Exercise session #9 24 Nov 2014

- Notes on kth.people.se/~turri

Robustness

So far we have considered we know exactly the system under control... BUT this is not always true.

↓

Robustness: how much tolerant is the control system to model errors in $G(s)$

Relative model error: $\underbrace{G^\circ(s)}_{\text{real model}} = \underbrace{G(s)}_{\text{know model}}\left(1 + \underbrace{\Delta_G(s)}_{\text{relative error model}}\right)$

Robustness criteria

Assumptions: 1. $G(s)$ and $G^\circ(s)$ same number of poles in RHP
2. $F(s)$ stabilies $G(s)$ (i.e. $G(s) = \frac{F(s)G(s)}{1+F(s)G(s)}$ stable)
3. $F(s)G(s)$ and $F(s)G^\circ(s)$ tend to 0 as $s \to \infty$

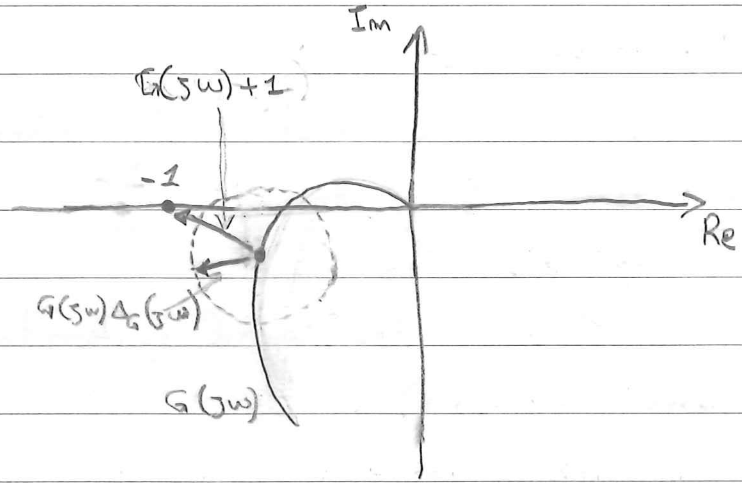
Then, IF $\left(|G_c(j\omega)| = \right) G_c(j\omega) < \frac{1}{|\Delta_G(j\omega)|} \quad \forall \omega$

THEN $G_c^\circ(s) = \frac{F(s)G^\circ(s)}{1 + F(s)G^\circ(s)}$ is stable

Note: the critiria is only sufficient and may be conservative

Intuitive explanation

$$G^{\circ}(s) = G(s) + G(s)\,\Delta_G(s)$$

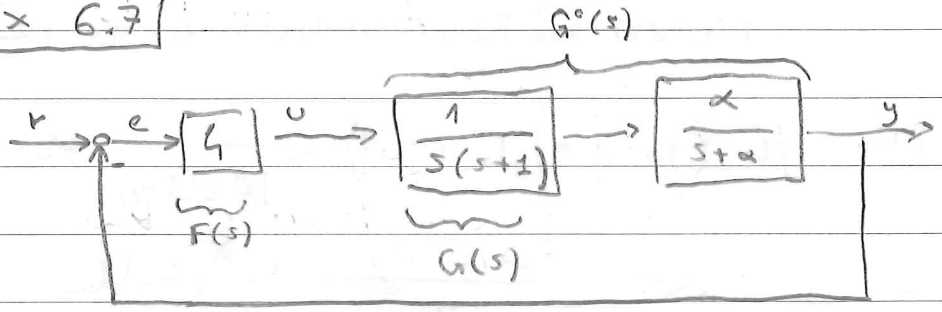


$$|G(j\omega)\,\Delta_G(j\omega)| < |G(j\omega)+1| \quad , \forall \omega$$

$$|G(j\omega)|\,|\Delta_G(j\omega)| < |G(j\omega)+1|$$

$$\frac{|G(j\omega)|}{|G(j\omega)+1|} < \frac{1}{\Delta_G(j\omega)}$$

Ex 6.7



$$G^{\circ}(s) = \frac{1}{s(s+1)} \cdot \frac{\alpha}{s+\alpha} = \underbrace{\frac{1}{s(s+1)}}_{G(s)} \left[1 + \underbrace{\frac{-s}{s+\alpha}}_{\Delta_G(s)} \right]$$

point a

$$G_c^\circ(s) = \frac{G^\circ(s)\,F(s)}{1+G^\circ(s)F(s)} = \frac{4\alpha}{s(s+1)(s+\alpha)+4\alpha}$$

pole eq: $s(s+1)(s+\alpha)+4\alpha=0 \implies \underbrace{s^2(s+1)}_{P(s)} + \underbrace{(s^2+s+4)}_{Q(s)}\alpha=0$

Root locus: (see figure)

- 3 starting points: 0 ; 0 ; -1
- 2 ending points: $-0,5 \pm \frac{\sqrt{15}}{2}$
- 1 asymptote: negative real axis
- intersection with immaginari axis: $\pm 1,73 j$ $(\alpha=3)$

$G_c^\circ(s)$ stable for $\alpha>3$

point b

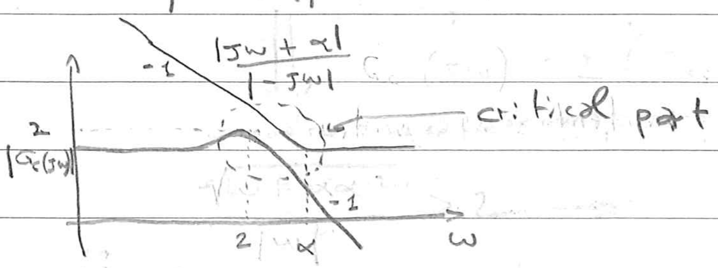
Verify assumptions: 1. $\alpha>0$, $G(s)$ and $G^\circ(s)$ don't have poles in RHP

2. $G_c(s) = \frac{4}{s(s+1)+4}$ is stable

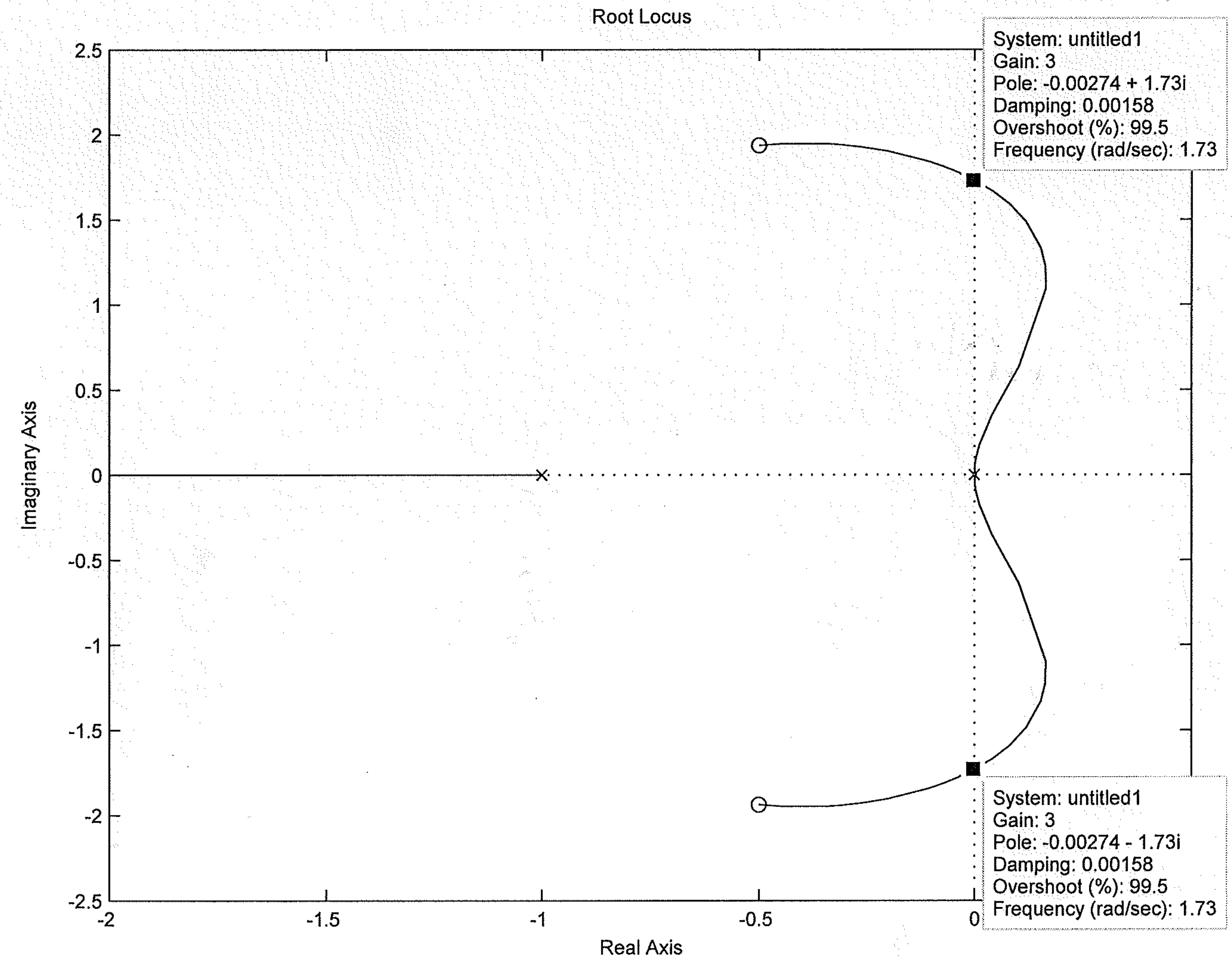
3. Yes & Yes

Real system is stable if:

$$\frac{1}{|\Delta_G(j\omega)|} = \frac{|j\omega+\alpha|}{|1-j\omega|} > |G_c(j\omega)| \quad , \forall\omega$$



$|G_c(j\omega)|$ above $\frac{|j\omega+\alpha|}{|1-j\omega|}$ ensures stability.



Root Locus
System: untitled1
Gain: 3
Pole: -0.00274 + 1.73i
Damping: 0.00158
Overshoot (%): 99.5
Frequency (rad/sec): 1.73
System: untitled1
Gain: 3
Pole: -0.00274 - 1.73i
Damping: 0.00158
Overshoot (%): 99.5
Frequency (rad/sec): 1.73
Imaginary Axis
Real Axis
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0

For critical area ...

$$2 < \frac{\sqrt{2^2+\alpha^2}}{2} \Rightarrow 4 < \sqrt{4+\alpha^2} \Rightarrow \alpha > \sqrt{12}$$

$\boxed{\text{point c}}$ The robustness criteria only provide sufficient condition

→ may be conservative!

State-space model and linearization

Any linear ODE of order n can be rewritten as a system of first order differential equations

$$\begin{cases} \dot{x}(t) = \overset{n\times n}{A}\, x(t) + \overset{n\times 1}{B}\, u(t) \\ y(t) = \overset{1\times n}{C}\, x(t) + \overset{1\times 1}{D}\, u(t) \end{cases}$$

- $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$: state vector
- $u(t)$ = input signal
- $y(t)$ = output signal

If the differential equations are not linear, we can linearize the system around an equibrium point.

Given $\dot{x}(t) = f(x(t), u(t))$, (1)

1- compute equilibrium $0 = f(x_0, u_0)$

2- linearize (1) around equilibrium with Taylor expantion

$$\dot{x}(t) = \underbrace{f(x_0,u_0)}_{\to 0} + \underbrace{\left.\frac{\partial f(x,u)}{\partial x}\right|_{x=x_0}}_{A} \underbrace{(x-x_0)}_{x'} + \underbrace{\left.\frac{\partial f(x,u)}{\partial u}\right|_{u=u_0}}_{B} \underbrace{(u-u_0)}_{u'}$$

$\boxed{\text{Ex } 8.2}$

Pendulum : $l\ddot{\theta} + g\sin\theta + \ddot{z}\cos\theta = 0$

Defining $x_1 = \theta$; $x_2 = \dot{\theta}$; $u = \ddot{z}/l$; $y = \theta$, $w_0^2 = g/l$

$$\dot{x}_2 + w_0^2 \sin(x_1) + u\cos(x_1) = 0$$

<u>state-space model</u>

$$\begin{cases} \dot{x}_1 = x_2 & f_1(x_1, x_2, u) \\ \dot{x}_2 = -w_0^2\sin(x_1) - u\cos(x_1) & f_2(x_1, x_2, u) \\ y = x_1 & h(x_1, x_2, u) \end{cases}$$

<u>Linearization</u>

(1) Equilibrium given $x_{1,0} = \pi$; $x_{2,0} = 0$; $u_0 = 0$; $y_0 = \pi$

(2) Linearize around equilibrium

$$\begin{cases} x_1 = x_{1,0} + \Delta x_1 \\ x_2 = x_{2,0} + \Delta x_2 \\ u = u_0 + \Delta u \\ y = y_0 + \Delta y \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \Delta\dot{x}_1 \\ \dot{x}_2 = \Delta\dot{x}_2 \end{cases}$$

$$\begin{cases} \Delta\dot{x}_1 = \Delta x_2 \\ \Delta\dot{x}_2 = f_2(x_{1,0}, x_{2,0}, u_0) + \left.\dfrac{\partial f_2}{\partial x_1}\right|_{x_{1,0}} \Delta x_1 + \left.\dfrac{\partial f_2}{\partial x_2}\right|_{x_{2,0}} \Delta x_2 + \left.\dfrac{\partial f_2}{\partial u}\right|_{u_0} \Delta u \\ \quad = 0 + \left[-w_0^2 \overset{-1}{\cancel{\cos(\pi)}} + 0\,\overset{0}{\cancel{\sin(\pi)}}\right]\Delta x_1 + 0\,\Delta x_2 + -\overset{-1}{\cancel{\cos(\pi)}}\,\Delta u \\ \quad = -w_0^2\Delta x_1 + \Delta u \\ \Delta y = \Delta x_1 \end{cases}$$

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$$\Delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$\boxed{\text{Ex. 8.3}}$

When the blocks represent first-order system we can use their output as a state

$$\begin{cases} x_1 = z \\ x_2 = \theta \\ x_3 = y \end{cases} \qquad \begin{aligned} u &= i \\ y &= y \\ V &= M_1 \end{aligned}$$

$$\dot{x}_1 = M_i - M_2 = k_1 i - k_2 \theta = -k_2 x_2 - k_1 u$$

$$\dot{x}_2 = z - y = x_1 - x_3$$

$$\dot{x}_3 = M_1 + k_2 \theta = k_2 x_2 + V$$

In state space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -k_2 & 0 \\ 1 & 0 & -1 \\ 0 & k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -k_1 \\ 0 \\ 0 \end{bmatrix} \cdot u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot V$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{\text{Ex } 8.6}$

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} -1 & 2 \end{bmatrix}}_{C} x$$

How to go from state - space model to tranfer function?

$\Longrightarrow$ <u>Laplace trasformation</u>

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} (sI - A)X(s) = BU(s) \quad \longrightarrow \quad X(s) = (sI-A)^{-1}BU(s) \\ Y(s) = CX(s) \end{cases}$$

$$\Downarrow$$

$$Y(s) = \underbrace{C(sI-A)^{-1}B}_{G(s)}U(s)$$

$$sI - A = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+1)(s+3)} \cdot \begin{bmatrix} s+3 & 1 \\ 0 & s+2 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{(s+2)(s+3)}$$

$$= \frac{1}{(s+2)(s+3)} \cdot \left[(-1)(s+4) + 2(s+2)\right]$$

$$= \frac{s}{(s+2)(s+3)}$$

$\boxed{\text{Ex 8.4}}$ from transfer function to state-space model

diff equation

note: not just one solution; there are more degrees of freedom

$\boxed{\text{point a}}$

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}$$

$$\begin{cases} \begin{cases} \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 + 6u \\ \dot{x}_2 = x_3 \\ \dot{x}_1 = x_2 \end{cases} \\ y = x_1 \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{\text{point b}}$

We use one of the canonical forms (Results at pag 158, 159)

$$s^3 Y(s) + s^2 Y(s) + 5s\,Y(s) + 3\,Y(s) = 4s^2 U(s) + s\,U(s) + 2U(s)$$

$$\left[s^3 + s^2 + 5s + 3\right] Y(s) = \left[4s^2 + s + 2\right] U(s)$$

$$Y(s) = \frac{4s^2 + s + 2}{s^3 + s^2 + 5s + 3} \cdot U(s)$$

Let use observer canonical form (pag 159)

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ -5 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$\boxed{\text{point c}}$

$$G(s) = \frac{2s+3}{s^2+5s+6}$$

Use one of the canonical forms or partial fraction expantion

$$Y(s) = \frac{2s+3}{s^2+5s+6} U(s) = \underbrace{-\frac{1}{s+2} U(s)}_{:= X_1(s)} + \underbrace{\frac{3}{s+3} U(s)}_{:= X_2(s)}$$

$$\dot{x}_1 = -2x_1 - u$$

$$x_2 = -3x_2 + 3u$$

$$y = x_1 + x_2$$

matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Resultat 8.1 (Styrbar kanonisk form)
Systemet med överföringsfunktionen

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

kan beskrivas på tillståndsform som

$$\dot{x} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} x$$

Resultat 8.2 (Observerbar kanonisk form)
Systemet med överföringsfunktionen

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

kan beskrivas på tillståndsform som

$$\dot{x} = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} x$$