Partiella differentialekvationer, PDE.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad k > 0. \text{ Värmeledningsekvationen .}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad . \text{ Vågekvationen.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad . \text{ Laplace ekvation .}$$

Variabelseparation.

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$. Vågekvationen.

Ansats: $u(x,t) = X(x)T(t)$.

$$a^2 X''(x)T(t) = X(x)T''(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = \text{konstant} = \lambda$$

Ett system av okopplade ODE erhålles.

$$X''(x) - \lambda X(x) = 0$$

$$T''(t) - \lambda a^2 T(t) = 0$$

Linjära med konstanta koefficienter.

Tre olika fall: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

$$\lambda > 0,\ \lambda = \mu^2,\ \mu \quad R.$$

$$X''(x) - \mu^2 X(x) = 0$$

Lösningarna ges av $X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$.

Motsvarande för "T - ekvationen" ger : $\mathrm{T(t)} = \mathrm{C}_1 e^{a\mu t} + D_1 e^{-a\mu t}$.

$$\lambda = 0$$

$$X''(x) = 0$$

$$X(x) = A_2 x + B_2$$

$$T(t) = C_2 t + D_2$$

$$\lambda < 0,\ \lambda = -\mu^2,\ \ \mu \quad R.$$

$$X''(x) + \mu^2 X(x) = 0$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

$$T(t) = C_3 \cos a\mu t + D_3 \sin a\mu t$$

12.4.1.

Vi söker den lösning som uppfyller de givna randvillkoren.

$$u(0,t) = u(L,t) = 0$$

Därefter anpassar vi lösningen till begynnelsevillkoren.

$$u(x,0) = \frac{1}{4}x(L-x), \quad \frac{\partial u}{\partial t}(x,0) = 0$$

Substitutionen ger att randvillkoren kan skrivas

$0 = u(0,t) = X(0)T(t)$

$0 = u(L,t) = X(L)T(t)$

Dessa samband skall gälla för alla t.

Detta innebär att : $0 = X(0),\ 0 = X(L).$

Vi studerar de tre olika fallen.

$$\lambda > 0,\ \lambda = \mu^2,\ \ \mu \quad R.$$

Lösningarna ges av $X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$.

$$0 = \mathrm{X}(0) = \mathrm{A}_1 + B_1$$

$$0 = X(L) = \mathrm{A}_1 e^{\mu L} + B_1 e^{-\mu L}$$

$$B_1 = -A_1$$

$$\mathrm{A}_1 (e^{\mu L} - e^{-\mu L}) = 0$$

Endast den triviala lösningen $A_1 = B_1 = 0$.

$$\lambda = 0$$

$$X(x) = A_2 x + B_2$$

$$T(t) = C_2 t + D_2$$

$$0 = X(0) = B_2$$

$$0 = X(L) = A_2 L + B_2$$

Endast den triviala lösningen $A_2 = B_2 = 0$.

$$\lambda < 0,\ \lambda = -\mu^2,\ \ \mu \quad R.$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

$$0 = X(0) = A_3$$

$$0 = X(L) = A_3 \cos \mu L + B_3 \sin \mu L$$

$$B_3 \sin \mu L = 0$$

$B_3 = 0$ ger endast den triviala lösningen.

Däremot ger $\sin \mu L = 0$ följande: $\mu L = n\pi,\ \ n \quad Z.$

$$X(x) = B_3 \sin \frac{n\pi}{L} x$$

Motsvarande för "T-lösningen": $T(t) = C_3 \cos a \frac{n\pi}{L} t + D_3 \sin a \frac{n\pi}{L} t.$

En lösning som satisfierar differentialekvationen

och de givna randvillkoren är :

$$\mathrm{u}_{\mathrm{n}}(x,t) = B_3 \sin \frac{n\pi}{L} x \ \{ \ C_3 \cos a \frac{n\pi}{L} t + D_3 \sin a \frac{n\pi}{L} t \ \}$$

Varje linjärkombination av lösningar är en lösning.

$$u(x,t) = \sum_{n=1}^{\infty} \{ \ a_n \cos a \frac{n\pi}{L} t + b_n \sin a \frac{n\pi}{L} t \ \} \ \sin \frac{n\pi}{L} x$$

Det återstår nu att bestämma konstanterna a_n och b_n.

Begynnelsevillkoren ger oss dessa.

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty} a\frac{n\pi}{L}\{-a_n \sin a\frac{n\pi}{L}t + b_n \cos a\frac{n\pi}{L}t\} \sin\frac{n\pi}{L}x$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x = \frac{1}{4}x(L-x)$$

$$\frac{\partial}{\partial t}u(x,0) = \sum_{n=1}^{\infty} a\frac{n\pi}{L}b_n \sin\frac{n\pi}{L}x = 0$$

$$b_n = 0$$

$$a_n = \frac{2}{L}\int_0^L \frac{1}{4}\, x(L-x)\sin\frac{n\pi}{L}x \; dx =$$

$$= \frac{2}{L}\left\{ \left[\frac{x(L-x)}{4}\,\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_0^L - \int_0^L \frac{L-2x}{4}\,\frac{-L}{n\pi}\cos\frac{n\pi}{L}x \; dx \right\} =$$

$$= \frac{2}{L}\left\{0 + \frac{L}{n\pi}\left\{ \left[\frac{L-2x}{4}\,\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_0^L - \int_0^L \frac{-2}{4}\,\frac{L}{n\pi}\sin\frac{n\pi}{L}x \; dx \right\}\right\} =$$

$$= \frac{2}{L}\left\{0 + \frac{L}{n\pi}\left\{ \left[\frac{-2}{4}\,\frac{L}{n\pi}\,\frac{L}{n\pi}\cos\frac{n\pi}{L}x\right]_0^L \right\}\right\} = \frac{L^2}{n^3\pi^3}(1-\cos n\pi) =$$

$$
= \begin{cases} 0 & , \quad n = 2m \\ \dfrac{2L^2}{(2m+1)^3 \pi^3} & , \quad n = 2m+1 \end{cases}
$$

$$
u(x,t) = \sum_{n=1}^{\infty} \left\{ \; a_n \cos a \frac{n\pi}{L} t + b_n \sin a \frac{n\pi}{L} t \; \right\} \; \sin \frac{n\pi}{L} x
$$

$$
u(x,t) = \sum_{m=0}^{\infty} \left\{ \; \frac{2L^2}{(2m+1)^3 \pi^3} \cos a \frac{(2m+1)\pi}{L} t \right\} \; \sin \frac{(2m+1)\pi}{L} x
$$