

F13 Riemannsummer och variabelbyten

Riemannsummer:

Sats: Om f är kontinuerlig på $D \subset \mathbb{R}^2$ och D är kompakt, $(\alpha_k, \beta_k) \in D_k$, $\bigcup_k D_k = D$, D och D_k kvadratforma så gäller att

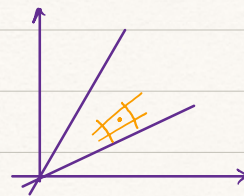
$$\sum_k f(\alpha_k, \beta_k) \mu(D_k) \longrightarrow \iint_D f(x, y) dx dy$$

då $\max \mu(D_k) \rightarrow 0$.

Ex: Låt $D = \{(x, y) : x = r \cos \theta, y = r \sin \theta, 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}\}$.

Vi vill beräkna $\iint_D f(x, y) dx dy$.

Låt oss dela in D i små sektioner



$$D_{i,j} = \{(x, y) : x = r \cos \theta, y = r \sin \theta, r_{i-1} \leq r < r_i, \theta_{j-1} \leq \theta < \theta_j\}.$$

Låt $\rho_i = \frac{r_i + r_{i-1}}{2}$ och $\varphi_j = \frac{\theta_j + \theta_{j-1}}{2}$, vilket ger punkten

$$(\alpha_{i,j}, \beta_{i,j}) = (\rho_i \cos \varphi_j, \rho_i \sin \varphi_j)$$

Från satsen får vi att $\sum_{i,j} f(\alpha_{i,j}, \beta_{i,j}) \mu(D_{i,j}) \longrightarrow \iint_D f(x, y) dx dy$
då $\max \mu(D_{i,j}) \rightarrow 0$.

$$\text{Notera att } \mu(D_{i,j}) = \frac{\pi(r_i^2 - r_{i-1}^2) \cdot (\theta_j - \theta_{j-1})}{2\pi} = \rho_i (r_i - r_{i-1}) (\theta_j - \theta_{j-1})$$

$$\sum_{i,j} f(\alpha_{i,j}, \beta_{i,j}) \mu(D_{i,j}) = \sum_{i,j} f(\rho_i \cos \varphi_j, \rho_i \sin \varphi_j) \rho_i (r_i - r_{i-1}) (\theta_j - \theta_{j-1})$$

Notera nu att HL är en Riemannsumma för funktionen

$(r, \theta) \mapsto f(r \cos \theta, r \sin \theta) \cdot r$ över rektanglarna

$\{(r, \theta) : r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$.

När indelningen förfäras får vi

$$HL \rightarrow \iint_E f(r \cos \theta, r \sin \theta) r \, dr \, d\theta, \text{ där } E = \{(r, \theta) : 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$$

Variabelbyte då $d=1$.

Låt $g: [\alpha, \beta] \rightarrow [a, b]$ vara monoton, deriverbar och bijektiv
då gäller

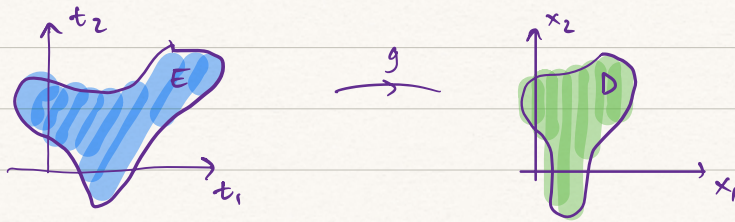
$$\begin{aligned} \text{Om } g \text{ växande: } \int_a^b f(x) \, dx &= \left\{ \begin{array}{l} x = g(t) \\ dx = g'(t) \, dt \end{array} \right\} = \int_{\alpha}^{\beta} f(g(t)) \cdot g'(t) \, dt \\ (g(\alpha) = a, g(\beta) = b) & \\ &= \int_{\alpha}^{\beta} f(g(t)) |g'(t)| \, dt \end{aligned}$$

$$\begin{aligned} \text{Om } g \text{ avtagande: } \int_a^b f(x) \, dx &= \left\{ \begin{array}{l} x = g(t) \\ dx = g'(t) \, dt \end{array} \right\} = \int_{\beta}^{\alpha} f(g(t)) \cdot g'(t) \, dt \\ (g(\alpha) = b, g(\beta) = a) & \\ &= \int_{\alpha}^{\beta} f(g(t)) \cdot |g'(t)| \, dt. \quad (\text{eftersom } |g'(t)| = -g'(t)) \end{aligned}$$

Variabelbytet $d=2$:

Låt $g: E \rightarrow D$, $E \subset \mathbb{R}^2$, $D \subset \mathbb{R}^2$, $g \in C^1(E)$ och bijektiv.

$$x = g(t) \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} g_1(t_1, t_2) \\ g_2(t_1, t_2) \end{pmatrix}$$



Dela in $D = \bigcup_k D_k$, där D_k är parvis disjunkta,
dvs $D_j \cap D_k = \emptyset$ om $j \neq k$.

Låt E_k vara sådan att

$$D_k = g(E_k) = \{x \in D : x = g(t) \text{ för något } t \in E_k\}$$

Area förstoringen i en punkt ges av funktionsdeterminanten
i den punkten.



$$q_k = g(p_k)$$

Anså $\mu(D_k) \approx \left| \frac{d(x)}{d(t)}(p_k) \right| \mu(E_k)$

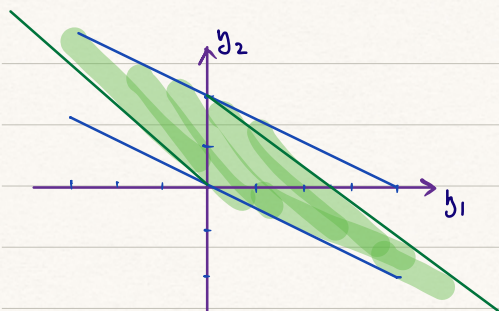
$$\text{och } \sum_k f(q_k) \mu(D_k) \approx \sum_k f(g(p_k)) \left| \frac{d(x)}{d(t)}(p_k) \right| \mu(E_k)$$

↓
Riemannsumma

$$\iint_D f(x) dx_1 dx_2 = \iint_E f(g(t)) \left| \frac{d(x)}{d(t)}(t) \right| dt_1 dt_2.$$

Ex: Beräkna $\iint_D (y_1^2 + y_2) dy_1 dy_2$ om $D = \{(y_1, y_2) : 0 < y_1 + 2y_2 < 1, 0 < 3y_1 + 4y_2 < 2\}$

Lösning: $y_2 = -\frac{y_1}{2}$, $y_2 = \frac{1-y_1}{2}$, $y_2 = -\frac{3y_1}{4}$, $y_2 = \frac{2-3y_1}{4}$



Ansätt $\begin{cases} x_1 = y_1 + 2y_2 \\ x_2 = 3y_1 + 4y_2 \end{cases} \Leftrightarrow \begin{cases} y_1 = -2x_1 + x_2 \\ y_2 = \frac{3x_1}{2} - \frac{x_2}{2} \end{cases}$

$$\frac{d(y)}{d(x)} = \begin{vmatrix} -2 & 1 \\ 3/2 & -1/2 \end{vmatrix} = 1 - \frac{3}{2} = -\frac{1}{2}.$$

$$\iint_D (y_1^2 + y_2) dy_1 dy_2 = \int_0^1 \int_0^2 \left((-2x_1 + x_2)^2 + \frac{3x_1}{2} - \frac{x_2}{2} \right) \left| -\frac{1}{2} \right| dx_2 dx_1$$

$$= \frac{1}{2} \int_0^1 \left[4x_1^2 x_2 - 2x_1 x_2^2 + \frac{x_2^3}{3} + \frac{3x_1 x_2}{2} - \frac{x_2^2}{4} \right]_0^2 dx_1$$

$$= \frac{1}{2} \int_0^1 \left(8x_1^2 - 8x_1 + \frac{8}{3} + 3x_1 - 1 \right) dx_1 = \frac{1}{2} \left[\frac{8x_1^3}{3} - \frac{5x_1^2}{2} + \frac{5x_1}{3} \right]_0^1 = \frac{1}{2} \left(\frac{8}{3} - \frac{5}{2} + \frac{5}{3} \right)$$

$$= \frac{1}{12} (16 - 15 + 10) = \frac{11}{12}.$$

Ex: Beräkna $I = \iint_E y^2 dx dy$ över $E = \{(x, y) : \frac{x^2}{9} + y^2 < 1\}$.

Variabelbyte: $\begin{cases} x = 3r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{matrix} 0 < r < 1 \\ 0 < \theta < 2\pi. \end{matrix}$

$$\frac{d(x, y)}{d(r, \theta)} = \begin{vmatrix} 3 \cos \theta & -3r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = 3r \cos^2 \theta + 3r \sin^2 \theta = 3r.$$

$$dx dy = 3r dr d\theta.$$

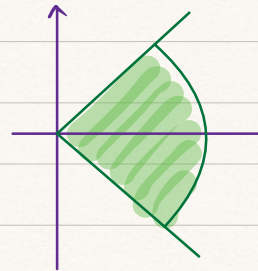
$$I = \int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta \cdot 3r dr d\theta = 3 \int_0^{2\pi} \sin^2 \theta d\theta \int_0^1 r^3 dr$$

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2\sin^2 \theta \end{aligned}$$

$$= 3 \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta \cdot \left[\frac{r^4}{4} \right]_0^1 = \frac{3}{2} (2\pi) \cdot \frac{1}{4} = \frac{3\pi}{4}.$$

Ex: Beräkna $I = \iint_{\mathcal{A}} (x^2 - y^2) e^{2xy} dx dy$, där

$$\mathcal{A} = \{(x, y) : x^2 + y^2 < 1, -x < y < x, x > 0\}.$$



Lösning:
$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}, \quad 0 < r < 1, \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}.$$

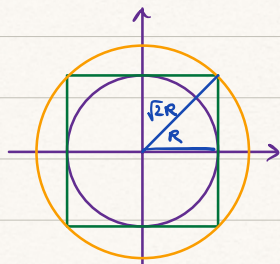
$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^2 (\cos^2 \theta - \sin^2 \theta) e^{2r^2 \cos \theta \sin \theta} r dr d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^3 \cos 2\theta e^{r^2 \sin 2\theta} dr d\theta = \int_0^1 \left[\frac{e^{r^2 \sin 2\theta}}{2} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dr$$

$$= \int_0^1 \frac{1}{2} (e^{r^2} - e^{-r^2}) dr = \frac{1}{4} \left[e^{r^2} + e^{-r^2} \right]_0^1 = \frac{1}{4} (e + e^{-1} - 2)$$

Ex: Beräkna $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

Eftersom $e^{-x^2} > 0$ gäller $I = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-x^2} dx = \lim_{R \rightarrow \infty} I_R$



$$K_R = \{(x, y) : |x| < R, |y| < R\}$$

$$C_R = \{(x, y) : x^2 + y^2 < R\}.$$

$$I_R^2 = I_R \cdot I_R = \int_{-R}^R e^{-x^2} dx \cdot \int_{-R}^R e^{-y^2} dy = \int_{-R}^R \int_{-R}^R e^{-x^2 - y^2} dx dy.$$

$$\iint_{C_R} e^{-x^2-y^2} dx dy \leq I_R^2 \leq \iint_{C_{\sqrt{2}R}} e^{-x^2-y^2} dx dy$$

$$\int_0^{2\pi} \int_0^R e^{-r^2} r dr d\theta \leq I_R^2 \leq \int_0^{2\pi} \int_0^{\sqrt{2}R} e^{-r^2} r dr d\theta$$

$$\int_0^{2\pi} \int_0^M e^{-r^2} r dr d\theta = 2\pi \left[\frac{e^{-r^2}}{-2} \right]_0^M = \pi (-e^{-M^2} + 1) \rightarrow \pi, \text{ da } M \rightarrow \infty.$$

Also ist $\lim_{R \rightarrow \infty} I_R^2 = \pi$, d.h., $\int_{-\infty}^{\infty} e^{-x^2} dx = \underline{\underline{\sqrt{\pi}}}$.