F23 Stokes sats

Definition: Låt $u = (u_1, u_2, u_3)$ vara ett C^1-fält.
Vi definierar rotationen av u som vektorfältet

$$\text{rot } u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Fältet u kallas virvelfritt om $\text{rot } u = 0$

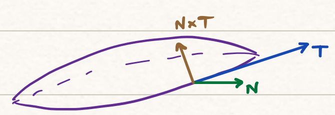
Notera att vi symboliskt kan se

$$\text{rot } u = \nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Stokes sats:
Låt $u : \Omega \to \mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$ öppen, $u \in C^1$. Om Y är ett orienterat ytstycke i Ω med orienterad rand ∂Y så gäller

$$\int_{\partial Y} u \cdot dr = \iint_Y (\text{rot } u) \cdot N \, dS,$$

där $N \times T$ pekar in mot ytan i varje randpunkt.



Anmärkning: Om ytan är en plan yta i xy-planet och $u = (u_1, u_2, u_3)$
Låt $r(t) = (x(t), y(t), 0)$ vara en parametrisering av randen, där $a \leq t \leq b$.

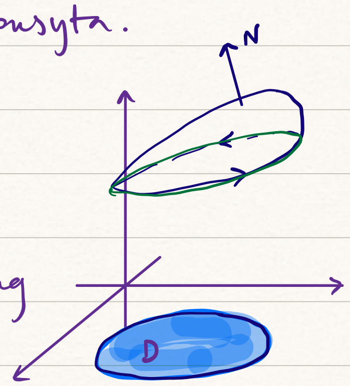
$$\int_{\partial Y} u \cdot dr = \int_a^b \left(u_1(r(t))x'(t) + u_2(r(t))y'(t)\right) dt$$

$$= \int_{\partial Y} u_1 dx + u_2 dy .$$

$$\iint_Y \text{rot } u \cdot N \, dS = \iint_Y \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) dx dy, \quad \text{eftersom } N = (0,0,1).$$

Alltså $\displaystyle\int_{\partial Y} u \cdot dr = \iint_Y \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) dx_1 dx_2$ (Greens formel)

Bevis: I fallet att Y är en funktionsyta.
och $\partial Y \in C^2$.



$$Y = \{(x,y,z) : z = f(x,y), \ (x,y) \in D\}.$$

Låt $r : [\alpha, \beta] \to \partial Y$ vara en parametrisering av randen ∂Y.

$$r(t) = (x(t), y(t), f(x(t), y(t))) .$$

$$\int_{\partial Y} u \cdot dr = \int_\alpha^\beta u(r(t)) \cdot r'(t) dt = \int_\alpha^\beta u(r(t)) \cdot (x', y', f'_x(x,y)x' + f'_y(x,y)y') dt$$

$$= \int_\alpha^\beta (u_1 x' + u_2 y' + u_3 f'_x \cdot x' + u_3 f'_y y') dt = \int_\alpha^\beta ((u_1 + u_3 f'_x) x' + (u_2 + u_3 f'_y) y') dt$$

$$= \int_\alpha^\beta (u_1 + u_3 f'_x, u_2 + u_3 f'_y) \cdot (x'(t), y'(t)) dt = \int_{\partial D} (u_1 + u_3 f'_x, u_2 + u_3 f'_y) \cdot dr = \circledast$$

eftersom $s(t) = (x(t), y(t))$ är en parametrisering av ∂D.

Greens formel ger nu

$$\circledast = \iint_D \left(\frac{\partial(u_2 + u_3 f_y')}{\partial x} - \frac{\partial(u_1 + u_3 f_x')}{\partial y} \right) dxdy = \circledast$$

$$\frac{\partial}{\partial x}\left(u_2(x,y,f(x,y)) + u_3(x,y,f(x,y)) f_y'(x,y) \right)$$

$$= \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial z} \cdot f_x' + \left(\frac{\partial u_3}{\partial x} + \frac{\partial u_3}{\partial z} \cdot f_x' \right) f_y' + u_3 f_{xy}''$$

$$\frac{\partial}{\partial y}\left(u_1(x,y,f(x,y)) + u_3(x,y,f(x,y)) f_x'(x,y) \right)$$

$$= \frac{\partial u_1}{\partial y} + \frac{\partial u_1}{\partial z} \cdot f_y' + \left(\frac{\partial u_3}{\partial y} + \frac{\partial u_3}{\partial z} f_y' \right) f_x' + u_3 f_{xy}''$$

$$\circledast = \iint_D \left(\frac{\partial u_2}{\partial x} + f_x' \cdot \frac{\partial u_2}{\partial z} + f_y' \frac{\partial u_3}{\partial x} + \cancel{f_x' f_y' \frac{\partial u_3}{\partial z}} + \cancel{f_{xy}'' u_3} \right.$$

$$\left. - \frac{\partial u_1}{\partial y} - f_y' \frac{\partial u_1}{\partial z} - f_x' \frac{\partial u_3}{\partial y} - \cancel{f_x' f_y' \frac{\partial u_3}{\partial z}} - \cancel{f_{xy}'' u_3} \right) dxdy$$

$$= \iint_D \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) (-f_x', -f_y', 1)\, dxdy$$

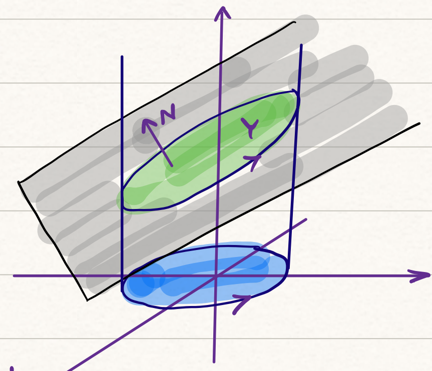
$$= \iint_Y (\text{rot } u) \cdot N\, dS.$$

Ex: Beräkna $\int_\gamma u \cdot dr$, där $u = (-y^3, x^3, -z^3)$

och γ är skärningen av cylindern $x^2+y^2=1$ och planet $2x+2y+z=3$ orienterad moturs sett ovanifrån

$r(x,y) = (x, y, 3-2x-2y)$

är en parametrisering av Y.



$r'_x \times r'_y = (2,2,1)$.

$$\int_\gamma u \cdot dr \overset{\text{Stokes sats}}{=} \iint_Y (\text{rot } u)\, N\, dS = \iint_D (\text{rot } u)\cdot(2,2,1)\, dx\,dy = \circledast$$

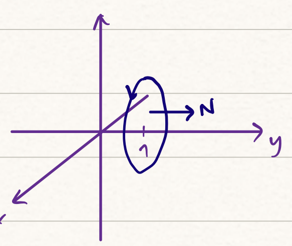
$$\text{rot } u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2+3y^2)$$

$$\circledast = \iint_D (3x^2+3y^2)\,dx\,dy = 3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = 3\cdot 2\pi\left[\frac{r^4}{4}\right]_0^1 = \frac{3\pi}{2}.$$

Ex: Bestäm kurvintegralen av $F(x,y,z) = (x^2y^3, e^{xy-x+z}, x+z^2)$ längs cirkeln $x^2+z^2=1$ i planet $y=1$, där cirkeln är orienterad medurs sedd från origo.

Lösning:

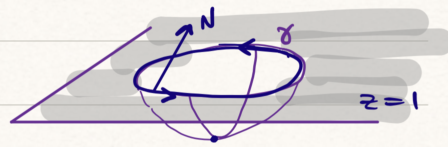
$$\text{rot } F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & e^{xy-x+z} & x+z^2 \end{vmatrix} = (-e^{xy-x+z}, -1, e^{xy-x+z}-3x^2y^2)$$



$$\int_\gamma F\cdot dr = \iint_Y (\text{rot } F)\,(0,1,0)\,dS = -\iint_Y dS = \underline{-\pi}$$

Ex: Ytan S består av den delen av paraboloiden $z = x^2 + 4y^2$ som ligger under planet $z=1$ orienterad så att normalen N får positiv z-komponent. Bestäm flödet av rot F genom S, där $F(x,y,z) = (y, -xz, xz^2)$

Lösning: $\iint_S (\text{rot } F)\cdot N \, dS = \int_\gamma F \cdot dr$



En parametrisering av γ är

$r(\theta) = (\cos\theta, \frac{1}{2}\sin\theta, 1)$, $0 \le \theta \le 2\pi$.

$r'(\theta) = (-\sin\theta, \frac{1}{2}\cos\theta, 0)$

$$\int_\gamma F \cdot dr = \int_0^{2\pi} \left(\tfrac{1}{2}\sin\theta, -\cos\theta, -\cos\theta\right)\cdot\left(-\sin\theta, \tfrac{1}{2}\cos\theta, 0\right) d\theta$$

$$= \int_0^{2\pi} \left(-\tfrac{1}{2}\sin^2\theta - \tfrac{1}{2}\cos^2\theta\right) d\theta = -\tfrac{1}{2}\cdot 2\pi = -\pi.$$