Klassen C^1 & Kedjeregeln

Definition: Låt $f: D \subset \mathbb{R}^n \to \mathbb{R}$, D öppen mängd.
Vi säger att f är av klass C^1 eller att $f \in C^1(D)$ om f är partiellt deriverbar och om alla partiella derivator är kontinuerliga.

Sats: Om $f \in C^1(D)$ så är den differentierbar i D.

Bevis: (Utför beviset i $d=3$).

f differentierbar $\iff$ $f(a+h) - f(a) = A_1h_1 + A_2h_2 + \dots + A_nh_n + |h|\rho(h)$
där $\rho(h) \to 0$, då $h \to 0$.

Enligt medelvärdessatsen för derivator $(d=1)$

$$f(a_1+h_1,\, a_2+h_2,\, a_3+h_3) - f(a_1, a_2, a_3)$$

$$= f(a_1+h_1, a_2+h_2, a_3+h_3) - \underline{f(a_1+h_1, a_2+h_2, a_3)}$$

$$+ \underline{f(a_1+h_1, a_2+h_2, a_3)} - \underline{f(a_1+h_1, a_2, a_3)}$$

$$+ \underline{f(a_1+h_1, a_2, a_3)} - f(a_1, a_2, a_3)$$

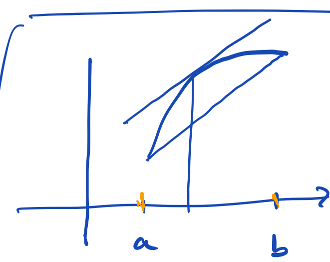
$$[\theta_j \in (0,1)]$$

$$= f'_z(a_1+h_1, a_2+h_2, a_3+\theta_3 h_3)\cdot h_3$$

$$+ f'_y(a_1+h_1, a_2+\theta_2 h_2, a_3)\cdot h_2$$

$$+ f'_x(a_1+\theta_1 h_1, a_2, a_3)\cdot h_1$$

$$= (*)$$



$$f(b) - f(a) = f'(p)(b-a)$$

$p \in (a,b)$

Nu vet vi att f'_x, f'_y, f'_z är kontinuerliga $(f \in C'(D))$.

$b = a + h$.

$p = a + h \cdot \theta$, $\theta \in (0,1)$

$f(a+h) - f(a) = f'(a + h\theta) \cdot h$

$$f'_z(a_1+h_1, a_2+h_2, a_3+\theta_3 h_3)$$
$$= f'_z(a_1,a_2,a_3) + \rho_3(h).$$

för någon funktion ρ_3.

Alltså får vi

$$\circledast = \left(f'_z(a_1,a_2,a_3) + \rho_3(h)\right) h_3$$
$$+ \left(f'_y(a_1,a_2,a_3) + \rho_2(h)\right) h_2$$
$$+ \left(f'_x(a_1,a_2,a_3) + \rho_1(h)\right) h_1$$

$a = (a_1, a_2, a_3)$

$$= f'_x(a)h_1 + f'_y(a)h_2 + f'_z(a)h_3 + \underbrace{\rho_1(h)h_1 + \rho_2(h)h_2 + \rho_3(h)h_3}_{= |h|\rho(h)}.$$

där $\rho(h) = \dfrac{1}{|h|}\left(h_1\rho_1(h) + h_2\rho_2(h) + h_3\rho_3(h)\right)$

Vi vill visa att $\rho(h) \to 0$, då $h \to 0$.

$|h_j| \le |h|$

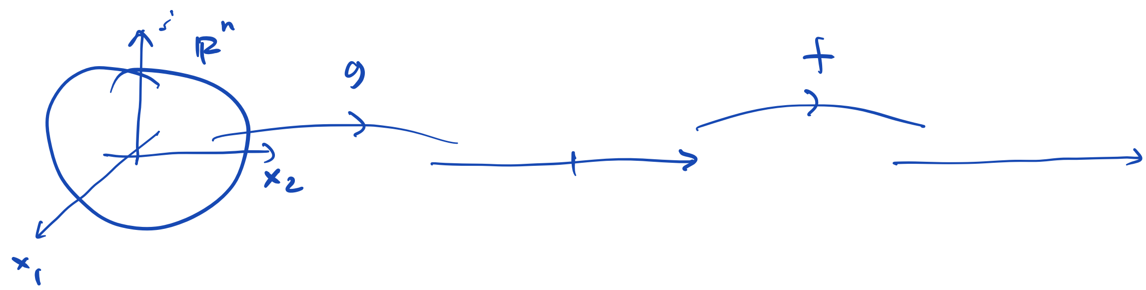
$$0 \le |\rho(h)| \le \frac{|h||\rho_1(h)| + |h||\rho_2(h)| + |h| \cdot |\rho_3(h)|}{|h|}$$

$$= |\rho_1(h)| + |\rho_2(h)| + |\rho_3(h)|$$

$$\rho_1(h) = f'_x(a_1+h_1, a_2, a_3) - f'_x(a_1,a_2,a_3) \to 0, \text{ då } h \to 0.$$

Kedjeregeln:

d=1: $$\frac{d}{dx}(f(g(x))) = f'(g(x))\cdot g'(x).$$

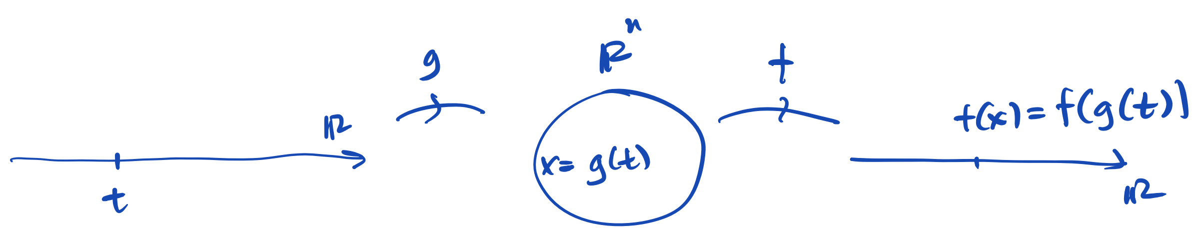


Om $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}^n\to\mathbb{R}$.

$$\frac{\partial}{\partial x_j}(f(g(x))) = f'(g(x))\cdot g'_{x_j}(x)$$

Frys de andra variablerna.

Låt nu $f:\mathbb{R}^n\to\mathbb{R}$ och $g:\mathbb{R}\to\mathbb{R}^n$



Sats: (Kedjeregeln)

Låt $f(x)=f(x_1,x_2,\dots,x_n)$ vara differentierbar och antag att funktionerna $g_1,g_2,g_3,\dots,g_n$ är deriverbara. Då gäller att

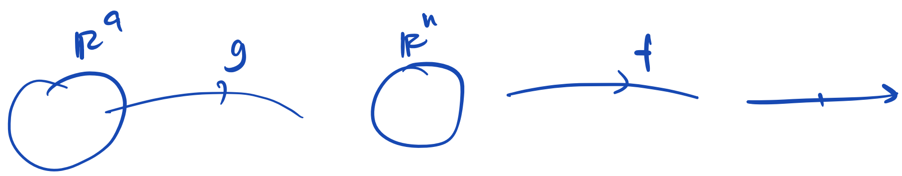
$$g(t)=(g_1(t),\dots,g_n(t))$$

$$x = (x_1, x_2, \dots, x_n)$$

$$\frac{d}{dt}(f(g(t)) = f'_{x_1}(g(t))\, g'_1(t) + \dots + f'_{x_n}(g(t)) \cdot g'_n(t).$$

$$= \frac{\partial f}{\partial x_1} \cdot \frac{dg_1}{dt} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{dg_n}{dt}.$$

I allmänna fallet $\quad g: \mathbb{R}^q \to \mathbb{R}^n, \; f: \mathbb{R}^n \to \mathbb{R}.$



$$\frac{\partial f}{\partial t_j}(x) = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}.$$

Ex: $\quad z = \sin(x^2 y)$, där $x = x(s,t) = st^2$, $y = s^2 + \frac{1}{t}$.

Beräkna $\frac{\partial z}{\partial s}$ och $\frac{\partial z}{\partial t}$.

Lösning:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = \cos(x^2 y) \cdot 2xy \cdot t^2$$

$$+ \cos(x^2 y) \cdot x^2 \cdot 2s$$

$$= \cos\left(s^2 t^4 \left(s^2 + \tfrac{1}{t}\right)\right) \cdot 2st^2\left(s^2 + \tfrac{1}{t}\right) t^2$$

$$+ \cos\left(s^2 t^4 \left(s^2 + \tfrac{1}{t}\right)\right) \cdot s^2 t^4 \cdot 2s$$

$$= \cos\left(s^4 t^4 + s^2 t^3\right) \left(2s^3 t^4 + 2st^3 + 2s^3 t^4\right)$$

$$= \left(4s^3 t^4 + 2st^3\right) \cos\left(s^4 t^4 + s^2 t^3\right).$$

$\sin(x^2y)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = \cos(x^2y) \cdot 2xy \cdot 2st$$

$$+ \cos(x^2y) \cdot x^2\left(-\frac{1}{t^2}\right) = \dots = \cos(s^4t^4 + s^2t^3)\,(4s^4t^3 + 4s^2t^2 - s^2t^3).$$

__Ex:__ Transformera uttrycket $f: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}$

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

genom övergång till polära koordinater.

__Lösning:__ $\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases}$

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial r} = \cos\theta \dfrac{\partial f}{\partial x} + \sin\theta \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial \theta} = -r\sin\theta \dfrac{\partial f}{\partial x} + r\cos\theta \dfrac{\partial f}{\partial y} \end{cases}$$

$(1) \cdot r\cos\theta - (2) \cdot \sin\theta$:

$$r\cos\theta \cdot \frac{\partial f}{\partial r} - \sin\theta \frac{\partial f}{\partial \theta} = \underbrace{r\cos^2\theta \frac{\partial f}{\partial x} + r\sin^2\theta \frac{\partial f}{\partial x}}_{\text{trigonometriska ettan}} + 0$$

$\cos^2\theta + \sin^2\theta = 1$

$\Longleftrightarrow$

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta}.$$

$(1) \cdot r\sin\theta + (2)\cos\theta$:

$$r\cdot\sin\theta\cdot\frac{\partial f}{\partial r}+\cos\theta\,\frac{\partial f}{\partial \theta}=0+\underbrace{r\sin^2\theta\,\frac{\partial f}{\partial y}+r\cos^2\theta\,\frac{\partial f}{\partial y}}_{=r\frac{\partial f}{\partial y}}$$

$$\Longleftrightarrow\ \frac{\partial f}{\partial y}=\sin\theta\,\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial f}{\partial \theta}.$$

Alltså

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\cos\theta\,\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial f}{\partial \theta}\right)^2+\left(\sin\theta\,\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial f}{\partial \theta}\right)^2$$

$$=\cos^2\theta\left(\frac{\partial f}{\partial r}\right)^2\cancel{-\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial f}{\partial r}\,\frac{\partial f}{\partial \theta}}+\frac{\sin^2\theta}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

$$+\sin^2\theta\left(\frac{\partial f}{\partial r}\right)^2\cancel{+\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial f}{\partial r}\cdot\frac{\partial f}{\partial \theta}}+\frac{\cos^2\theta}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

$$=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Ex: $y\cdot\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0$, $x>0$, $y>0$.

inför variablerna $\begin{cases} s=x^2+y^2 \\ t=x^2-y^2. \end{cases}$ $\quad s=s(x,y)$, $t=t(x,y)$

Vi söker den lösning som uppfyller att $u(0,y)=e^y$.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}=\frac{\partial u}{\partial s}\cdot 2x+\frac{\partial u}{\partial t}\cdot 2x$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}=\frac{\partial u}{\partial s}\cdot 2y-\frac{\partial u}{\partial t}\cdot 2y.$$

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = \cancel{2xy\frac{\partial u}{\partial s}} + 2xy\frac{\partial u}{\partial t} - \cancel{2xy\frac{\partial u}{\partial s}} + 2xy\frac{\partial u}{\partial t}$$

$$= 4xy\frac{\partial u}{\partial t} = 0 \iff \frac{\partial u}{\partial t} = 0.$$

Alltså är $u(s,t) = \varphi(s)$, för någon funktion φ.

$$= \varphi(x^2+y^2)$$

$$u(0,y) = e^y = e^{\sqrt{y^2}} = \varphi(y^2).$$

Alltså $\varphi(s) = e^{\sqrt{s}}$ och $u(x,y) = \varphi(x^2+y^2) = e^{\sqrt{x^2+y^2}}$