F15 Riemannsummor och variabelbyten

Riemannsummor:

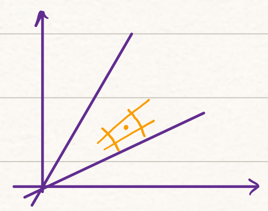
Sats: Om f är kontinuerlig på $D \subset \mathbb{R}^2$ och D är kompakt, $(\alpha_k, \beta_k) \in D_k$, $\bigcup_k D_k = D$ så gäller att

$$\sum_k f(\alpha_k, \beta_k)\, \mu(D_k) \longrightarrow \iint_D f(x,y)\,dx\,dy$$

då $\max \mu(D_k) \to 0$.

Ex: Låt $D = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta,\ 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$.

Vi vill beräkna $\iint_D f(x,y)\,dx\,dy$.



Låt oss dela in D i små sektioner

$$D_{i,j} = \{(x,y) : x = r\cos\theta,\ y = r\sin\theta,\ r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}.$$

Låt $\rho_i = \frac{r_i + r_{i-1}}{2}$ och $\varphi_j = \frac{\theta_j + \theta_{j-1}}{2}$, vilket ger punkten $(\alpha_{i,j}, \beta_{i,j}) = (\rho_i \cos\varphi_j, \rho_i \sin\varphi_j)$

Notera att $\mu(D_{i,j}) = \pi(r_i^2 - r_{i-1}^2) \cdot \frac{(\theta_j - \theta_{j-1})}{2\pi} = \rho_i (r_i - r_{i-1})(\theta_j - \theta_{j-1})$

Riemannsumman blir nu

$$\sum_{i,j} f(\alpha_{i,j}, \beta_{i,j})\,\mu(D_{i,j}) = \sum_{i,j} f(\rho_i\cos\theta_j, \rho_i\sin\theta_j)\,\rho_i (r_i - r_{i-1})(\theta_j - \theta_{j-1})$$

$$\text{VL} \longrightarrow \iint_D f(x,y)\,dx\,dy \quad \text{enligt sats.}$$

Notera nu att HL är en Riemannsumma för funktionen

$(r,\theta) \longmapsto f(r\cos\theta, r\sin\theta)\cdot r$ över rektanglarna

$\{(r,\theta) : r_{i-1} \le r \le r_i \,,\ \theta_{j-1} \le \theta \le \theta_j\}$.

När indelningen går mot noll får vi

$$HL \longrightarrow \iint_E f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta \quad , \text{ där } E = \{(r,\theta) : 1 \le r \le 2,\ \tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{2}\}$$

<u>Variabelbyte då $d=1$.</u>

Låt $g : [\alpha,\beta] \longrightarrow [a,b]$ vara monoton, deriverbar och surjektiv då gäller

Om g växande: $(g(\alpha)=a,\ g(\beta)=b)$

$$\int_a^b f(x)\,dx = \left\{\begin{matrix} x = g(t) \\ dx = g'(t)\,dt \end{matrix}\right\} = \int_\alpha^\beta f(g(t))\cdot g'(t)\,dt$$

$$= \int_\alpha^\beta f(g(t))\,|g'(t)|\,dt$$

Om g avtagande: $(g(\alpha)=b,\ g(\beta)=a)$

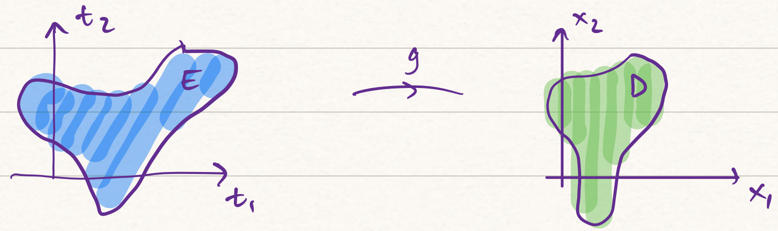
$$\int_a^b f(x)\,dx = \left\{\begin{matrix} x = g(t) \\ dx = g'(t)\,dt \end{matrix}\right\} = \int_\beta^\alpha f(g(t))\cdot g'(t)\,dt$$

$$= \int_\alpha^\beta f(g(t))\cdot |g'(t)|\,dt . \qquad (\text{eftersom } |g(t)| = -g(t))$$

<u>Variabelbyten $d=2$:</u>

Låt $g : E \longrightarrow D$, $E \subset \mathbb{R}^2$, $D \subset \mathbb{R}^2$, $g \in C^1(E)$ och surjektiv. $x = g(t)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} g_1(t_1,t_2) \\ g_2(t_1,t_2) \end{pmatrix}$$

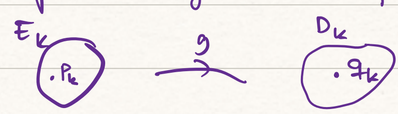


Dela in $D = \bigcup_k D_k$, där D_k är parvis disjunkta,
dvs $D_j \cap D_k = \emptyset$ om $j \neq k$.

Låt E_k vara sådan att

$$D_k = g(E_k) = \{x \in D : x = g(t) \text{ för något } t \in E_k\}$$

Area förstoringen i en punkt ges av funktionaldeterminanten
i den punkten.



$q_k = g(p_k)$

Antag $\mu(D_k) \approx \left| \frac{d(x)}{d(t)}(p_k) \right| \mu(E_k)$

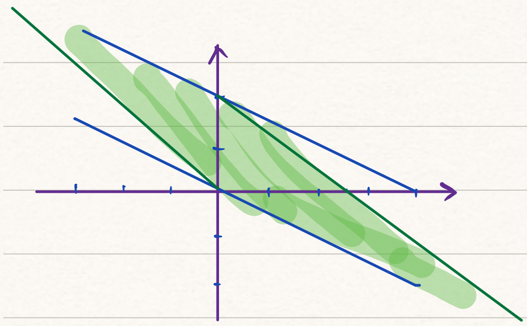
och $\sum_k f(q_k)\,\mu(D_k) \approx \sum_k f(g(p_k)) \left| \frac{d(x)}{d(t)}(p_k) \right| \mu(E_k)$

$\downarrow$ Riemannsumma

$$\iint_D f(x)\,dx_1 dx_2 = \iint_E f(g(t)) \left| \frac{d(x)}{d(t)}(t) \right| dt_1 dt_2 .$$

Ex: Beräkna $\iint_D (y_1^2 + y_2)\,dy_1 dy_2$ om $D = \{(y_1, y_2) : 0 < y_1 + 2y_2 < 1,\ 0 < 3y_1 + 4y_2 < 2\}$

Lösning: $y_2 = -\frac{y_1}{2}$, $y_2 = \frac{1 - y_1}{2}$, $y_2 = -\frac{3y_1}{4}$, $y_2 = \frac{2 - 3y_1}{4}$



Ansätt $\begin{cases} x_1 = y_1 + 2y_2 \\ x_2 = 3y_1 + 4y_2 \end{cases} \iff \begin{cases} y_1 = -2x_1 + x_2 \\ y_2 = \frac{3x_1}{2} - \frac{x_2}{2} \end{cases}$

$$\frac{d(y)}{d(x)} = \begin{vmatrix} -2 & 1 \\ 3/2 & -1/2 \end{vmatrix} = 1 - \frac{3}{2} = -\frac{1}{2}.$$

$$\iint_D (y_1^2 + y_2)\,dy_1 dy_2 = \int_0^1 \int_0^2 \left((-2x_1 + x_2)^2 + \frac{3x_1}{2} - \frac{x_2}{2}\right) \left|-\frac{1}{2}\right| dx_2\,dx_1$$

$$= \frac{1}{2}\int_0^1 \left[4x_1^2 x_2 - 2x_1 x_2^2 + \frac{x_2^3}{3} + \frac{3x_1 x_2}{2} - \frac{x_2^2}{4}\right]_0^2 dx_1$$

$$= \frac{1}{2}\int_0^1 \left(8x_1^2 - 8x_1 + \frac{8}{3} + 3x_1 - 1\right) dx_1 = \frac{1}{2}\left[\frac{8x_1^3}{3} - \frac{5x_1^2}{2} + \frac{5x_1}{3}\right]_0^1 = \frac{1}{2}\left(\frac{8}{3} - \frac{5}{2} + \frac{5}{3}\right)$$

$$= \frac{1}{12}(16 - 15 + 10) = \frac{11}{12}.$$

Ex: Beräkna $I = \iint_E y^2\,dx\,dy$ över $E = \{(x,y) : \frac{x^2}{9} + y^2 < 1\}$.

Variabelbyte: $\begin{cases} x = 3r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad 0 < r < 1$, $\quad 0 < \theta < 2\pi$.

$$\frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 3r\cos^2\theta + 3r\sin^2\theta = 3r.$$

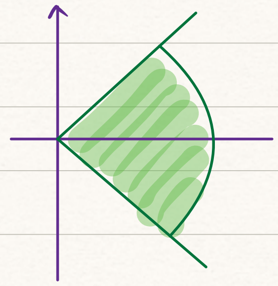
$$dx\,dy = 3r\,dr\,d\theta.$$

$$I = \int_0^{2\pi}\int_0^1 r^2\sin^2\theta \cdot 3r\,dr\,d\theta = 3\int_0^{2\pi} \sin^2\theta\,d\theta \int_0^1 r^3\,dr$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

$$= 3\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta \cdot \left[\frac{r^4}{4}\right]_0^1 = \frac{3}{2}(2\pi)\cdot\frac{1}{4} = \frac{3\pi}{4}.$$

Ex: Beräkna $I = \iint_{\Omega} (x^2-y^2)\, e^{2xy}\, dx\,dy$, där



$\Omega = \{(x,y) : x^2+y^2<1,\ -x<y<x,\ x>0\}$.

Lösning: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $0<r<1$, $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$.

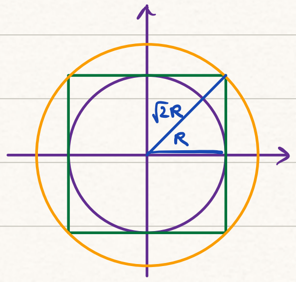
$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^2(\cos^2\theta - \sin^2\theta)\, e^{2r^2\cos\theta\sin\theta}\, r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^1 r^3 \cos 2\theta\, e^{r^2\sin 2\theta}\, dr\,d\theta = \int_0^1 r \left[\frac{e^{r^2\sin 2\theta}}{2} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dr$$

$$= \int_0^1 \frac{r}{2}\left(e^{r^2} - e^{-r^2}\right) dr = \frac{1}{4}\left[e^{r^2} + e^{-r^2} \right]_0^1 = \frac{1}{4}\left(e + e^{-1} - 2\right)$$

Ex: Beräkna $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$

Eftersom $e^{-x^2} > 0$ gäller $I = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\,dx = \lim_{R\to\infty} I_R$



$K_R = \{(x,y) : |x|<R,\ |y|<R\}$

$C_R = \{(x,y) : x^2+y^2<R\}$.

$$I_R^2 = I_R \cdot I_R = \int_{-R}^{R} e^{-x^2}dx \cdot \int_{-R}^{R} e^{-y^2}dy = \int_{-R}^{R}\int_{-R}^{R} e^{-x^2-y^2}\,dx\,dy.$$

$$\int_0^{2\pi}\int_0^{R} e^{-r^2}\, r\,dr\,d\theta \le I_R^2 \le \int_0^{2\pi}\int_0^{\sqrt{2}R} e^{-r^2}\, r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{M} e^{-r^2}\, r\,dr\,d\theta = 2\pi\left[\frac{e^{-r^2}}{-2}\right]_0^M = \pi\left(-e^{-M^2}+1\right) \longrightarrow \pi, \text{ då } M\to\infty.$$

Alltså är $\lim_{R\to\infty} I_R^2 = \pi$, dvs $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.