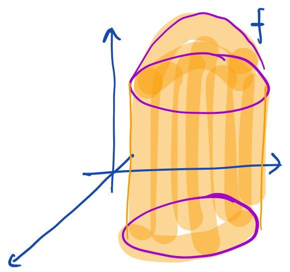
Volymberäkningar: Låt $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$

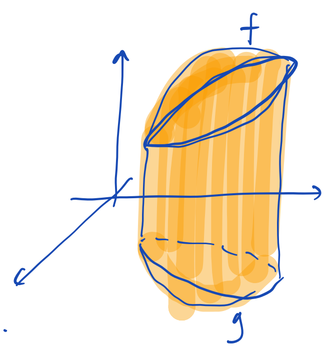
$$\iint_D f(x,y)\,dx\,dy$$



Volymen under grafen

Om $g: D \to \mathbb{R}$ så ger $\iint_D (f(x,y) - g(x,y))\,dx\,dy$

volymen mellan funktionerna.

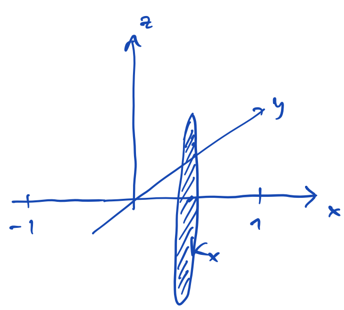


Om vi kallar området i $\mathbb{R}^3$ för K.

$$\mu(K) = \iiint_K dx\,dy\,dz = \iint_D (f(x,y) - g(x,y))\,dx\,dy.$$

Ex: Beräkna volymen av $K = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z - x^2)^2 \leq 1\}$.

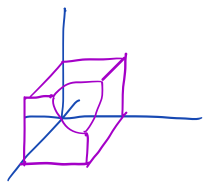
Lösning: För givet $x \in [-1,1]$ får vi ytan $K_x = \{(y,z) : y^2 + (z - x^2)^2 \leq 1 - x^2\}$



$$\mu(K_x) = \pi \cdot (1 - x^2).$$

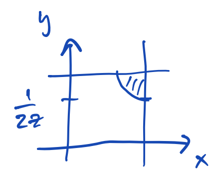
$$V = \iiint dz\,dy\,dx = \int_{-1}^{1} \left(\iint_{K_x} dy\,dz \right) dx = \pi \int_{-1}^{1} (1 - x^2)\,dx = \pi 2 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{4\pi}{3}$$

Ex: Beräkna volymen av området i $\mathbb{R}^3$ som definieras genom $2xzy \le 1$, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.



$V = 1 - \mu(\Omega)$, där $\Omega = \{(x,y,z) : 2xyz \ge 1$
$0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

För givet $\frac{1}{2} \le z \le 1$ gäller att $1 \ge y \ge \frac{1}{2xz}$



$$\mu(\omega) = \int_{1/2}^{1} \int_{1/2z}^{1} \int_{1/2yz}^{1} dx\, dy\, dz = \int_{1/2}^{1} \int_{1/2z}^{1} \left(1 - \frac{1}{2yz}\right) dy\, dz$$

$$= \int_{1/2}^{1} \left[y - \frac{1}{2z} \log y\right]_{1/2z}^{1} dz = \int_{1/2}^{1} \left(1 - \frac{1}{2z} + \frac{1}{2z} \log\left(\frac{1}{2z}\right)\right) dz$$

$$= \left\{ \begin{array}{l} \frac{1}{2z} = t \\ z = \frac{1}{2t} \\ dz = \frac{1}{2} \log t\, dt \end{array} \right\} = \int_{1}^{1/2} (1 - t + t \ln t) \frac{1}{2} \ln t\, dt$$

$$= \frac{-1}{2} \int_{1/2}^{1} \left(t \ln t - \ln t - t \cdot (\ln t)^2\right) dt$$

$$I_1 = \int_{1/2}^{1} t \ln t\, dt = \left[\frac{t^2}{2} \ln t\right]_{1/2}^{1} - \int_{1/2}^{1} t \cdot \frac{1}{t}\, dt = 0 - \frac{\ln\left(\frac{1}{2}\right)}{8} - \frac{1}{2}.$$

$$I_2 = -\int_{1/2}^{1} \ln t\, dt = -\left[t \ln t - t\right]_{1/2}^{1} = -\left(-1 - \frac{1}{2} \ln\left(\frac{1}{2}\right) + \frac{1}{2}\right)$$

$$= 1 + \frac{1}{2} \ln\left(\frac{1}{2}\right) - \frac{1}{2}.$$

$$I_3 = -\int_{1/2}^{1} t(\ln t)^2\,dt = -\left[\frac{t^2}{2}(\ln t)^2\right]_{1/2}^{1} + \int_{1/2}^{1} \frac{t^2}{2}\, 2\ln t \cdot \frac{1}{t}\,dt$$

$$= +\frac{1}{8}\left(\ln \frac{1}{2}\right)^2 + \int_{1/2}^{1} t \ln t\,dt = -\frac{\ln\left(\frac{1}{2}\right)}{8} - \frac{1}{2} \cdot + \underline{\frac{1}{8}(\ln 2)^2}$$

$$I = -\frac{1}{2}\left(I_1 + I_2 + I_3\right) = \cancel{\frac{1}{4}} - \frac{\ln 2}{16} - \cancel{\frac{1}{4}} + \frac{1}{4}\ln 2 - \frac{\ln 2}{16} + \frac{1}{4} + \frac{1}{8}(\ln 2)^2$$

$$= \frac{1}{4} - \frac{\ln 2}{8} + \frac{1}{4}\ln 2 + \frac{1}{8}(\ln 2)^2 = \frac{1}{4} + \frac{1}{8}\ln 2 + \frac{1}{8}(\ln 2)^2.$$

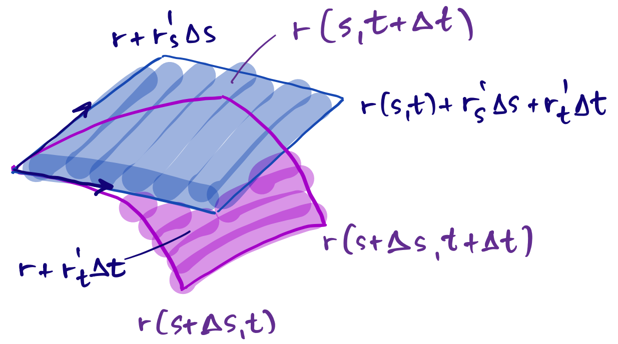
Area av buktig yta:

Låt Y vara en yta i $\mathbb{R}^3$ som har parameterframställningen

$$r = r(s,t), \quad \text{där} \quad (s,t) \in D$$

Tangentplanet för Y i en punkt (s,t) spänns upp av vektorerna r'_s och r'_t.

Vi kan approximera en liten del av ytan med en liten del av tangentplanet.



Arean av parallellogrammet ges av

$$|r'_s \Delta s \times r'_t \Delta t| = |r'_s \times r'_t| \Delta s \Delta t.$$

Vi definierar areaelementet på Y som

$$dS = |r'_s \times r'_t| \, ds dt.$$

Arean av Y. $\mu(Y) = \iint_Y dS = \iint_D |r'_s \times r'_t| \, ds dt$

(Jämför med kurvintegralen $\int_\gamma ds = \int_\alpha^\beta |r'(t)| dt$.)

<u>Sats</u>: Area integralen $\iint_Y dS$ är oberoende av parameterframställning.

<u>Bevis</u>: Antag att Y ges av $r = r(s,t)$, $(s,t) \in D$.
och av $x = x(u,v)$, $(u,v) \in E$ och att
det finns en bijektion $T: E \to D$, alltså
$(s,t) = T(u,v) = (T_1(u,v), T_2(u,v))$, $r(s,t) = r(T(u,v)) = x(u,v)$

$$x'_u(u,v) = \frac{\partial}{\partial u}(x(u,v)) = \left(\frac{\partial}{\partial s} \cdot \frac{\partial s}{\partial u} + \frac{\partial}{\partial t} \cdot \frac{\partial t}{\partial u}\right)(x(u,v)) =$$

Note!

$$= \left(\frac{\partial s}{\partial u} \frac{\partial}{\partial s} + \frac{\partial t}{\partial u} \cdot \frac{\partial}{\partial t}\right)(r(s,t))$$

$$x'_u = r'_s \cdot \frac{\partial s}{\partial u} + r'_t \cdot \frac{\partial t}{\partial u} \quad , \quad r'_v = r'_s \cdot \frac{\partial s}{\partial v} + r'_t \cdot \frac{\partial t}{\partial v} .$$

$$x'_u \times x'_v = \left(r'_s \cdot \frac{\partial s}{\partial u} + r'_t \frac{\partial t}{\partial u}\right) \times \left(r'_s \frac{\partial s}{\partial v} + r'_t \frac{\partial t}{\partial v}\right)$$

$$= \frac{\partial s}{\partial u}\frac{\partial t}{\partial v}\left(r'_s \times r'_t\right) + \frac{\partial t}{\partial u}\frac{\partial s}{\partial v}\left(r'_t \times r'_s\right) = \left(\frac{\partial s}{\partial u} \cdot \frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u}\right)\left(r'_s \times r'_t\right)$$

$$= \frac{d(s,t)}{d(u,v)}\left(r'_s \times r'_t\right) .$$ Alltså $$|r'_u \times r'_v| = \left|\frac{d(s,t)}{d(u,v)}\right| \cdot |r'_s \times r'_t|$$

Alltså

$$\iint_E |x'_u \times x'_v| \, du\, dv = \iint_E |r'_s \times r'_t| \cdot \left|\frac{d(s,t)}{d(u,v)}\right| du\, dv = \iint_D |r'_s \times r'_t| \, ds\, dt$$

Ex: Beräkna arean av en sfär Σ med radien R.

Ytan kan ges av $Y(\theta,\varphi) = R(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$0 \le \theta \le \pi \quad , \quad 0 \le \varphi \le 2\pi .$

Då blir

$$r'_\theta = R(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$r'_\varphi = R(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\frac{1}{R^2} r'_\theta \times r'_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{vmatrix} = \left(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \substack{\cos^2\varphi\cos\theta\sin\theta \\ \sin^2\varphi\sin\theta\cos\theta}\right.$$

$$= (\sin^2\theta\cos\varphi,\ \sin^2\theta\sin\varphi,\ \sin\theta\cos\theta)$$

$$\frac{1}{R^2}|r'_\theta \times r'_\varphi| = \sqrt{\sin^4\theta\cos^2\varphi + \sin^4\theta\sin^2\varphi + \sin^2\theta\cos^2\theta} = |\sin\theta|\cdot\sqrt{\sin^2\theta+\cos^2\theta} = \sin\theta$$

$$\iint_Y dS = R^2 \int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\varphi \, d\theta = 2\pi R^2 \Big[-\cos\theta\Big]_0^{\pi} = \underline{\underline{4\pi R^2}}$$

Ex: Härled en formel för beräkning av arean av en funktionsyta $z = f(x,y)$, $(x,y) \in D$.

Lösning: $(x,y) \in D$ kan användas som parametrar.

$$r(x,y) = (x,\ y,\ f(x,y))$$

$$r'_x = (1,\ 0,\ f'_x) \qquad r'_y = (0,\ 1,\ f'_y)$$

$$r'_x \times r'_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{vmatrix} = (-f'_x,\ -f'_y,\ 1)$$

$|r'_x \times r'_y| = \sqrt{f'^2_x + f'^2_y + 1}$. Alltså $\displaystyle\iint_Y dS = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy$

Alt uppgift: Beräkna volymen av området givet av olikheterna $x^2+4y^2+z^2 \le 5$ och $z \geq 1$.

Lösning: $V = \iiint_Q dx\,dy\,dz$, där $Q = \{(x,y,z): x^2+4y^2+z^2 \le 5,\ z \geq 1\}$

$$V = \int_1^{\sqrt{5}} \left(\iint_{x^2+4y^2 \le 5-z^2} dx\,dy \right) dz$$

$$\frac{x^2}{\sqrt{5-z^2}^2} + \frac{y^2}{\left(\frac{\sqrt{5-z^2}}{2}\right)^2} = 1$$

$$= \int_1^{\sqrt{5}} \frac{\pi}{2}(5-z^2)\,dz = \frac{\pi}{2}\left[5z - \frac{z^3}{3}\right]_1^{\sqrt{5}}$$

$$\text{Area} = \pi \sqrt{5-z^2} \cdot \frac{\sqrt{5-z^2}}{2} = \frac{\pi}{2}(5-z^2)$$

$$= \frac{\pi}{2}\left(\frac{5\sqrt{5}\cdot 2}{3} - 5 + \frac{1}{3}\right) = \frac{\pi}{6}(10\sqrt{5} - 14) = \frac{\pi}{3}(5\sqrt{5} - 7)$$

Om man inte kan formeln $\pi \cdot ab$ = arean av $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\iint_{x^2+4y^2 \le 5-z^2} dx\,dy = \left\{ \begin{array}{l} x = r\cos\varphi \\ 2y = r\sin\varphi \\ \text{abs}\left(\frac{d(x,y)}{d(r,\varphi)}\right) = \text{abs}\left(\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \frac{\sin\varphi}{2} & \frac{r\cos\varphi}{2} \end{vmatrix}\right) = \frac{1}{2}r \end{array} \right\}$$

$$= \int_0^{2\pi} \int_0^{\sqrt{5-z^2}} \frac{r}{2}\,dr\,d\varphi = \pi\left[\frac{r^2}{2}\right]_0^{\sqrt{5-z^2}} = \frac{\pi}{2}(5-z^2).$$