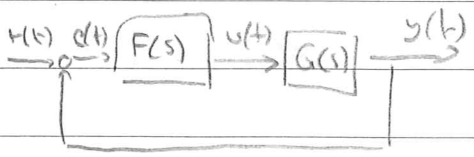
Exercite session #10

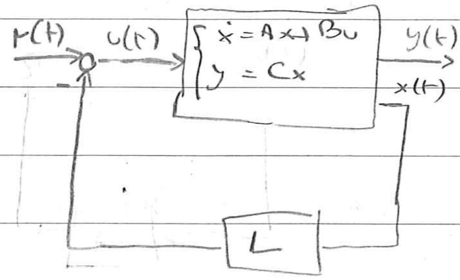
<u>state-space design</u>

Until now:
- frequency-domain design



$U(s) = F(s)\left[R(s) - Y(s)\right]$

Today
- $\boxed{\text{state space design}}$



$u(t) = -Lx(t) + r(t)$

Controlled system

$$\begin{cases} \dot{x}(t) = Ax(t) + B\left[-Lx(t) + r(t)\right] \\ y(t) = Cx(t) \end{cases}$$

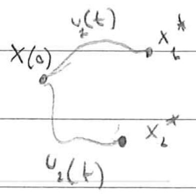
$$\Downarrow$$

$$\begin{cases} \dot{x}(t) = \overbrace{(A - BL)}^{A'}x(t) + Br(t) \\ y(t) = Cx(t) \end{cases}$$

poles of closed-loop system corresponds to the eigenvalues of A'

Tuning L, can we place the closed-loop poles where we want?
Yes, if the system is <u>CONTROLLABLE</u>

system is CONTROLLABLE if there exists a signal $u(t)$ that can drive the state $x(t)$ from $X(0) = 0$ to any arbitrary state x^*.



When is controllable?

Defined the controllability matrix $S = [B \; AB \; \dots \; A^{n-1}B]$,

$\boxed{\text{controllable if } \quad \mathrm{rank}(S) = n}$

The controllable subspace defined as the state space that can be reached is the range space of S, defined as $R(S) = \{y \in \mathbb{R}^n : y = Sx, x \in \mathbb{R}^n\}$

$\boxed{\text{Ex 8.13}}$ $\boxed{\text{draw pendulums}}$

Two pendulum in the unstable position with same input

$$\begin{cases} \ddot{z}\cos\varphi_1 + \ddot{\varphi}_1 = \sin\varphi_1 \\ \ddot{z}\cos\varphi_2 + \alpha\ddot{\varphi}_2 = \sin\varphi_2 \end{cases}$$

$$\begin{cases} x_1 = \varphi_1 \\ x_2 = \dot{\varphi}_1 \\ x_3 = \varphi_2 \\ x_4 = \dot{\varphi}_2 \end{cases} \qquad \begin{cases} u = \ddot{z} \\ y = \varphi_1 = x_1 \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin x_1 - u\cos x_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{\sin x_3}{\alpha} - \dfrac{u\cos x_1}{\alpha} \end{cases}$$

Linearize around $x_1, x_2, x_3, x_4 = 0 \longrightarrow$ don't need to introduce Δx

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - u \\ \dot{x}_3 = x_4 \\ x_4 = \frac{1}{\alpha}x_3 - \frac{1}{\alpha}u \end{cases} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\alpha} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -\frac{1}{\alpha} \end{bmatrix} u$$

$$
\begin{array}{c}
\begin{array}{cccc} B & AB & A^2B & A^3B \end{array} \\
S = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -\frac{1}{\alpha} & 0 & -\frac{1}{\alpha^2} \\ -\frac{1}{\alpha} & 0 & \frac{1}{\alpha^2} & 0 \end{bmatrix}
\end{array}
$$

$$
\det(S) = \frac{1}{\alpha^2}\left(\frac{1}{\alpha^2} - \frac{1}{\alpha}\right) + \left(-\frac{1}{\alpha^3} + \frac{1}{\alpha^2}\right) = \frac{1 - \alpha - \alpha + \alpha^2}{\alpha^4} = \frac{(\alpha-1)^2}{\alpha^4}
$$

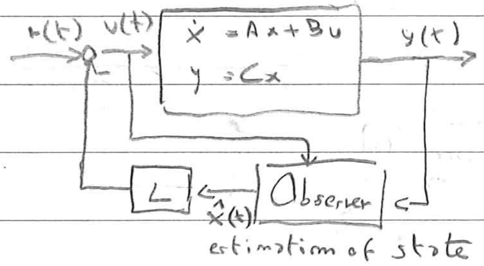
$\det(S) = 0$ (i.e. $\text{rank}(S) < n$) if $\boxed{\alpha = 1}$

$\boxed{\text{Not controllable if } \alpha = 1}$

Explanation: in this case both pendulum are the same. If we start from position $x(t) = 0$ we cannot take the pendulum to 2 different positions.

State observer

If we don't have access to state? STATE OBSERVER



State observer design? We reproduce the system behavior + correction term

$$
\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}
$$

$$
\dot{\hat{x}} = \underbrace{(A - KC)}_{A'}\hat{x} + Bu + Ky
$$

the state error $\left(\tilde{x}(t) = x(t) - \hat{x}(t)\right)$ have dynamics:

$$
\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - (A - KC)\hat{x} - \cancel{Bu} - \overbrace{Ky}^{-KCx} = (A - KC)(x - \hat{x})
$$

$$
\dot{\tilde{x}} = (A - KC)\tilde{x}
$$

Eigenvalues of $A-KC$ define poles of observer
Can poles can place everywhere? if system is OBSERVABLE

Let $u(t)=0\ \forall(t)$. the system is OBSERVABLE if there does not exist any intial condition $x(0)=x^*\neq 0$ for which $y(t)=0$, $\forall t\geq 0$

When is Observable?
Defined the observability matrix $O=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix}$

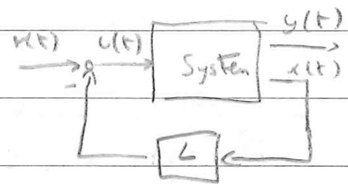
$\boxed{\text{System is OBSERVABLE if } \operatorname{rank}(O)=n}$

The unobservable subspace is the null space of O $N(O)=\{x\in\mathbb{R}^n : Ox=0\}$
with dimension $n-\operatorname{rank}(O)$.

Ex 9.1

$$\dot{x}=\begin{bmatrix} -2 & -1 \\ 1 & 0\end{bmatrix}x+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}x$$



Is controllable? $R=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$
Yes $\leftarrow$ $\det(R)=2$
$\Rightarrow$ we can place the pole where we want!

I) eigenvalues of $A-BL$ in $-3, -5$.
$L=\begin{bmatrix}l_1 & l_2\end{bmatrix}$

$\det\left[sI-(A-BL)\right]=0$

$$\det\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} l_1 & l_2 \\ 0 & 0 \end{bmatrix}\right] = 0$$

$$\det\left(\begin{bmatrix} s+2+l_1 & +1+l_2 \\ -1 & s \end{bmatrix}\right) = 0$$

$$s^2 + 2s + s l_1 + 1 + l_2 = 0$$

$$s^2 + (2+l_1)s + (1+l_2) = 0 \quad (=) \quad (s+3)(s+5) = s^2 + 8s + 15 = 0$$

$$\begin{cases} 2+l_1 = 8 \\ 1+l_2 = 15 \end{cases} \Rightarrow \begin{cases} l_1 = 6 \\ l_2 = 14 \end{cases} \Rightarrow L = \begin{bmatrix} 6 & 14 \end{bmatrix}$$

II) poles in -10 and -15

same calculation .... $\Rightarrow L = \begin{bmatrix} 23 & 149 \end{bmatrix}$

Note that to have "fast poles" we need large gain L and control signal become large

$\boxed{\text{point b}}$

No access to $x(t) \Rightarrow$ design observer

Observer: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - Cx)$, $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$= (A-KC)\hat{x} + Bu + Ky$

Is Observable? $\det(O) = \det\begin{bmatrix} C \\ CA \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \neq 0 \Rightarrow$ Yes

The poles of the observer can be place arbitrarly!

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

Where to place the poles?

- Ideally in RHP far from origin, BUT in this case K really large and therefore the system become iper sensible to noise.
- We still want that they are not dominant compare to poles computed in first part.

  $\Rightarrow$ more on the left then poles designed with L

Example. poles in -20

$$\det\left[sI - (A - KC)\right] = \det\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ k_2 & 0 \end{bmatrix}\right] = 0$$

$$\det\left(\begin{bmatrix} s+2+k_1 & 1 \\ -1+k_2 & s \end{bmatrix}\right) = s^2 + (2+k_1)s + (1-k_2) = 0$$

same solution of $s^2 + 40s + 400 = 0$

$$\begin{cases} 2 + k_1 = 40 \\ 1 - k_2 = 400 \end{cases} \Rightarrow \begin{cases} k_1 = 38 \\ k_2 = -399 \end{cases} \qquad K = \begin{bmatrix} 38 \\ -399 \end{bmatrix}$$