R

Δ ff F

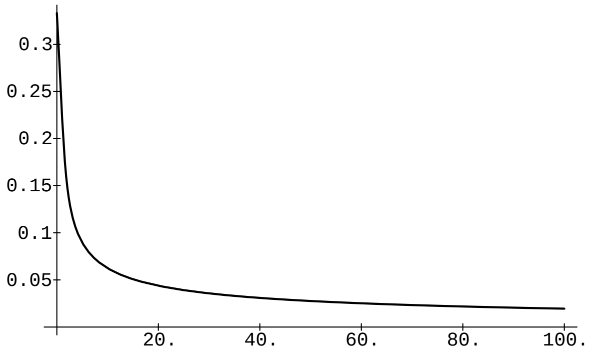
ff

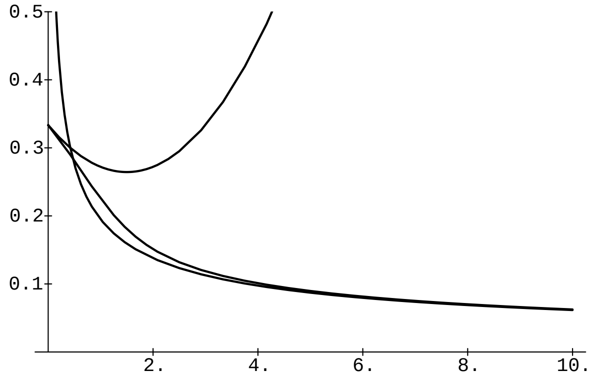
ff

F

$\alpha \approx$

x α 0 5 1 0
: A (A x A 2) G
, $\sigma = \sqrt{t}$, , x, N
t M , d x
x , x $d^N \sim x^{N} \sim \sigma^{N}$,
$t^{N/2}$


0.3
0.25
0.2
0.15
0.1
0.05
20.
40.
60.
80.
100.



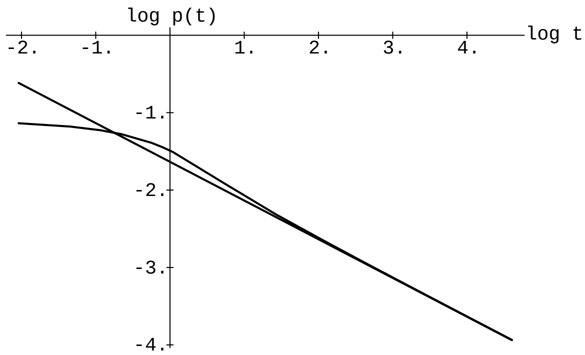
0.5
0.4
0.3
0.2
0.1
2.
4.
6.
8.
10.

Figur 2

$\in$, $\in$,

p

p



Figur 3
∈ , p p
p p
p p $\bar{3}$

ff

ff ff z

$_{diff}$ $_{c}$

4 Su a y a D u

W ff

ff

function of the ordinary scale parameter .

A p] A A 5 6

$$p_d = \frac{}{\pi}\sqrt{\frac{3}{}}\frac{}{\sqrt{}}\left(+ \frac{}{} + O \frac{}{2} \right) \quad 3$$

3

ff $_d$ $p = \frac{1}{3}$

$$_d = \left(\frac{p_d}{p_d}\right) = \left(\frac{\pi}{3}\sqrt{\frac{}{3}}\right) + - \quad + \left(- \frac{}{} + O \frac{}{2} \right) \quad 33$$

Corollary 8 (Effective scale at coarse scales (1D))

At coarse scales the effective scale for one-dimensional discrete signals is approximately (up to an arbitrary affine transformation) a logarithmic function of the ordinary scale parameter .

$-\frac{1}{t} + O\ \frac{1}{t^2}$ ff

ff ff

A B

3 4 B w D R

ff

$_c$ p p_d F

F 3 A ff

ff W

$$_{diff} = \frac{_d \quad - \ _c}{_c \quad - \ _c} = \frac{_d \quad - \ _c}{\frac{\log(2)}{2}} \quad 34$$

$$_c\ _d = \frac{7}{\pi^2} \approx \quad 34$$

σ 5 4

P 5 P

Corollary 6 (Effective scale for discrete signals (1D))

For discrete one-dimensional signals the effective scale parameter $_d$ *as function of the ordinary scale parameter* *is given by*

$$_d = A' + B'' \left(\frac{4\pi}{3\pi + 6arctan\left(\frac{a\ (t)}{\sqrt{a^2(t)\ \ a^2(t)}}\right)} \right)$$

for some arbitrary constants A'' and $B'' >$ with a_0 and a_1 are given by (22) and (23).

W ff $_d$ =

$_d =$ 'A z W A = $B =$

3 3 y B v F

A M $_d p$] A A 5 5

$$p_d = \frac{}{3} - \frac{}{\sqrt{3\pi}} + \frac{}{6\sqrt{3\pi}}\ ^2 + O\ ^3 \quad 3$$

ff $_d$ M] A A 5 5

$$_d = \left(\frac{p_d}{p_d}\right) = \frac{\sqrt{3}}{\pi} + \left(\frac{}{\sqrt{3\pi}} + \frac{3}{\pi^2}\right)\ ^2 + O\ ^3 \quad 3$$

Corollary 7 (Effective scale at fine scales (1D))

At fine scales the effective scale for one-dimensional discrete signals is approximately an affine

$$a_1 \quad = C \quad \eta_1, \eta_2 \quad = C \quad \eta_2, \eta_1 \quad = T \quad ; \qquad - \quad T \quad ; \qquad + T \quad ; \qquad (3)$$

F $a_0 \quad - a_1 \quad = T \quad ; \quad - T \quad ;$ $T \quad T \quad i; \quad \geq T \quad j;$

$|i| > |j|$ $_0 a \quad > a_1$ $_0 a \quad >$ $_d p$

$$p_d \quad = \int\int_{\{\eta=(\eta \ ,\eta_2):(\eta \ \geq 0)\wedge(\eta_2\geq 0)\}} \frac{}{\sqrt{\quad \pi \ ^2 |C_{2D}|}} e^{\ \frac{}{2}\eta^T C_{2D}\eta} d\eta_1 d\eta_2 \qquad (4)$$

A] A A 5 4

$$p_d \quad = \frac{}{4} + \frac{}{\pi} \left(\frac{a_1}{\sqrt{a_0^2 \quad - a_1^2}} \right) \qquad (5)$$

0 a_1

$[\ , \frac{1}{2}]$ W $_0 a$ a_1

$= \ :$

$$p_d \quad = \frac{}{3} \qquad (6)$$

Proposition 5 (Density of local extrema in discrete scale-space (1D))

In the scale-space representation (17) of a one-dimensional discrete signal generated by a white noise stationary normal process, the expected density of local maxima (minima) in a smoothed signal at a certain scale is given by (25) with a_0 and a_1 according to (22) and (23).

5 3

$=$ $_c p \quad = p_d$

$$\frac{}{\pi}\sqrt{\frac{3}{}}\frac{}{\sqrt{\ }} = \frac{}{3} \qquad (7)$$

f L

i $1, L_i, L_{i+1}$ ^{T}

$\xi = \xi_1, \xi_2, \xi_3$ ^{T} j

ξ z f z f

L $_L$ C L

$$C_L \cdot; = T \cdot; * T \cdot; * f C \cdot = T \cdot; * C_f \cdot$$

C_f f

$T \cdot; s * T \cdot; = T \cdot; s +$

$_f$ C $_L$ $\cdot$ C $= T \cdot;$

$T -n; = T n;$ ξ j

m_{3D} C_{3D} :

$$m_{3D} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}; \quad C_{3D} = \begin{pmatrix} T ; & T ; & T ; \\ T ; & T ; & T ; \\ T ; & T ; & T ; \end{pmatrix}$$

$\eta_1 = \xi_2 - \xi_1$ $\eta_2 = \xi_2 - \xi_3$ $\eta = \eta_1, \eta_2$ ^{T}

j

$$m_{2D} = \begin{pmatrix} \\ \\ \end{pmatrix}; \quad C_{2D} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}$$

F $C \cdot, \cdot$

$$a_0 = C \eta_1, \eta_1 = C \eta_2, \eta_2 = T ; - T ;$$

normal process with spectral density $\omega^{\ \beta}$*, the expected density of local maxima (minima) in a smoothed signal at a certain scale decreases with scale as* $^{\ \ }_{2}$.

3 D y

F

z A

W

[] $f : Z \to R$

$L : Z \times R_+ \to R$

$$L\ x; \quad = \sum_{n=\ \infty}^{\infty} T\ n; \quad f\ x - n \qquad (7)$$

$T\ n; \quad = e^{\ t} I_n$ $_n$ I

] [

z ff] [

C

7 :

$$P\ x_i \qquad = $$

$$P\ \ L\ x_i; \quad \geq L\ x_{i\ 1}; \quad \wedge\ \ L\ x_i; \quad \geq L\ x_{i+1};$$

---

[7]A x ff “>” “≥”,

a graph showing the density of local maxima (minima) as function of scale can be expected[6] *to be a straight line in a log-log diagram*

$$p_c = - \frac{3}{-} - \pi -- = cons\ an\ — \quad 4$$

P P

Corollary 3 (Effective scale for continuous signals (1D))

For continuous one-dimensional signals the effective scale parameter $_c$ *as function of the ordinary scale parameter is (up to an arbitrary affine transformation, i.e., for some arbitrary constants* A' *and* $B' >$ *) given by a logarithmic transformation*

$$_c = A + B' \quad 5$$

A

A

] [[6] $S\ w = w^{\beta}$ F

β 3] [

$_f Sw$ z

7 $\beta < 3$ W

$$p_{c,\beta} = \frac{}{\pi} \sqrt{\frac{\int_{\infty}^{\infty} \omega^4 \frac{1}{4} e^{\omega^2 t} \omega^{\beta} d\omega}{\int_{\infty}^{\infty} \omega^2 \frac{1}{4} e^{\omega^2 t} \omega^{\beta} d\omega}} = \quad = \frac{}{\pi} \sqrt{\frac{3-\beta}{\sqrt{}}} \quad \beta < 3 \quad 6$$

Proposition 4 (Density of local extrema in scale-space (fractal noise, 1D))

In the scale-space representation of a one-dimensional continuous signal generated by a stationary

---

[6] O , x ,
q H ,
ff x

h g F G

$$g\ \xi; \quad = \frac{}{\sqrt{\ \pi}} e^{\ \xi^2/2t}; \quad G\ \omega; \quad = - e^{\ \omega^2 t/2}$$

A f $_f\ S\omega$ $=$

$$S_L\ \omega \ = \frac{}{4} e^{\ \omega^2 t}$$

] [5 77

$$\int_0^\infty x^m e^{\ ax^2} dx = \frac{\Gamma\ \frac{m+1}{2}}{a^{\frac{m+}{2}}}$$

$_c$: p

$$p_c \quad = \frac{}{\pi} \sqrt{\frac{\int_{\ \infty}^{\infty} \omega^4 \frac{1}{4} e^{\ \omega^2 t} d\omega}{\int_{\ \infty}^{\infty} \omega^2 \frac{1}{4} e^{\ \omega^2 t} d\omega}} = \frac{}{\pi} \sqrt{\frac{\frac{\Gamma(\frac{5}{2})}{2t^{\frac{5}{2}}}}{\frac{\Gamma(\frac{3}{2})}{2t^{\frac{3}{2}}}}} = \frac{}{\pi} \sqrt{\frac{3}{}} \frac{}{\sqrt{\ }} \qquad 3$$

[5] z

Proposition 2 (Density of local extrema in scale-space (white noise, 1D))

In the scale-space representation of a one-dimensional continuous signal generated by a white noise stationary normal process, the expected density of local maxima (minima) in a smoothed signal at a certain scale decreases with scale as $^{\ \ \overline{2}}$*.*

---

[5] O q $(p\ _c(t) \sim t^{\ \frac{1}{2}})$ x
spatial derivatives (j $H = G$ $H = (i\omega)^n G$)

3 D y L a E a a Fu S a

ffi

“ ”? /

?

W

W]R [7

P

[6] C [4

3 y

μ

R

S F R :

$$\mu = \frac{}{\pi}\sqrt{-\frac{R^{(4)}}{R''}} = \frac{}{\pi}\sqrt{\frac{\int_{\infty}^{\infty}\omega^4 S \quad \omega \quad d\omega}{\int_{\infty}^{\infty}\omega^2 S \quad \omega \quad d\omega}} \qquad 7$$

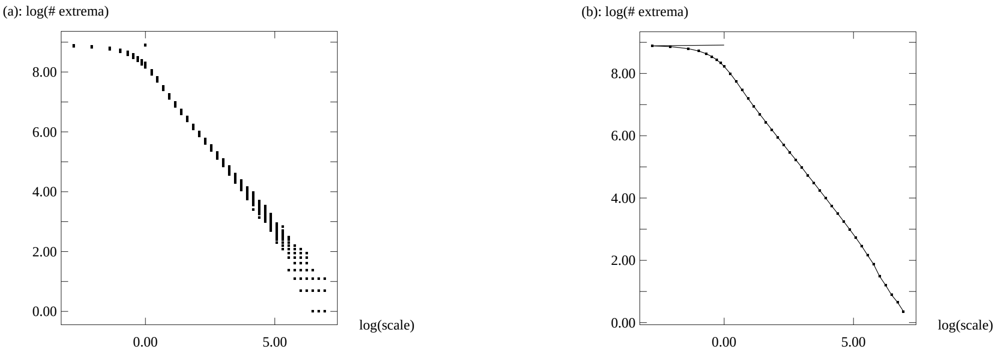
L f

L $_L$ S

$$S_L \quad \omega \quad = |H \quad \omega \quad|^2 S_f \quad \omega$$

S_f f $H \quad \omega$ F h

$$H \quad \omega \quad = \int_{\infty}^{\infty} h \qquad e^{\ i\omega t} d$$



Figur

A *inner scale*

outer] *scale* [7

z

ff

C

] [

ff j

ff

A B W A

z j ff ff

B j

ff

W

Ex **Ex** **R**

F

M¨] [5

3 p $=$ $/$

$$= A + B \quad p \quad = A + B \quad - \alpha B \qquad 6$$

ff R

ff

F

ff

F

ff ;

D **D** **v**

A

4

$$p \quad = \{ \quad \}$$

W h ff

$= h$:

Requirement 1 (Uniform relative decay rate for local extrema)

The probability that a certain extremum point (or equivalently a certain blob) disappears after a small increment d in effective scale should be independent of both the effective scale and the current number of local extrema in the signal. That is

$$\frac{\frac{dp}{d\tau}}{p} = \frac{d \quad p}{d} = C_1 = cons \; an \qquad 3$$

3 :

$$p = C_1 \quad + C_2 \qquad 4$$

C_2 A B

Proposition 1 (Effective scale)

Assume that we know how the expected density of local extrema p behaves as a function of scale and let be the effective scale parameter given by Req. 1. Then, for some arbitrary constants A and B > , the effective scale as function of the ordinary scale parameter is given by

$$= h \quad = A + B \quad p \qquad 5$$

[4] I $p(t)$ x
x t

A j “ ”

W ff

ff

j

[3]

2 T a a h S a Pa a : Eff v

A

W “ ff ”?

ff

z ?

M *certain* [3]

Δ ff

the relative decay rate of local extrema should be constant over scales

---

[3] F x
I , x
( ) x [11] H ,
x x , , x

] [6] [5]

discrete

continuous scale parameter[2] [ ]

inner scale [7]

1 2 $_{life} = \;_2 - \;_1$

F

z

$_{life} = \;_2 - \;_1$

“

”

A ff W

ff

W

ff ff

A

ff

[2] I , , x

(H , q z q)

1 I u

W W [ ] *scale-space* “

”

:

lifetime of structures[1]

ff

ff

$$\frac{\partial L}{\partial} = -\nabla^2 L$$

$L\ \cdot;\ = f$ $^N \times LR_+R \to R$

$f : R^N \to R$ [7]

[, 2] $h : R \to R$

$_{life} = h\ 2 - h\ 1$

effective scale

ff

F *continuous* A

A A ff

A x A 1

Eff v S a : Na u a U

M a u S a -S a L

Tony Lindeberg

C V A P CVAP

A C

R 44

: @

To appear in IEEE Transactions on Pattern Analysis and Machine Intelligence.

A str ct

W *w* *ff* *w*

w

fi *w*

x *w* *w* *w* *ff*

fi

x

w *w* *w*

x *x* *w*

ff

K *w* *:* *ff*

x