**Definition:** Låt $u=(u_1,u_2,u_3)$ vara ett $C^1$-fält. Vi definierar **rotationen** av $u$ som vektorfältet

$$\text{rot } u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

Fältet $u$ kallas **virvelfritt** om $\text{rot } u = 0$.

Notera att vi symboliskt kan se

$$\text{rot } u = \nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

**Sats:** (Stokes sats)

Låt $u=(u_1,u_2,u_3)$ vara ett $C^1$-fält definierat i en öppen mängd $\Omega \subset \mathbb{R}^3$. Om $Y$ är ett orienterat ytstycke i $\Omega$ med orienterad rand $\partial Y$ så gäller att

$$\int_{\partial Y} u \cdot dr = \iint_Y (\text{rot } u) \cdot N \, dS.$$

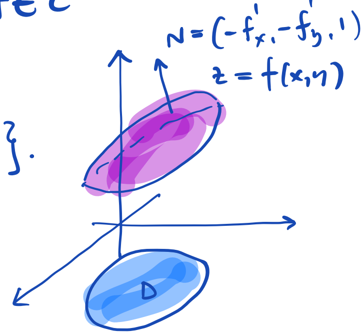
**Anmärkning:** Om $Y$ är en plan yta i xy-planet och $u=(u_1,u_2,0)$ så blir ovanstående

$$\int_{\partial Y} u_1 dx + u_2 dy = \iint_Y \left(-\frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) \cdot (0,0,1) \, dx\, dy$$

$$= \iint_Y \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dx\,dy \qquad \text{(Greens formel).}$$

Bevis: Återigen i ett specialfall, då $Y$ är en funktionsyta $z = f(x,y)$, med $f \in C^2$

$$Y = \{(x,y,z) : z = f(x,y),\ (x,y) \in D\}.$$




$$\int_{\partial Y} u \cdot dr = \int_{\partial Y} u_1\,dx + u_2\,dy + u_3\,dz$$

$$= \int_{\partial D} u_1\,dx + u_2\,dy + u_3 \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)$$

$$= \int_{\partial D} \underbrace{\left( u_1 + u_3 \frac{\partial f}{\partial x} \right)}_{P} dx + \underbrace{\left( u_2 + u_3 \frac{\partial f}{\partial y} \right)}_{Q} dy$$

Greens formel

$$\downarrow$$

$$= \iint_D \left( \frac{\partial}{\partial x} \left( u_2 + u_3 \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left( u_1 + u_3 \frac{\partial f}{\partial x} \right) \right) dx\,dy$$

Notera nu att $\frac{\partial}{\partial x}(u_2) = \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial z} \cdot \frac{\partial z}{\partial x}$ eftersom $u_2 = u_2(x, y, f(x,y))$

$$= \iint_D \left( \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial z} \cdot \frac{\partial z}{\partial x} + \left( \frac{\partial u_3}{\partial x} + \frac{\partial u_3}{\partial z} \cdot \frac{\partial z}{\partial x} \right) \left( \frac{\partial f}{\partial y} \right) + u_3 \frac{\partial^2 f}{\partial x \partial y} \right.$$

$$\left. - \frac{\partial u_1}{\partial y} - \frac{\partial u_1}{\partial z} \cdot \frac{\partial z}{\partial y} - \left( \frac{\partial u_3}{\partial y} + \frac{\partial u_3}{\partial z} \cdot \frac{\partial z}{\partial y} \right) \left( \frac{\partial f}{\partial x} \right) - u_3 \frac{\partial^2 f}{\partial x \partial y} \right) dx\,dy$$

$$= \iint_D \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} + \left(\frac{\partial f}{\partial x}\right)\left( \frac{\partial u_2}{\partial z} - \frac{\partial u_3}{\partial y}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right)\right) dx\,dy$$

Normal till funktionsyta är $(-f'_x, -f'_y, 1)$.

$$= \iint_D \left( \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \cdot (-f'_x, -f'_y, 1)\, dx\,dy$$

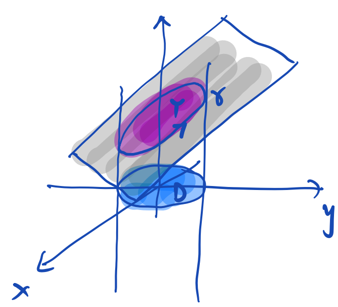
$$= \iint_Y ((\operatorname{rot} u) \cdot N)\, dS \qquad \blacksquare$$

<u>Ex:</u> Beräkna $\int_\gamma (-y^3, x^3, -z^3) \cdot dr$, där $\gamma$ är skärningen

av cylindern $x^2+y^2=1$ och planet $2x+2y+z=3$

orienterad moturs (vid projektionen på $xy$-planet).

<u>Lösning:</u> $N\,dS = (2,2,1)\,dx\,dy$




$$I = \int_\gamma u \cdot dr = \iint_Y (\operatorname{rot} u) \cdot N\, dS$$

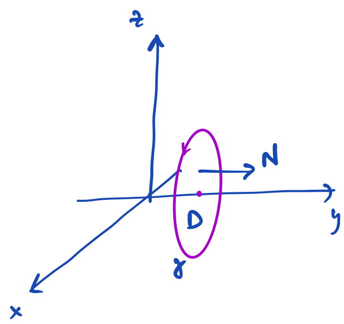
$$\operatorname{rot} u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2+3y^2)$$

$$I = \iint_D (0,0,3x^2+3y^2)\cdot(2,2,1)\,dx\,dy = 3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\varphi = 3\cdot 2\pi \,\frac{1}{4} = \frac{3\pi}{2}.$$

6. (4 poäng) Bestäm kurvintegralen av vektorfältet

$$\mathbf{F}(x,y,z) = (x^2y^3, e^{xy-x+z}, x+z^2)$$

längs cirkeln $x^2+z^2=1$ i planet $y=1$, där cirkeln är orienterad medurs sedd från origo.



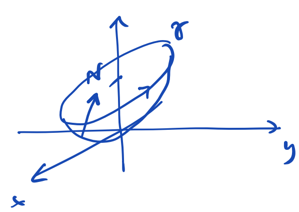

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ x^2y^3 & e^{xy-x+z} & x+z^2 \end{vmatrix} =$$

$$= \left(-e^{xy-x+z}, -1, (y-1)e^{xy-x+z} - 3x^2y^2\right)$$

$$N\,dS = (0,1,0)\,dx\,dz$$

$$\int_\gamma F\cdot dr = \iint_D (\operatorname{rot} F)\, N\, dS = -\iint_D dx\,dz = \underline{\underline{-\pi}}.$$

8. (4 poäng) Ytan $S$ består av den delen av paraboloiden $z = x^2 + 4y^2$ som ligger under planet $z=1$ orienterad så att normalvektorn $\mathbf{N}$ får positiv $z$-komponent. Bestäm flödet av vektorfältet $\nabla \times \mathbf{F}$ genom $S$ där $\mathbf{F} = (y, -xz, xz^2)$.




$$\iint_S (\operatorname{rot} F)\, N\, dS = \int_\gamma F\cdot dr$$

$$= \int_0^{2\pi} \left(\frac{\sin\theta}{2}, -\cos\theta, \cos^2\theta\right)\cdot\left(-\sin\theta, \frac{\cos\theta}{2}, 0\right) d\theta$$

$$= \int_0^{2\pi}\left(-\frac{\sin^2\theta}{2} - \frac{\cos^2\theta}{2}\right) d\theta = -\frac{1}{2}\int_0^{2\pi} d\theta = \underline{-\pi}$$

$x = \cos\theta$

$y = \frac{\sin\theta}{2}$

$z = 1$

$r(\theta) = \left(\cos\theta, \frac{\sin\theta}{2}, 1\right)$

$r'(\theta) = \left(-\sin\theta, \frac{\cos\theta}{2}, 0\right)$