F4 Kedjeregeln och partiella differentialekvationer

Definition: Låt $f : D \subset \mathbb{R}^n \to \mathbb{R}$, D öppen mängd.

Vi säger att f är av klass C^k eller att $f \in C^k(D)$ om f är k gånger partiellt deriverbar och alla partiella derivatorna är kontinuerliga.

Sats: Om $f \in C^1(D)$ så är f differentierbar.

Bevis: $(d=2)$ $\qquad f(a+h) - f(a) = f'(a+\theta h)\theta$

$$f(a_1+h_1, a_2+h_2) - f(a_1, a_2) =$$

$$= f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2) + f(a_1+h_1, a_2) - f(a_1, a_2)$$

$$= f_2'(a_1+h_1, a_2+\theta_2 h_2) \cdot h_2 + f_1'(a_1+\theta h_1, a_2) \cdot h_1 \qquad \theta_j \in (0,1)$$

$$= f_1'(a_1,a_2)h_1 + f_2'(a_1,a_2)h_2 + \left(f_1'(a_1+\theta h_1, a_2) - f_1'(a_1,a_2)\right)h_1$$
$$+ \left(f_2'(a_1+h_1, a_2+\theta_2 h_2) - f_2'(a_1,a_2)\right)h_2.$$

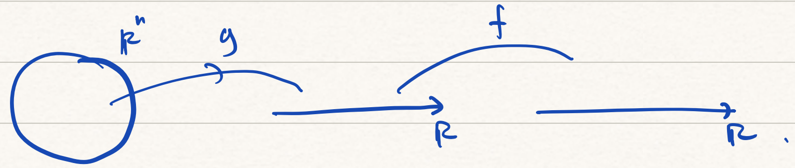
Alltså är $\quad \rho(h) = \frac{1}{|h|}\Big(\left(f_1'(a_1+\theta h_1, a_2) - f_1'(a_1,a_2)\right)h_1$
$$+ \left(f_2'(a_1+h_1, a_2+\theta_2 h_2) - f_2'(a_1,a_2)\right)h_2\Big)$$

$$|\rho(h)| \leq |f_1'(a_1+\theta h_1, a_2) - f_1'(a_1,a_2)| + |f_2'(a_1+h_1, a_2+\theta_2 h_2) - f_2'(a_1,a_2)|$$

$\longrightarrow 0$ då f_j' är kontinuerlig. $\blacksquare$

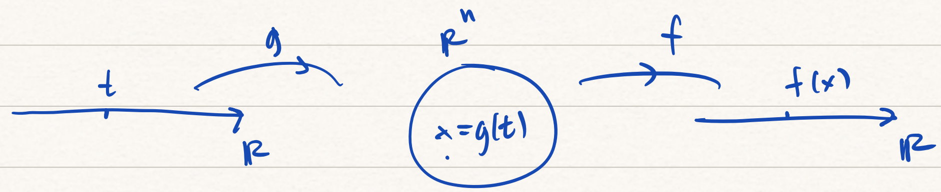
Kedjeregeln:

d=1: $$\frac{d}{dt}\left(f(g(t))\right) = f'(g(t))\cdot g'(t).$$



Om $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}^n\to\mathbb{R}$. Genom att frysa variablerna $t_2,\dots,t_n$ får vi att $\frac{\partial}{\partial t_1}\left(f(g(t))\right) = f'(g(t))\, g'_{t_1}(t)$.

Låt nu $f:\mathbb{R}^n\to\mathbb{R}$ och $g:\mathbb{R}\to\mathbb{R}^n$.



$g(t)\in\mathbb{R}^n$ $\quad$ $f(g(t))\in\mathbb{R}$. $\quad$ $f(g(t)) = f(g_1(t), g_2(t),\dots,g_n(t))$

Sats: (kedjeregeln) Låt $f(x) = f(x_1,x_2,x_3,\dots,x_n)$ vara en differentierbar funktion av n variabler och antag att funktionerna $g_1(t), g_2(t),\dots,g_n(t)$ är deriverbara. Då gäller att

$$\frac{d}{dt} f(g(t)) = f'_{x_1}(g(t))\,g'_1(t) + f'_{x_2}(g(t))\,g'_2(t) + \dots + f'_{x_n}(g(t))\,g'_n(t).$$

Bevis: $(d=2)$

Eftersom f är differentierbar gäller för varje $h\in\mathbb{R}^2$

$$f(x+h) - f(x) = f'_1(x)h_1 + f'_2(x)h_2 + |h|\rho(h), \quad \text{där } \rho(h)\to 0, \text{ då } h\to 0.$$

Låt nu $x = g(t)$. Tag $k \in \mathbb{R}$ och låt $h = g(t+k) - g(t)$.
Eftersom likheten ovan gäller för varje h så gäller

$$f(g(t) + g(t+k) - g(t)) - f(g(t)) = f_1'(g(t))(g_1(t+k) - g_1(t)) + f_2'(g(t))(g_2(t+k) - g_2(t)) + |h| \varrho(h).$$

Alltså
$$\frac{f(g(t+k)) - f(g(t))}{k} = f_1'(g(t))\left(\frac{g_1(t+k) - g_1(t)}{k}\right) + f_2'(g(t))\left(\frac{g_2(t+k) - g_2(t)}{k}\right) + \frac{|h|}{k}\varrho(h)$$

$$\longrightarrow f_1'(g(t)) \cdot g_1'(t) + f_2'(g(t)) \cdot g_2'(t), \text{ då } k \to 0.$$

Eftersom $\left|\frac{|h|}{k}\varrho(h)\right| = \left|\frac{g(t+k) - g(t)}{k}\right| |\varrho(h)| \to |g'(t)| \cdot \lim_{k \to 0} |\varrho(h)|$.

Då $k \to 0$ gäller att $h = g(t+k) - g(t) \to 0$. $\blacksquare$

I allmänna fallet:

Låt $f = f(x)$ och $x = x(t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}^q$.

$$\frac{\partial}{\partial t_j} f(x) = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}.$$

Ex: a) Visa att variabelbytet $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

leder till att $\frac{\partial f}{\partial \theta} = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}$.

b) Bestäm den lösning till $-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = x$
som uppfyller att $f(x,0) = \sin(x^2)$.

Lösning: a) $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta} = (-r \cdot \sin\theta) \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y}$

$= -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y}$.

b) $-y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} = x \iff \frac{\partial f}{\partial \theta} = r\cos\theta$.

$f = r\sin\theta + g(r) = y + g(\sqrt{x^2+y^2})$.

$f(x,0) = g(|x|) = \sin |x|^2$

$f(x,y) = y + \sin(x^2+y^2)$.

Kontroll: $-y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} = -y\, 2x \cos(x^2+y^2) + x\,(1 + 2y \cos(x^2+y^2)) = x$.

Sats: Om $f \in C^2(D)$, $D \subset \mathbb{R}^n$ så gäller att

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Bevis: Vi kan anta att $n = 2$. Låt $(a,b) \in D$.

Vi vill då visa att $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$.

Bilda $g(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$.

Vi kommer att visa att $\lim_{(h,k)\to(0,0)} \frac{g(h,k)}{h \cdot k} = \frac{\partial^2 f}{\partial x_1 \partial x_2}$

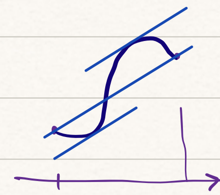
och $\lim_{(h,k)\to(0,0)} \frac{g(h,k)}{h \cdot k} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$.

Bilda $\varphi(t) = f(a+h,t) - f(a,t)$
och $\psi(s) = f(s,b+k) - f(s,b)$.

$q(h,k) = \psi(a+h) - \psi(a) = \varphi(b+k) - \varphi(b)$.

Medelvärdessatsen för derivator ger att de finns ett $\theta \in [0,1]$ sådant att

$\psi(a+h) - \psi(a) = \psi'(a+\theta h)\cdot h$



$$\frac{q(h,k)}{h} = \psi'(a+\theta h) = f'_x(a+\theta h, b+k) - f'_x(a+\theta h, b)$$

Återigen använder vi medelvärdessatsen för derivator

$= f''_{xy}(a+\theta h, b+\varepsilon k)\cdot k$, där $\varepsilon \in [0,1]$.

Alltså är $\frac{q(h,k)}{h\cdot k} = f''_{xy}(a+\theta h, b+\varepsilon k) \to f''_{xy}(a,b)$, då $(h,k)\to(0,0)$

eftersom f''_{xy} är kontinuerlig.

På likvande sätt kan vi visa att

$\frac{q(h,k)}{h\cdot k} \to f''_{yx}(a,b)$ genom att nyttja

$$q(h,k) = \varphi(b+k) - \varphi(b).$$

Ex: Bestäm alla lösningar till $f \in C^2(\mathbb{R}^2)$ till vågekvationen

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \quad , \quad c \text{ utbredningshastigheten.}$$

genom att införa variablerna $\begin{cases} u = x + ct \\ v = x - ct . \end{cases}$

Lösning:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2} .$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} \cdot c - \frac{\partial f}{\partial v} c = c\left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right)$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(c\left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right)\right)$$

$$= \frac{\partial}{\partial u}\left(c\frac{\partial f}{\partial u} - c\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(c\frac{\partial f}{\partial u} - c\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial t}$$

$$= \left(c\frac{\partial^2 f}{\partial u^2} - c\frac{\partial^2 f}{\partial u \partial v}\right)c + \left(c\frac{\partial^2 f}{\partial u \partial v} - c\frac{\partial^2 f}{\partial v^2}\right)(-c)$$

$$= c^2\left(\frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right) .$$

Alltså

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial f^2}{\partial x^2} = c^2\left(\frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right) - c^2\left(\frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right)$$

$$= -c^2 4\frac{\partial^2 f}{\partial u \partial v} = 0. \quad \Longleftrightarrow \quad \frac{\partial^2 f}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) = 0$$

$$\Longleftrightarrow \quad \frac{\partial f}{\partial v} = g_1(v) \qquad \Longleftrightarrow \quad f = g_2(v) + g_3(u) \, .$$

$$f(x,t) = g_2(x-ct) + g_3(x+ct) \, .$$