F4 Kedjeregeln och partiella differentiationer

Definition: Låt $f : D \subset \mathbb{R}^n \to \mathbb{R}$, D öppen mängd.

Vi säger att $f \in C^k(D)$ om f är k gånger partiellt deriverbar och alla derivatorna är kontinuerliga.

Sats: Om $f \in C^1(D)$ så är f differentierbar i D.

Bevis: Fallet då $D \subset \mathbb{R}^2$.

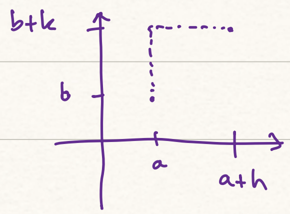
Vi vill visa att

$$f(a+h, b+k) - f(a,b) = f_1'(a,b)h + f_2'(a,b)k + |(h,k)|\,\rho(h,k)$$

där $\rho(h,k) \to 0$ då $(h,k) \to \bar{0}$.

Medelvärdessatsen

$$\boxed{h(x+t) - h(x) = h'(x+\theta t)\cdot t}$$



$$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b)$$

$$= f_1'(a+\theta_1 h, b+k)\cdot h + f_2'(a, b+\theta_2 k)k$$

$$= f_1'(a,b)h + f_2'(a,b)k + |(h,k)|\cdot\left(\frac{1}{|(h,k)|}\Big(\big(f_1'(a+\theta_1 h, b+k) - f_1'(a,b)\big)h\right.$$

$$\left. + \big(f_2'(a, b+\theta_2 k) - f_2'(a,b)\big)k\Big)\right)$$

$$g(h,k) = \frac{1}{|(h,k)|}\Big[\big(f_1'(a+\theta_1 h, b+k) - f_1'(a,b)\big)h + \big(f_2'(a, b+\theta_2 k) - f_2'(a,b)\big)k\Big]$$

$|g(h,k)| \to 0$, då $(h,k) \to \bar{0}$ ty $|h| \leq |(h,k)|$

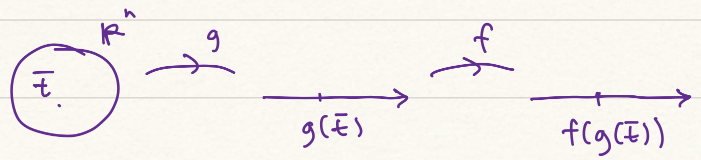
och $|f_1'(a+\theta_1 h, b+k) - f_1'(a,b)| \to 0$ då $(h,k) \to \bar{0}$

eftersom f_1' är kontinuerlig.

<u>Kedjeregeln:</u>

Dimension $= 1$ $\quad \frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t)$.

<u>Fall 1:</u>



$g : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$.

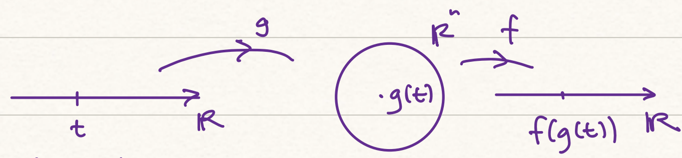
Fixera variablerna $t_2, t_3, \ldots, t_n$ $\quad (h(t_1) = g(t_1, t_2, \ldots, t_n))$

$$\frac{d}{dt_1}(f(h(t_1)) = f'(h(t_1)) \cdot h'(t_1)$$

$$= f'(g(\bar{t})) \cdot g_1'(\bar{t}).$$

<u>Fall 2:</u>

<u>Sats:</u> (Kedjeregeln)



Låt $f : \mathbb{R}^n \to \mathbb{R}$ vara differentierbar och låt $g_1, g_2, \ldots, g_n$ vara deriverbara. Då gäller

$$\frac{d}{dt}(f(g(t))) = \sum_{j=1}^{n} f_j'(g(t)) \cdot g_j'(t).$$

Bevis: $(n=2)$.

$$\frac{d}{dt}\left(f(g(t))\right) = \lim_{h\to 0} \frac{f(g(t+h)) - f(g(t))}{h}$$

$$= \lim_{h\to 0} \frac{f\big(g(t) + \overbrace{g(t+h) - g(t)}^{\bar{k}}\big) - f(g(t))}{h}$$

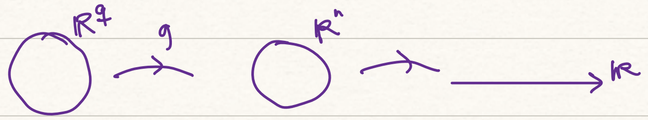
$$= \lim_{h\to 0} \frac{1}{h}\Big(f_1'(g(t))\big(g_1(t+h) - g_1(t)\big) + f_2'(g(t))\big(g_2(t+h) - g_2(t)\big) + |k|\,\rho(k)\Big)$$

$$= f_1'(g(t))\, g_1'(t) + f_2'(g(t))\cdot g_2'(t) + \lim_{h\to 0} \frac{|k|}{h}\,\rho(k)$$

$$\left|\frac{|k|}{h}\right|\cdot|\rho(k)| \longrightarrow |(g_1'(t), g_2'(t))| \cdot \lim_{h\to 0} |\rho(k)| = 0$$

ty $k = g(t+h) - g(t) \longrightarrow 0$, då $h \longrightarrow 0$. ∎

Fall 3:



$\bar{x} = g(\bar{t})$.

$$\frac{\partial(f(g(\bar{t})))}{\partial t_j} = \frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j}.$$

Ex: a) Visa att variabelbytet $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

leder till att $\dfrac{\partial f}{\partial \theta} = x\dfrac{\partial f}{\partial y} - y\dfrac{\partial f}{\partial x}$.

b) Bestäm den lösning till $-y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y} = x$

som uppfyller $f(x,0) = \sin(x^2)$

Lösning: a) $\dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial \theta} = \dfrac{\partial f}{\partial x}\cdot(-r\sin\theta) + \dfrac{\partial f}{\partial y}\cdot(r\cos\theta)$

$$= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

b) $-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = x \iff \frac{\partial f}{\partial \theta} = r\cos\theta$

$\iff f = r\sin\theta + g(r) = y + g\left(\sqrt{x^2+y^2}\right)$

$f(x,0) = g\left(\sqrt{x^2}\right) = \sin(x^2) = \sin\left(\sqrt{x^2}^{\,2}\right)$

Alltså är $g(r) = \sin r^2$

Vi har $f(x,y) = y + \sin(x^2+y^2)$

Sats: Om $f \in C^2(D)$ så gäller att $f''_{ij} = f''_{ji}$.

Bevis: Det räcker att visa att $f''_{12} = f''_{21}$ och $D \subset \mathbb{R}^2$.

$$f''_{21}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \lim_{h\to 0}\left(\frac{f'_2(x+h,y) - f'_2(x,y)}{h}\right)$$

$$= \lim_{h\to 0}\lim_{k\to 0} \boxed{\frac{f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y)}{h\cdot k}}$$

$$\frac{f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y)}{h\cdot k}$$

medelvärdessatsen på $f(x,y+k) - f(x,y)$

$$= \frac{\left(f'_1(x+\theta h, y+k) - f'_1(x+\theta h, y)\right)\cdot h}{hk} = \frac{f''_{12}(x+\theta h, y+\varphi k)\cdot k}{k} = f''_{12}(x+\theta h, y+\varphi k)$$

$\to f''_{12}(x,y)$ ty f'' är kontinuerlig.

Ex: a) Antag $u \in C^2(\Omega)$ där $\Omega = \{(x,y) \in \mathbb{R}^2 : x > y > 0\}$.

Låt $\begin{cases} s = \sqrt{y} \\ t = \sqrt{x-y} \end{cases}$.

Uttryck $\frac{\partial^2 u}{\partial x^2}$ i de nya variablerna

Lösning: $\frac{\partial^2}{\partial x^2} u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{x-y}} \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{x-y}}\right)\frac{\partial u}{\partial t} + \frac{1}{2\sqrt{x-y}} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$

$= -\frac{1}{4(x-y)^{3/2}} \frac{\partial u}{\partial t} + \frac{1}{2\sqrt{x-y}}\left(\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial t}\right) \cdot \frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\frac{\partial t}{\partial x}\right)$

$= \frac{-1}{4(x-y)^{3/2}} \frac{\partial u}{\partial t} + \frac{1}{4(x-y)} \frac{\partial^2 u}{\partial t^2} = \frac{-1}{4t^3}\frac{\partial u}{\partial t} + \frac{1}{4t^2}\frac{\partial^2 u}{\partial t^2}$.

b) Uttryck differentialekvationen $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 1$ i de nya variablerna.

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{2\sqrt{x-y}} \frac{\partial u}{\partial t}\right) = \frac{1}{4(x-y)^{3/2}}\frac{\partial u}{\partial t} + \frac{1}{2\sqrt{x-y}} \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)$

$= \frac{1}{4t^3}\frac{\partial u}{\partial t} + \frac{1}{2t}\left(\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial t}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\frac{\partial t}{\partial y}\right)$

$= \frac{1}{4t^3}\frac{\partial u}{\partial t} + \frac{1}{2t}\left(\frac{1}{2\sqrt{y}}\frac{\partial^2 u}{\partial s \partial t} + \frac{-1}{2\sqrt{x-y}}\frac{\partial^2 u}{\partial t^2}\right)$

$= \frac{1}{4t^3}\frac{\partial u}{\partial t} + \frac{1}{4st}\frac{\partial^2 u}{\partial s \partial t} - \frac{1}{4t^2}\frac{\partial^2 u}{\partial t^2}$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 1 \iff \frac{1}{4st} \frac{\partial^2 u}{\partial s \partial t} = 1$$

c) Bestäm alla lösningar till differentialekvationen.

$$\frac{\partial^2 u}{\partial s \partial t} = 4st \iff \frac{\partial u}{\partial t} = 2s^2 t + f(t)$$

$$\iff u = s^2 t^2 + F(t) + G(s) = y(x-y) + F(\sqrt{y}) + G(\sqrt{x-y})$$