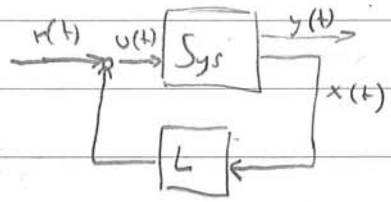
Exercise session #12 3 dec 2014

$\boxed{\text{Ex } 9.4}$

$$\dot{x} = \underset{A}{\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}} x + \underset{B}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} u$$

$$y = \underset{C}{[1 \; -1]} x$$



$L = [l_1 \;\; l_2]$

place poles in $\{-2, -3\}$

- Controllable ?

$$S = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{rank}(S) = 2 \Rightarrow Sys \text{ is controllable}$$

- Desired poles / eigenvalues characteristic equation:

$$(s+2)(s+3) = 0$$

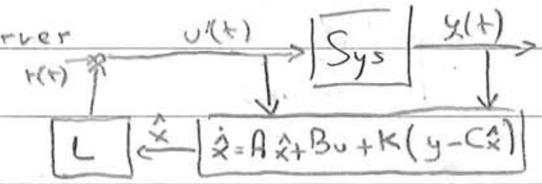
$$\det\left(sI - (A - BL)\right) = 0 \iff (s+2)(s+3) = 0$$

$$\det\left(\begin{bmatrix} s+l_1 & l_2 \\ l_1 & s+1+l_2 \end{bmatrix}\right) = (s+l_1)(s+1+l_2) - l_1 l_2 = s^2 + (l_1+l_2+1)s + l_1 = 0$$

$$\iff s^2 + 5s + 6$$

$$\rightarrow \begin{cases} l_1 = 6 \\ l_1 + l_2 + 1 = 5 \end{cases} \rightarrow \begin{cases} l_1 = 6 \\ l_2 = -2 \end{cases} \Rightarrow L = [6 \;\; -2]$$

If state non accessible $\rightarrow$ observer



Is Sys Observable?

$$O = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(O) = 0 \Rightarrow \underline{\text{Sys is observable}}$$

Rule of thumbs: we place the poles of the observer 2 times faster than the poles of the controlled system

poles in $\{-4 ; -4\}$

Desired eigenvalue / poles characteristic equation

$$(s+4)^2 = 0$$

$$\det\left(sI - (A - \underbrace{K}_{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}} C)\right) = 0$$

$$\det\left(\begin{bmatrix} s+k_1 & -k_1 \\ k_2 & s+1-k_2 \end{bmatrix}\right) = 0 \Rightarrow s^2 + (k_1 - k_2 + 1)s + k_1 = 0$$

$$s^2 + 8s + 16 = 0$$

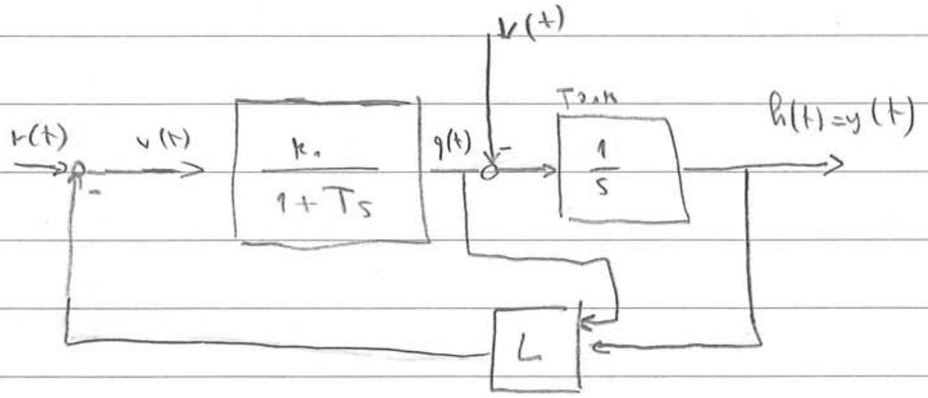
$$\Rightarrow \begin{cases} k_1 = 16 \\ k_1 - k_2 + 1 = 8 \end{cases} \Rightarrow \begin{cases} k_1 = 16 \\ k_2 = 9 \end{cases} \Rightarrow k = \begin{bmatrix} 16 \\ 9 \end{bmatrix}$$

Ex 9.8

part a

$v(t) \rightarrow$ input

$y(t) = h(t) \rightarrow$ output



. State variable: $q(t)$, $h(t)$

$$Q(s) = \frac{k_1}{1+Ts} U(s) \rightarrow Q(s) + Ts\,Q(s) = k_1 U(s)$$

$$\rightarrow q(t) + T\dot{q}(t) = k_1 u(t)$$

$$\begin{cases} \dot{q}(t) = -\frac{1}{T} q(t) + \frac{k_1 u(t)}{T} \\ \dot{h}(t) = \frac{1}{A} q(t) - \frac{1}{A} v(t) \end{cases}$$

Substitute $T = 0.5$; $A = 1$; $k_1 = 1$

$$\begin{bmatrix} \dot{q} \\ \dot{h} \end{bmatrix} = \overset{A}{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} q \\ h \end{bmatrix} + \overset{B}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} u + \overset{E}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} v$$

$$y = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} q \\ h \end{bmatrix}$$

Controllable?

$$S = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \Rightarrow \text{rank}(S) = 2 \Rightarrow \text{Sys is controllable}$$

poles in $\{-2 ; -2\}$

$$\det\left(\left(sI - (A - BL)\right)\right) = \det\left(\begin{bmatrix} s+2+2l_1 & 2l_2 \\ -1 & s \end{bmatrix}\right) = s^2 + (2+2l_1)s + 2l_2$$

$(=)$

$s^2 + 4s + 4$

$$\begin{cases} 2 + 2l_1 = 4 \\ 2l_2 = 4 \end{cases} \Rightarrow \begin{cases} l_1 = 1 \\ l_2 = 2 \end{cases} \Rightarrow L = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$\boxed{\text{part b}}$

- Close-loop system

$$\begin{cases} \dot{x} = Ax + B(r - Lx) + Ev \\ y = Cx \end{cases}$$

$\Downarrow$

$$\begin{cases} \dot{x} = (A - BL)x + Br + Ev \\ y = Cx \end{cases}$$

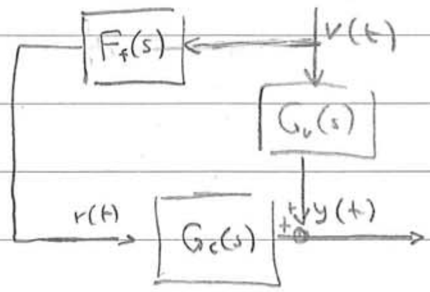
$$\begin{bmatrix} \dot{q} \\ \dot{h} \end{bmatrix} = \underbrace{\begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}}_{A'} \begin{bmatrix} q \\ h \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{B} r + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{E} v$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} q \\ h \end{bmatrix}$$

- steady-state for $v = 0{,}1$; $r = 0$

$$\begin{cases} -4q - 4h = 0 \\ q - 0{,}1 = 0 \end{cases} \Longrightarrow \begin{cases} q = 0{,}1 \\ h = -0{,}1 \end{cases}$$

point c



We want to design $F_f(s)$ such that tf from $V(t)$ to $y(t) = 0$

$$Y(s) = G_v(s)V(s) + F_f(s)G_c(s)V(s)$$

$$Y(s) = \underbrace{\left[G_v(s) + F_f(s)G_c(s)\right]}_{=0} V(s)$$

$$\boxed{F_f(s) = -\frac{G_v(s)}{G_c(s)}}$$

from close-loop state-space model

$$G_c(s) = C(sI - A')^{-1} B$$

$$G_c(s) = C(sI - A')^{-1} E$$

$$(sI - A')^{-1} = \left(\begin{bmatrix} s+4 & 4 \\ -1 & s \end{bmatrix}\right)^{-1} = \frac{1}{s^2+4s-4}\begin{bmatrix} s & -4 \\ 1 & s+4 \end{bmatrix}$$

$$G_c(s) = \frac{1}{s^2+4s-4}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s & -4 \\ 1 & s+4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2}{s^4+4s-4}$$

$$G_v(s) = \frac{1}{s^2+4s-4}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s & -4 \\ 1 & s+4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\frac{s+4}{s^4+4s-4}$$

$$F_f(s) = \frac{s+4}{2} = 2 + \cancel{\frac{s}{2}}$$ remove the derivate term otherwise not implementable

$\boxed{r(t) = 2v(t)}$

What happens at steady state?

$$\begin{cases} -4q - 4h + 4V = 0 \\ q - V = 0 \end{cases} \Rightarrow \begin{array}{l} q = V \\ -\cancel{4q} - 4h + \cancel{4q} = 0 \longrightarrow h = 0 \end{array}$$

No steady state error

$\boxed{\text{point d}}$

k_1 only affect $B = \begin{bmatrix} \frac{k_1}{p} \\ 0 \end{bmatrix} = \begin{bmatrix} 2k_1 \\ 0 \end{bmatrix}$

same control law

$$\boxed{u = -\begin{bmatrix} 1 & 2 \end{bmatrix} x(t) + 2v(t)}$$

· Close-loop system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x \underbrace{- \begin{bmatrix} 2k_1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} x + \begin{bmatrix} 2k_1 \\ 0 \end{bmatrix} 2v}_{= Bu(t)} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} V$$

$$\dot{x} = \begin{bmatrix} -2-2k_1 & -4k_1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 4k_1 \\ -1 \end{bmatrix} V$$

At steady state $\dot{x} = 0$

$$\begin{cases} (-2-2k_1)q - 4k_1 h + 4k_1 V = 0 \\ q = V \end{cases} \Rightarrow \begin{cases} (-2-2k_1)V - 4k_1 h + 4k_1 V = 0 \\ q = V \end{cases}$$

$$h = \frac{-2+2k_1}{4k_1} V = \frac{k_1 - 1}{2k_1} V \neq 0 \text{ if } k_1 \neq 1$$

$\boxed{\text{point e}}$

We introduce the integral of $h(t)$ as a new state

$$z(t) = \int_0^t h(s)\, ds \implies \dot{z} = h$$

Why? Because at steady state $\dot{x} = 0 \implies \dot{z} = 0 \implies h = 0$

$$x = \begin{bmatrix} q \\ h \\ z \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2k_1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} v$$

$$L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}$$

- We define L such that close poles loop (eigenvalue of $(A - BL)$) are stable