilde{D}(\mathcal{G}) \mathbb{F}_p(\tilde{p}) \Sigma_e(\xi_e, t) \varepsilon_e + \\ &\varepsilon_e^\top \Sigma_e(\xi_e, t) \left[ \mathbb{F}_p(\tilde{p})^\top \hat{R} D_R(R, \mathcal{G})^\top (e_{v_p} - n_p(x, t)) - \dot{\rho}_e(t) \xi_e \right], \end{aligned}$$

where  $\tilde{D}(\mathcal{G}) := \hat{R} D_R(R, \mathcal{G})^\top D_R(R, \mathcal{G}) \hat{R}^\top = (D(\mathcal{G})^\top \otimes I_3) \Delta (D(\mathcal{G}) \otimes I_3) \in \mathbb{R}^{3K \times 3K}$  (by employing (20)), and  $e_{v_p}$ ,  $n_p(x, t)$  are the linear parts of  $e_v$  and  $n(x, t)$  (i.e., the stack vector of the first three components of every  $e_{v_i}$ ,  $n_i(x_i, t)$ ), respectively. Note first that, due to (23c), the function  $e_{v_p}(t)$  is bounded for all  $t \in [0, t_{\max})$ . Moreover, note that (23a) implies that  $0 < d_{k,\text{col}} < \|p_{k_1}(t) - p_{k_2}(t)\| < d_{k,\text{con}}$ ,  $\forall t \in [0, t_{\max})$ . Therefore, it holds that  $\text{rank}(\mathbb{F}_p(\tilde{p}(t))) = K$ ,  $\forall t \in [0, t_{\max})$ . In addition, since  $G$  is a connected tree graph and  $\delta_i \in \mathbb{R}_{>0}$ ,  $\forall i \in \mathcal{N}$ ,  $\tilde{D}(\mathcal{G})$  is positive definite and  $\text{rank}(\tilde{D}(\mathcal{G})) = 3K$ . Hence, we conclude that  $\text{rank}([\mathbb{F}_p(\tilde{p}(t))]^\top \tilde{D}(\mathcal{G}) \mathbb{F}_p(\tilde{p}(t))) = K$  and the positive definiteness of  $[\mathbb{F}_p(\tilde{p}(t))]^\top \tilde{D}(\mathcal{G}) \mathbb{F}_p(\tilde{p}(t))$ ,  $\forall t \in [0, t_{\max})$  is deduced. In addition, since  $\|p_{k_2}(t) - p_{k_1}(t)\| < d_{k,\text{con}}$ , we also conclude that the term  $\mathbb{F}_p(\tilde{p})^\top \hat{R} D_R(R, \mathcal{G})^\top$  is upper bounded,  $\forall t \in [0, t_{\max})$ . Finally,  $\dot{\rho}_e(t)$  and  $n_p(x, t)$  are bounded by definition and assumption, respectively,  $\forall x \in \text{SE}(3)^N$ ,  $t \in \mathbb{R}_{\geq 0}$ . We obtain

$$\dot{V}_e \leq -\lambda_{\tilde{D}} \|\Sigma_e(\xi_e, t) \varepsilon_e\| \left[ \|\Sigma_e(\xi_e, t) \varepsilon_e\| - \frac{\bar{B}_e}{\lambda_{\tilde{D}}} \right], \quad \forall t \in [0, t_{\max}), \text{ where}$$

$$\begin{aligned} \lambda_{\tilde{D}} &:= \min_{p(t), t \in [0, t_{\max})} \left\{ \lambda_{\min} \left( [\mathbb{F}_p(\tilde{p}(t))]^\top \tilde{D}(\mathcal{G}) \mathbb{F}_p(\tilde{p}(t)) \right) \right\} \\ &\geq d_{k,\text{col}}^2 \lambda_{\min}(\tilde{D}(\mathcal{G})) > 0, \end{aligned}$$

and  $\bar{B}_e$  is a positive constant, independent of  $t_{\max}$ , satisfying  $\bar{B}_e \geq \|\mathbb{F}_p(\tilde{p})^\top \hat{R} D_R(R, \mathcal{G})^\top (e_{v_p}(t) - n_p(x, t)) - \dot{\rho}_e(t) \xi_e(t)\|$ ,  $\forall t \in [0, t_{\max})$ . Note that, in view of the aforementioned discussion,  $\bar{B}_e$  is finite.

Hence, we conclude that  $\dot{V}_e < 0 \Leftrightarrow \|\Sigma_e(\xi_e, t) \varepsilon_e\| > \frac{\bar{B}_e}{\lambda_{\tilde{D}}}$ .

It holds that  $r_{e_k}(x) = \frac{\partial T_{e_k}(x)}{\partial x} = \frac{\frac{1}{C_{k,\text{col}}} + \frac{1}{C_{k,\text{con}}}}{\left(1 + \frac{x}{C_{k,\text{col}}}\right) \left(1 - \frac{x}{C_{k,\text{con}}}\right)}$   $> \frac{1}{C_{k,\text{col}}} + \frac{1}{C_{k,\text{con}}}$ ,  $\forall x \in (-C_{k,\text{col}}, C_{k,\text{con}})$ , and  $\rho_{e_k}(t) \leq 1$ ,  $\forall t \in \mathbb{R}_{\geq 0}$ ,  $k \in \mathcal{K}$ , and thus we conclude that  $\|\Sigma_e(\xi_e(t), t) \varepsilon_e(t)\| = \sqrt{\sum_{k \in \mathcal{K}} \frac{[r_{e_k}(\xi_{e_k}(t))]^2}{[\rho_{e_k}(t)]^2} [\varepsilon_{e_k}(t)]^2} \geq \bar{C} \|\varepsilon_e(t)\|$ ,  $\forall t \in [0, t_{\max})$ , where  $\bar{C} := \max_{k \in \mathcal{K}} \left\{ \frac{C_{k,\text{col}} + C_{k,\text{con}}}{C_{k,\text{col}} C_{k,\text{con}}} \right\}$ .

Hence, we conclude that  $\dot{V}_e(\varepsilon_e) < 0$ ,  $\forall \|\varepsilon_e\| \geq \frac{\bar{B}_e}{\lambda_{\tilde{D}} \bar{C}}$ ,  $\forall t \in [0, t_{\max})$ . Therefore, by invoking Theorem 4.8 in [53] we conclude that

$$\|\varepsilon_e(t)\| \leq \bar{\varepsilon}_e := \max \left\{ \varepsilon_e(0), \frac{\bar{B}_e}{\lambda_{\tilde{D}} \bar{C}} \right\}, \quad (24)$$

$t \in [0, t_{\max})$ , and by taking the inverse logarithm function:

$$-C_{k,\text{col}} < -\xi_e \leq \xi_{e_k}(t) \leq \bar{\xi}_e < C_{k,\text{con}}, \quad (25)$$

$\forall t \in [0, t_{\max})$ , where  $\bar{\xi}_e := \frac{\exp(\bar{\varepsilon}_e) - 1}{\exp(\bar{\varepsilon}_e) + 1} C_{k,\text{con}}$ , and  $\xi_e := \frac{\exp(-\bar{\varepsilon}_e) - 1}{\exp(-\bar{\varepsilon}_e) + 1} C_{k,\text{col}}$ . Note that  $\varepsilon_e(0)$  is finite due to the assumption  $d_{k,\text{col}} < \|p_{k_2}(0) - p_{k_1}(0)\| < d_{k,\text{con}}$ . Therefore, since  $\lambda_{\tilde{D}}$  is strictly positive and  $\bar{B}_e$  is also finite,  $\bar{\varepsilon}_e$  is well defined. Hence, (24) and (25) imply the boundedness of  $\varepsilon_{e_k}(t)$ ,  $r_{e_k}(\xi_{e_k}(t))$ ,  $\tilde{p}(t)$ , and  $p(t)$  in compact sets,  $\forall k \in \mathcal{K}$ , and therefore, through (15), the boundedness of  $v_{i,L\text{des}}(t)$ ,  $\forall i \in \mathcal{N}$ ,  $t \in [0, t_{\max})$ .

Similarly, consider the positive definite function  $V_\psi = 2 \sum_{k \in \mathcal{K}} \varepsilon_{\psi_k}$ , whose derivative is  $\dot{V}_\psi = 2 \sum_{k \in \mathcal{K}} \frac{r_{\psi_k}(\xi_{\psi_k})}{\rho_{\psi_k}(t)} (\dot{\psi}_k - \dot{\rho}_{\psi_k} \xi_{\psi_k})$ . After substituting (7b), (19), we obtain

$$\begin{aligned} \dot{V}_\psi &= -2 \sum_{k \in \mathcal{K}} \frac{r_{\psi_k}(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \dot{\rho}_{\psi_k}(t) \xi_{\psi_k} + \\ &e_R^\top [\Sigma_\psi(\xi_\psi, t) \otimes I_3] D_R(R, \mathcal{G})^\top [\omega_{\text{des}} + e_{v_R} - n_R(x, t)], \end{aligned}$$

where  $e_{v_R}$  and  $n_R(x, t)$  are the angular parts of  $e_v$  and  $n(x, t)$  (i.e., the stack vector of the last three components of every  $e_{v_i}$ ,  $n_i(x_i, t)$ ), respectively. By substituting (21b) and defining  $\tilde{\Sigma}_\psi(\xi_\psi, t) := \Sigma_\psi(\xi_\psi, t) \otimes I_3 \in \mathbb{R}^{3K \times 3K}$ ,  $\tilde{D}_R(R, \mathcal{G}) := D_R(R, \mathcal{G})^\top \Delta D_R(R, \mathcal{G}) \in \mathbb{R}^{3K \times 3K}$ , we ob-

tain

$$\begin{aligned}\dot{V}_\psi &= -e_R^\top \tilde{\Sigma}_\psi(\xi_\psi, t) \tilde{D}_R(R, \mathcal{G}) \tilde{\Sigma}_\psi(\xi_\psi, t) e_R \\ &\quad + e_R^\top \tilde{\Sigma}_\psi(\xi_\psi, t) D_R(R, \mathcal{G})^\top [e_{v_R} - n_R(x, t)] \\ &\quad - 2 \sum_{k \in \mathcal{K}} \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \dot{\rho}_{\psi_k}(t) \xi_{\psi_k}.\end{aligned}$$

According to (20),  $D_R(R, \mathcal{G}) = \bar{R}^\top [D(\mathcal{G}) \otimes I_3] \hat{R}$ . Since  $\bar{R}$  and  $\hat{R}$  are rotation (and thus unitary) matrices, the singular values of  $D_R(R, \mathcal{G})$  are identical to the ones of  $D(\mathcal{G})$ , and hence  $\lambda_{\min}(\tilde{D}_R(R, \mathcal{G})) = \lambda_{\min}(\tilde{D}(\mathcal{G})) > 0$ . Indeed, let  $D(\mathcal{G}) \otimes I_3 = U \Sigma_D V^\top$  be a singular value decomposition of  $D(\mathcal{G}) \otimes I_3$ , where  $U, V$  are unitary matrices, and  $\Sigma_D$  is a diagonal matrix containing the singular values of  $D(\mathcal{G}) \otimes I_3$ . Then  $D_R(R, \mathcal{G}) = \bar{R}^\top U \Sigma_D V^\top \hat{R} = \tilde{U} \Sigma_D \tilde{V}^\top$  where  $\tilde{U} := \bar{R}^\top U$ , and  $\tilde{V} = \hat{R}^\top V$  are unitary matrices (being products of unitary matrices). Thus,  $\tilde{U} \Sigma_D \tilde{V}^\top$  is the singular value decomposition of  $D_R(R, \mathcal{G})$ , and hence its singular values are the diagonal values of  $\Sigma_D$ . By further defining  $\beta := [\beta_1^\top, \dots, \beta_K^\top]^\top := D_R(R, \mathcal{G})^\top (e_{v_R} - n_R(x, t)) \in \mathbb{R}^{3M}$ , with  $\beta_k \in \mathbb{R}^3$ ,  $\forall k \in \mathcal{K}$ ,  $\dot{V}_\psi$  becomes

$$\begin{aligned}\dot{V}_\psi &\leq -\lambda_{\min}(\tilde{D}(\mathcal{G})) \|\tilde{\Sigma}_\psi(\xi_\psi, t) e_R\|^2 \\ &\quad + \sum_{k \in \mathcal{K}} \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} (e_{R_k})^\top \beta_k - 2 \sum_{k \in \mathcal{K}} \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \dot{\rho}_{\psi_k}(t) \xi_{\psi_k}.\end{aligned}$$

Note that, by construction,  $\xi_{\psi_k} \geq 0$ ,  $\forall k \in \mathcal{K}$ , and  $r_\psi(x) = \frac{\partial T_\psi(x)}{\partial x} = \frac{1}{1-x} > 1, \forall x < 1$ . Hence, in view of (23b), we conclude that  $r_\psi(\xi_{\psi_k}(t)) > 1, \forall t \in [0, t_{\max})$ . By noting also that  $\dot{\rho}_{\psi_k}(t) < 0, \forall t \in \mathbb{R}_{\geq 0}$  and after substituting (8),  $\dot{V}_\psi$  becomes

$$\begin{aligned}\dot{V}_\psi(\varepsilon_\psi) &\leq -\lambda_{\min}(\tilde{D}(\mathcal{G})) \sum_{k \in \mathcal{K}} \left[ \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \right]^2 \|e_{R_k}\|^2 \\ &\quad + \bar{B}_{\psi_1} \sum_{k \in \mathcal{K}} \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \|e_{R_k}\| \\ &\quad + 2 \max_{k \in \mathcal{K}} \{l_{\psi_k}(\rho_{\psi_k,0} - \rho_{\psi_k,\infty})\} \sum_{k \in \mathcal{K}} \frac{r_\psi(\xi_{\psi_k})}{\rho_{\psi_k}(t)} \xi_{\psi_k},\end{aligned}$$

where  $\bar{B}_{\psi_1}$  is a positive constant, independent of  $t_{\max}$ , satisfying  $\bar{B}_{\psi_1} \geq \max_{k \in \mathcal{K}} \{\|\beta_k(t)\|\}, \forall t \in [0, t_{\max})$ . Note that  $\bar{B}_{\psi_1}$  is finite,  $\forall t \in [0, t_{\max})$ , due to (23b) and the boundedness of the noise signals  $n(x, t)$ .

From (23b) and the definition of  $\psi_k$ , we conclude that  $0 \leq \psi_k(t) < \rho_{\psi_k}(t) \leq \rho_{\psi_k,0} < 2$ , and hence  $2 - \psi_k(t) \geq 2 - \rho_{\psi_k,0} =: \underline{\rho}_k > 0 \forall t \in [0, t_{\max}), k \in \mathcal{K}$ . Moreover, by noticing that  $2 - \psi_k \leq 2, \rho_{\psi_k}(t) \leq \rho_{\psi_k,0}$ , and  $\psi_k =$

$\xi_{\psi_k} \rho_{\psi_k}(t), \forall k \in \mathcal{K}$ ,  $\dot{V}_\psi$  becomes

$$\begin{aligned}\dot{V}_\psi &\leq -\tilde{\mu} \sum_{k \in \mathcal{K}} [r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k} \\ &\quad + \frac{2\bar{B}_{\psi_1}}{\max_{k \in \mathcal{K}} \{\sqrt{\rho_{\psi_k,0}}\}} \sum_{k \in \mathcal{K}} r_\psi(\xi_{\psi_k}) \sqrt{\xi_{\psi_k}} \\ &\quad + 2 \max_{k \in \mathcal{K}} \left\{ \frac{l_{\psi_k}(\rho_{\psi_k,0} - \rho_{\psi_k,\infty})}{\rho_{\psi_k,0}} \right\} \sum_{k \in \mathcal{K}} r_\psi(\xi_{\psi_k}) \xi_{\psi_k},\end{aligned}$$

where  $\tilde{\mu} := \frac{2\lambda_{\min}(\tilde{D}(\mathcal{G})) \min_{k \in \mathcal{K}} \{\rho_k\}}{\max_{k \in \mathcal{K}} \{\rho_{\psi_k,0}\}}$ . In view of (23b), it holds that  $\xi_{\psi_k}(t) < \sqrt{\xi_{\psi_k}(t)}, \forall k \in \mathcal{K}$ . By also employing  $\sum_{k \in \mathcal{K}} r_\psi(\xi_{\psi_k}) \sqrt{\xi_{\psi_k}} \leq \sqrt{K} \sqrt{\sum_{k \in \mathcal{K}} (r_\psi(\xi_{\psi_k}))^2 \xi_{\psi_k}}$ , we obtain

$$\dot{V}_\psi \leq -\sqrt{\sum_{k \in \mathcal{K}} [r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k}} \left\{ \tilde{\mu} \sqrt{\sum_{k \in \mathcal{K}} [r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k}} - \bar{B}_\psi \right\},$$

where

$$\bar{B}_\psi := 2\sqrt{K} \left[ \frac{\bar{B}_{\psi_1}}{\max_{k \in \mathcal{K}} \{\sqrt{\rho_{\psi_k,0}}\}} + \max_{k \in \mathcal{K}} \left\{ \frac{l_{\psi_k}(\rho_{\psi_k,0} - \rho_{\psi_k,\infty})}{\rho_{\psi_k,0}} \right\} \right].$$

Therefore,  $\dot{V}_\psi < 0 \Leftrightarrow \sqrt{\sum_{k \in \mathcal{K}} [r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k}} > \frac{\bar{B}_\psi}{\tilde{\mu}}$ . From (14), given  $y = T_\psi(x)$ , we obtain  $[r_\psi(x)]^2 x = \left[ \frac{\partial T(x)}{\partial x} \right]^2 T^{-1}(y) = \frac{1}{(1-x)^2} T^{-1}(y) = \frac{1}{[1-T^{-1}(y)]^2} T^{-1}(y) = \exp(y) [\exp(y) - 1], \forall x \in [0, 1)$ . Therefore,  $[r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k} = \exp(\varepsilon_{\psi_k}) [\exp(\varepsilon_{\psi_k}) - 1]$ , and according to Prop. 1,

$$\begin{aligned}\sqrt{\sum_{k \in \mathcal{K}} [r_\psi(\xi_{\psi_k})]^2 \xi_{\psi_k}} &= \sqrt{\sum_{k \in \mathcal{K}} \exp(\varepsilon_{\psi_k}) [\exp(\varepsilon_{\psi_k}) - 1]} \\ &\geq \sqrt{\sum_{k \in \mathcal{K}} \varepsilon_{\psi_k}^2} = \|\varepsilon_\psi\|.\end{aligned}$$

Hence, we conclude that  $\dot{V}_\psi < 0, \forall \|\varepsilon_\psi\| > \frac{\bar{B}_\psi}{\tilde{\mu}}$ . Therefore,

$$\|\varepsilon_\psi(t)\| \leq \bar{\varepsilon}_\psi := \max \left\{ \varepsilon_\psi(0), \frac{\bar{B}_\psi}{\tilde{\mu}} \right\}, \quad (26)$$

$\forall t \in [0, t_{\max})$ , and by taking the inverse logarithm:

$$0 \leq -\underline{\xi}_\psi \leq \xi_{\psi_k}(t) \leq \bar{\xi}_\psi < 1, \quad (27)$$

$\forall k \in \mathcal{K}$ , where  $\bar{\xi}_\psi := \frac{\exp(\bar{\varepsilon}_\psi)-1}{\exp(\bar{\varepsilon}_\psi)}$  and  $\underline{\xi}_\psi := \frac{\exp(-\bar{\varepsilon}_\psi)-1}{\exp(-\bar{\varepsilon}_\psi)}$ .

Note that  $\bar{B}_\psi$  as well as  $\varepsilon_\psi(0)$  are finite, due to the choice  $\psi_k(0) < \rho_{\psi_k}(0) < 2$ ,  $\forall k \in \mathcal{K}$ . Hence, since  $\tilde{\mu}$  is strictly positive,  $\bar{\varepsilon}_\psi$  is also finite. Therefore, we conclude the boundedness of  $\varepsilon_{\psi_k}, r_{\psi_k}(\xi_{\psi_k}(t)), e_v(t)$  in compact sets,  $\forall k \in \mathcal{K}$ , and therefore, through (15), the boundedness of  $\omega_{i,\text{des}}(t), \forall i \in \mathcal{N}, t \in [0, t_{\max}]$ . From the proven boundedness of  $p(t)$  and  $p_{i,\text{des}}(t)$ , we also conclude the boundedness of  $n(x(t), t)$  and invoking  $\tilde{v} = v + n(x, t) = e_v(t) - v_{\text{des}}(t)$  and (23c), the boundedness of  $v(t)$  and  $\dot{x}(t), \forall t \in [0, t_{\max}]$ . Moreover, in view of (24), (25), (22), (15), we also conclude the boundedness of  $\dot{v}_{\text{des}}(t)$ .

Proceeding along similar lines, we consider the positive definite Lyapunov candidate  $V_v : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  with  $V_v(\varepsilon_v) = \frac{1}{2} \varepsilon_v^\top \Gamma \varepsilon_v$ . By computing  $\dot{V}_v(\varepsilon_v) = \left[ \frac{\partial V_v(\varepsilon_v)}{\partial \varepsilon_v} \right]^\top \dot{\varepsilon}_v$  and using the dynamics  $\dot{\xi}_v = \rho_v(t)^{-1}(\dot{e}_v(t) - \dot{\rho}_v(t)\xi_v)$ , we obtain:

$$\begin{aligned} \dot{V}_v(\varepsilon_v) &= -\varepsilon_v^\top \Sigma_v(\xi_v, t) \Gamma M^{-1} \Gamma \Sigma_v(\xi_v, t) \varepsilon_v \\ &\quad - \varepsilon_v^\top \Sigma_v(\xi_v, t) \left\{ \Gamma M^{-1} \left[ C(v)v + g(x) + w(x, v, t) \right] \right. \\ &\quad \left. - \dot{n}(x, t) + \dot{v}_{\text{des}} + \dot{\rho}_v(t)\xi_v \right\}. \end{aligned} \quad (28)$$

Since we have proven the boundedness of  $v(t)$  and  $\dot{x}, \forall t \in [0, t_{\max}]$  the terms  $C(v)v, \dot{n}(x, t)$ , and  $w(x, v, t)$  are also bounded,  $t \in [0, t_{\max}]$ , due to the continuities of  $C(\cdot), w(\cdot)$ , and  $\dot{n}(\cdot)$  in  $v, \dot{x}$  and the boundedness of  $w(\cdot)$  and  $\dot{n}(\cdot)$  in  $x, t$ . Moreover,  $g(x), \xi_v(t)$ , and  $\dot{\rho}_v(t)$  are also bounded due to (2b), (23c), and by construction, respectively. By also using (2a), we obtain from (28):  $\dot{V}_v(\varepsilon_v) \leq -\underline{\lambda}_K \|\Sigma_v(\xi_v, t)\varepsilon_v\|^2 + \|\Sigma_v(\xi_v, t)\varepsilon_v\| \bar{B}_v$ , where  $\bar{B}_v$  is a positive term, independent of  $t_{\max}$ , satisfying  $\bar{B}_v \geq \left\| \frac{\max_{i \in \mathcal{N}} \{\gamma_i\}}{\min_{i \in \mathcal{N}} \{\bar{m}_i\}} \left[ C(v)v + g(x) + w(x, v, t) \right] - \dot{n}(x, t) + \dot{v}_{\text{des}}(t) + \dot{\rho}_v(t)\xi_v(t) \right\|, \forall t \in [0, t_{\max}]$  and  $\underline{\lambda}_K := \frac{\min_{i \in \mathcal{N}} \{\gamma_i\}^2}{\max_{i \in \mathcal{N}} \{\bar{m}_i\}} > 0$ . Hence,  $\dot{V}_v(\varepsilon_v) < 0 \Leftrightarrow \|\Sigma_v(\xi_v, t)\varepsilon_v\| > \frac{\bar{B}_v}{\underline{\lambda}_K}$ . By noting that  $r_v(x) = \frac{\partial T_v(x)}{\partial x} = \frac{2}{(1+x)(1-x)} > 2 > 1, \forall x \in (-1, 1)$ , as well as  $\rho_{v_{i,\ell}}(t) \leq \rho_{v_{i,\ell}}^0, \forall \ell \in \{1, \dots, 6\}, t \in \mathbb{R}_{\geq 0}$ , we conclude that  $\|\Sigma_v(\xi_v(t), t)\varepsilon_v(t)\| = \sqrt{\sum_{i \in \mathcal{N}} \sum_{\ell \in \{1, \dots, 6\}} \frac{[r_v(\xi_{v_{i,\ell}}(t))]^2}{[\rho_{v_{i,\ell}}(t)]^2} [\varepsilon_{v_{i,\ell}}(t)]^2} \geq \frac{1}{\rho} \|\varepsilon_v(t)\|, \forall t \in [0, t_{\max}]$ , where  $\tilde{\rho} := \max_{i \in \mathcal{N}} \{\rho_{v_{i,m}}^0\}$ . Hence, we

conclude that  $\dot{V}_v(\varepsilon_v) < 0, \forall \|\varepsilon_v\| \geq \frac{\tilde{\rho} \bar{B}_v}{\underline{\lambda}_K}, \forall t \in [0, t_{\max}]$ , and consequently that

$$\|\varepsilon_v(t)\| \leq \bar{\varepsilon}_v := \max \left\{ \varepsilon_v(0), \frac{\tilde{\rho} \bar{B}_v \max_{i \in \mathcal{N}} \{\gamma_i\}}{\underline{\lambda}_K \min_{i \in \mathcal{N}} \{\gamma_i\}} \right\}, \quad (29)$$

$\forall t \in [0, t_{\max}]$ , and by taking the inverse logarithm func-

tion:

$$-1 < -\bar{\xi}_v \leq \xi_{v_{i,\ell}}(t) \leq \bar{\xi}_v < 1, \quad (30)$$

$\forall \ell \in \{1, \dots, 6\}, t \in [0, t_{\max}]$  where  $\bar{\xi}_v := \frac{\exp(\bar{\varepsilon}_v)-1}{\exp(\bar{\varepsilon}_v)+1} = -\frac{\exp(-\bar{\varepsilon}_v)-1}{\exp(-\bar{\varepsilon}_v)+1}$ . Note that the terms  $\bar{B}_v$  finite,  $\forall t \in [0, t_{\max}]$ . Moreover, the term  $\varepsilon_v(0)$  is finite due to the choice  $\rho_{v_{i,\ell}}^0 > |e_{v_{i,\ell}}(0)|, \forall \ell \in \{1, \dots, 6\}, i \in \mathcal{N}$ . Hence, since  $\underline{\lambda}_K$  is strictly positive, the term  $\bar{\varepsilon}_v$  is also finite. Thus, the terms  $e_v(t), \tilde{r}_v(\xi_v(t))$  and hence the control laws (18) are also bounded in compact sets for all  $t \in [0, t_{\max}]$ . What remains to be shown is that  $t_{\max} = \infty$ . Towards that end, suppose that  $t_{\max}$  is finite, i.e.,  $t_{\max} < \infty$ . Then, according to Theorem 2, it holds that

$$L := \lim_{t \rightarrow t_{\max}^-} \left[ \|z(t)\| + \frac{1}{d_S((z(t), t), \partial\Omega)} \right] = \infty, \quad (31)$$

where  $\|z\| := \|p\| + \|v\| + \|R\|_T$  and, with a slight abuse of notation with respect to Section 2,  $d_S((z(t), t), \partial\Omega) := \inf_{(z', v', R') \in \partial\Omega} \{\|z_{p,v} - z'_{p,v}\| + \|R - R'\|_T\}$ , and  $z_{p,v} := [p^\top, v^\top]^\top \in \mathbb{R}^{3N} \times \mathbb{R}^{6N}$ . We now aim to prove that (31) is a contradiction. Firstly, it holds that  $\|R(t)\|_T = \sum_{i \in \mathcal{N}} \|R_i(t)\|_F \leq N \sup_{t \in [0, t_{\max}]} \{\max_{i \in \mathcal{N}} \{R_i(t)\}\}$ . However, according to Proposition 3, it holds that  $-1 \leq \text{tr}(R) \leq 3$  for any  $R \in \text{SO}(3)$ . Hence,  $\|R(t)\|_T \leq 3N, \forall t \in [0, t_{\max}]$ . Moreover, from (30) and (16) we obtain  $\|e_v(t)\| \leq \sqrt{6} \bar{\xi}_v \tilde{\rho}, \forall t \in [0, t_{\max}]$ . By invoking (24), (26), we can also conclude that there exists a finite  $\bar{v}_{\text{des}}$  such that  $\|v_{\text{des}}(t)\| \leq \bar{v}_{\text{des}}, \forall t \in [0, t_{\max}]$ . Therefore, since  $\|n_i(x_i, t)\| \leq \bar{n}_i, \forall x_i \in \mathbb{S}\mathbb{E}(3), t \in \mathbb{R}_{\geq 0}, i \in \mathcal{N}$ ,  $v = \tilde{v} - n(x, t) = e_v + v_{\text{des}} - n(x, t)$  implies that there exists a finite  $\bar{v}$  such that  $\|v(t)\| \leq \bar{v}, \forall t \in [0, t_{\max}]$ . Hence,  $\|p(t)\| = \left\| \int_0^{t_{\max}} \bar{R}(s)v(s)ds \right\| \leq \int_0^{t_{\max}} \|\bar{R}(s)v(s)\| ds = \int_0^{t_{\max}} \|v(s)\| ds \leq \int_0^{t_{\max}} \bar{v} ds \Rightarrow \|p(t)\| \leq t_{\max} \bar{v}, \forall t \in [0, t_{\max}]$ , which proves the boundedness of  $\|p(t)\|$ , since  $t_{\max}$  is bounded. Next, note that  $\partial\Omega = \{(p, v, R, t) \in \mathbb{R}^{3N} \times \mathbb{R}^{6N} \times \text{SO}(3)^N \times \mathbb{R}_{\geq 0} : (\exists k \in \mathcal{K} : \xi_{e_k}(p_{k_1}, p_{k_2}, t) = -C_{k,\text{col}} \text{ or } \xi_{e_k}(p_{k_1}, p_{k_2}, t) = C_{k,\text{con}} \text{ or } \xi_{\psi_k}(R_{k_1}, R_{k_2}, t) = 1) \text{ or } (\exists i \in \mathcal{N}, \ell \in \{1, \dots, 6\} : \xi_{v_{i,\ell}}(x, v_i, t) = -1 \text{ or } \xi_{v_{i,\ell}}(x, v_i, t) = 1)\}$ . We have proved, however, from (25), (27), and (30) that the maximal solution satisfies the strict inequalities  $-C_{k,\text{col}} < -\xi_e \leq \xi_{e_k}(p_{k_1}(t), p_{k_2}(t), t) \leq \bar{\xi}_e < C_{k,\text{con}}, \xi_{\psi_k}(R_{k_1}(t), R_{k_2}(t), t) \leq \bar{\xi}_\psi < 1$ , and  $|\xi_{v_{i,\ell}}(x(t), v_i(t), t)| \leq \bar{\xi}_v < 1, \forall k \in \mathcal{K}, \ell \in \{1, \dots, 6\}, i \in \mathcal{N}, t \in [0, t_{\max}]$ . Therefore, we conclude that there exists a strictly positive constant  $\varepsilon_z, \in \mathbb{R}_{>0}$  such that  $d_S((z(t), t), \partial\Omega) \geq \varepsilon_z$ . Therefore, we have proved that  $L \leq (t_{\max} + 1)\bar{v} + 3N + \varepsilon_z^{-1}$ , which is finite, since  $t_{\max}$  is finite. This contradicts (31) and hence, we conclude that  $t_{\max} = \infty$ .

We have proved the containment of the errors  $e_k(t), \psi_k(t)$  in the domain defined by the prescribed performance funnels:  $-C_{k,\text{col}}\rho_{e_k}(t) < e_k(t) < C_{k,\text{con}}\rho_{e_k}(t)$ ,

$0 \leq \psi_k(t) < \rho_{\psi_k}(t), \forall k \in \mathcal{K}, t \in \mathbb{R}_{\geq 0}$ , which also implies that:  $d_{k,\text{col}} < \|p_{k_1}(t) - p_{k_2}(t)\| < d_{k,\text{con}}, 0 \leq \psi_k(t) < 2, \forall k \in \mathcal{K}, t \in \mathbb{R}_{\geq 0}$ , i.e., avoidance of the singularity  $\psi_k = 2$  and satisfaction of the distance and connectivity constraints for the initially connected edge set  $\mathcal{E}$ . The closed loop signals and functions are also proven to be bounded for all  $t \in [0, \infty)$ , which leads to the conclusion of the proof.

**Remark 8 (Prescribed performance)** *We can deduce from the aforementioned proof that the proposed control scheme achieves its goals without resorting to the need of rendering  $\bar{\varepsilon}_e, \bar{\varepsilon}_\psi, \bar{\varepsilon}_v$  arbitrarily small by adopting extreme values of the control gains  $\delta_i, \gamma_i$ . Notice that (24), (26), and (29) hold no matter how large the finite bounds  $\bar{\varepsilon}_e, \bar{\varepsilon}_\psi, \bar{\varepsilon}_v$  are. Hence, the actual performance of the system is determined solely by the performance functions  $\rho_e(t), \rho_\psi(t), \rho_v(t)$  and the parameters  $C_{k,\text{col}}, C_{k,\text{con}}$ , as mentioned in Remark 6.*

## 5 Simulation Results

We considered  $N = 5$  spherical agents with  $\mathcal{N} = \{1, \dots, 5\}$ , with dynamics of the form (1), with  $r_i = 1\text{m}$ ,  $s_i = 4\text{m}$ , and dynamic parameters (mass and moment of inertia) randomly selected in  $(0, 1)$ ,  $i \in \mathcal{N}$ . We selected the exogenous disturbances and measurement noise as  $w_i = A_{w_i} \sin(\|p_1\|_1 \text{tr}(R_i) \omega_{w,i} t + \phi_{w,i}) v_i$ , and  $n_i = A_{n_i} \sin(\|p_1\|_1 \text{tr}(R_i) \omega_{n,i} t + \phi_{n,i}) v_i$ , where the parameters  $A_{w_i}, A_{n_i}, \omega_{w,i}, \omega_{n,i}, \phi_{w,i}, \phi_{n,i}$  are randomly chosen in  $(0, 0.1)$ ,  $\forall i \in \mathcal{N}$ . The initial conditions were taken as:  $p_1(0) = [0, 0, 0]^\top \text{m}$ ,  $p_2(0) = [-2.1, -2.3, 2]^\top \text{m}$ ,  $p_3(0) = [1.3, 1.3, 1.5]^\top \text{m}$ ,  $p_4(0) = [-2, 3.25, 2.2]^\top \text{m}$ ,  $p_5(0) = [2, 2.4, -0.15]^\top \text{m}$ ,  $R_1(0) = R_4(0) = R_5(0) = I_3$ ,  $R_2(0) = [-0.8253, 0, 0.5646; 0.1, 0.2562; -0.5646, 0, -0.8253]$ ,  $R_3(0) = [-0.3624, 0, 0.9320; 0.6591, 0.7071, 0.2562; -0.6591, 0.7071, -0.2562]$ ,  $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = 0_{6 \times 1}$ , which form the edge set  $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}\}$ . The desired graph formation was defined by the constants  $d_{k,\text{des}} = 2.5\text{m}$  and  $R_{k,\text{des}} = [0.5, -0.8660, 0; 0.6124, 0.3536, -0.7071; 0.6124, 0.3536, 0.7071]$ ,  $\forall k \in \{1, \dots, 4\}$ . We selected  $d_{k,\text{col}} = 2$ ,  $d_{k,\text{con}} = 4$ , and in view of (10),  $C_{k,\text{col}} = 2.25$  and  $C_{k,\text{con}} = 9.75$ . Moreover, the parameters of the performance functions were chosen as  $\rho_{e_k, \infty} = \rho_{\psi_k, \infty} = 0.1$ ,  $\rho_{\psi_k, 0} = 1.99 > \max\{\rho_{\psi_1}(0), \rho_{\psi_2}(0), \rho_{\psi_3}(0)\}$  and  $l_{e_k} = l_{\psi_k} = 1.5$ . In addition, we chose  $\rho_{v_{i,\ell}}^0 = 2|e_{v_{i,\ell}}(0)| + 1$ ,  $l_{v_{i,\ell}} = 0.2$  and  $\rho_{v_{i,\ell}, \infty} = 0.1$ , for every  $i \in \mathcal{N}, \ell \in \{1, \dots, 6\}$ . Finally, the control gains were set to  $\delta_i = 0.1$  and  $\gamma_i = 15, \forall i \in \mathcal{N}$ . The simulation results are shown in Figs. 1-3. In particular, Figs. 1 and 2 depict the distance and orientation errors  $e_k(t), \psi_k(t)$ , respectively, along with the corresponding performance functions  $\rho_k(t), \rho_{\psi_k}(t), \forall k \in \mathcal{K}$ . Moreover, Fig. 3 depict the control inputs of the agents,  $\forall t \in [0, 5]$  seconds. It can be observed that, although the initial errors  $e_k(0)$  and  $\psi_k(0)$  are very close to the performance

bounds, the proposed control algorithm achieves convergence to the desired formation configuration in a short time interval without significant control effort. A video illustrating the simulation results can be found in <http://y2u.be/Z4xLy01twvk>.

## 6 Conclusions and Future Work

In this paper we proposed a robust decentralized control protocol for distance- and orientation-based formation control of multiple rigid bodies with unknown dynamics in  $\mathbb{SE}(3)$ . The transient- and steady-state trajectories of the closed loop system are determined by pre-specified performance functions. Simulation examples have verified the proposed approach. Future efforts will be devoted towards extending the current results to collision avoidance among all the agents as well as relaxing the tree graph assumption.

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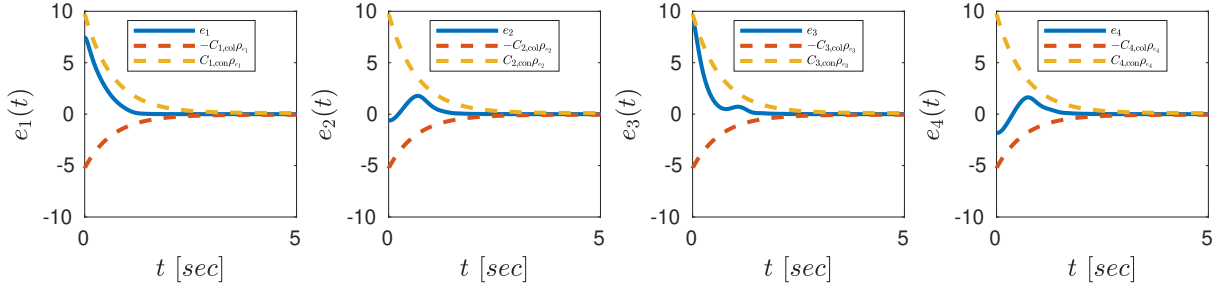


Fig. 1. The distance errors  $e_k(t)$  along with the performance functions  $-C_{k,col}\rho_{e_k}(t)$ ,  $C_{k,con}\rho_{e_k}(t)$ ,  $\forall k \in \mathcal{K}$ .

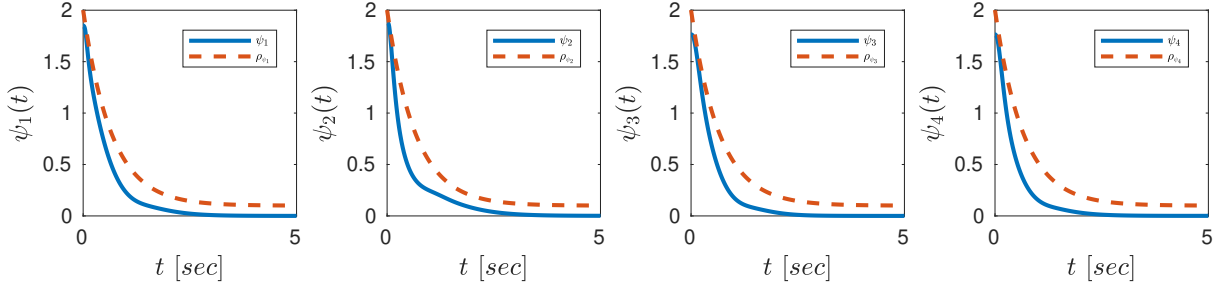


Fig. 2. The orientation errors  $\psi_k(t)$  along with the performance function  $\rho_{\psi_k}(t)$ ,  $\forall k \in \mathcal{K}$ .

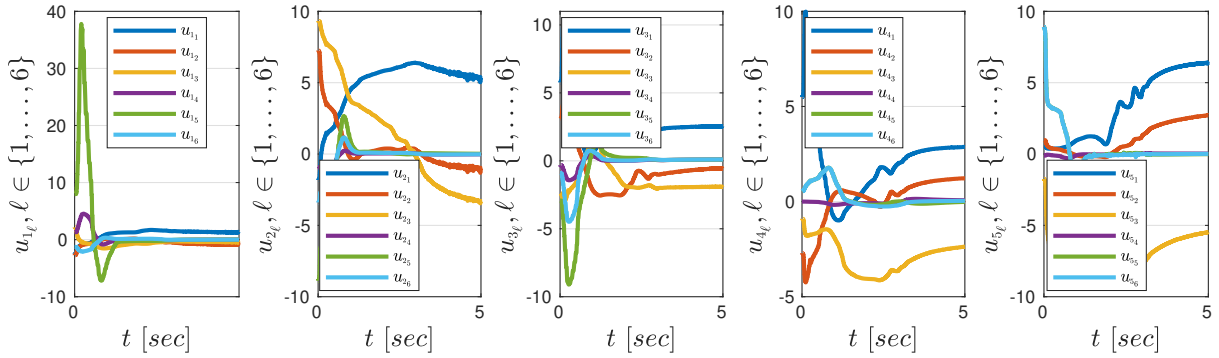


Fig. 3. The control inputs of the agents  $u_i(t)$ ,  $\forall i \in \mathcal{N}$ .

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