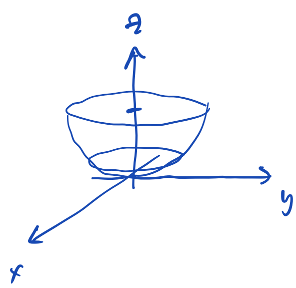
Multipelintegraler: 1) Börja med trappfunktioner Φ (konstanta på intervall $\Delta_{1,2,\dots,n}$)

2) Definiera $\displaystyle\int\int\cdots\int_{\Delta} \Phi(x)\,dx_1\,dx_2\cdots dx_n = \sum_{i_1,i_2,\dots,i_n} c_{1,2,\dots,n}\cdot\mu(\Delta_{i_1,i_2,\dots,i_n})$

3) Generalisera till $\displaystyle\int\int\cdots\int_{D} f(x)\,dx$.

Ex: Beräkna $\displaystyle I=\iiint_Q z\sqrt{x^2+y^2}\,dx\,dy\,dz$, $Q=\{(x,y,z): x^2+y^2\le z\le 1\}$



$$I=\int_0^1 z\left(\iint_{x^2+y^2\le z}\sqrt{x^2+y^2}\,dx\,dy\right)dz = \int_0^1 z\left(\int_0^{2\pi}\int_0^{\sqrt{z}} r\cdot r\,dr\,d\theta\right)dz$$

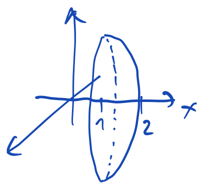
$$=\int_0^1 z\cdot 2\pi\left[\frac{r^3}{3}\right]_0^{\sqrt{z}}dz = \frac{2\pi}{3}\int_0^1 z^{5/2}\,dz = \frac{2\pi}{3}\cdot\left[\frac{2z^{7/2}}{7}\right]_0^1 = \frac{4\pi}{21}$$

Alternativ 2: $$I=\iint_{x^2+y^2\le 1}\left(\int_{x^2+y^2}^1 z\,dz\right)\sqrt{x^2+y^2}\,dx\,dy = \iint_{x^2+y^2\le 1}\left[\frac{z^2}{2}\right]_{x^2+y^2}^1\sqrt{x^2+y^2}\,dx\,dy$$

$$=\iint_{x^2+y^2\le 1}\frac{1}{2}\left(1-(x^2+y^2)^2\right)\sqrt{x^2+y^2}\,dx\,dy = \frac{2\pi}{2}\int_0^1(1-r^4)r^2\,dr$$

$$=\pi\left[\frac{r^3}{3}-\frac{r^7}{7}\right]_0^1 = \pi\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{4\pi}{21}$$

Ex: Beräkna $\displaystyle I=\iiint_Q x\,dx\,dy\,dz$, $Q=\{(x,y,z): x^2+y^2+z^2\le 4,\ x\ge 1\}$



$$y^2+z^2 \le 4-x^2.$$

$$I = \int_1^2 x \left(\iint_{y^2+z^2\le 4-x^2} dy\,dz \right) dx = \int_1^2 x \cdot \underbrace{\pi(4-x^2)}_{\text{arean av cirkelskivan}}\, dx$$

$$= \left[2\pi x^2 - \frac{\pi x^4}{4} \right]_1^2 = 8\pi - 4\pi - 2\pi + \frac{\pi}{4} = 2\pi + \frac{\pi}{4} = \underline{\underline{\frac{9\pi}{4}}}$$

Ex: Beräkna klotets volym. $B_R = \{ (x,y,z) : z^2+y^2+z^2 \le R^2 \}$

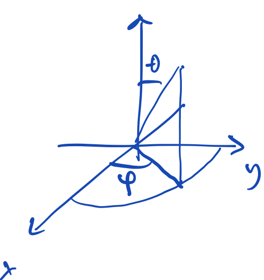
$$\mu(B_R) = \iiint_{B_R} dx\,dy\,dz = 2\int_0^R \left(\iint_{y^2+z^2\le R^2-x^2} dy\,dz \right) dx = 2\pi \int_0^R (R^2-x^2)\,dx$$

$$= 2\pi \left[R^2 x - \frac{x^3}{3} \right]_0^R = 2\pi \left(R^3 - \frac{R^3}{3} \right) = \underline{\underline{\frac{4\pi R^3}{3}}}.$$

~~Al~~ternativ 2

Sfäriskt koordinatbyte:

$$\begin{cases} x = r\cos\varphi \cdot \sin\theta \\ y = r\sin\varphi \cdot \sin\theta \\ z = r\cos\theta \end{cases}$$



$$\left| \frac{d(x,y,z)}{d(r,\varphi,\theta)} \right| \overset{\text{abs}}{=} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \overset{\text{abs}}{=} \begin{vmatrix} \cos\varphi\sin\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix}$$

$$= \left| -r^2\cos^2\varphi\sin^3\theta - r^2\sin^2\varphi\sin\theta\cos^2\theta - r^2\sin\theta\cos^2\theta\cos^2\varphi - r^2\sin^3\theta\sin^2\varphi \right|$$

$$= \left| r^2\sin^3\theta + r^2\sin\theta\cos^2\theta \right| = r^2\sin\theta\,|1| = r^2\sin\theta$$

$$\mu(B_R) = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\theta \, dr\, d\varphi\, d\theta = 2\pi \left[\frac{r^3}{3}\right]_0^R \cdot \int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{2\pi R^3}{3} \cdot \left[-\cos\theta\right]_0^{\pi} = \frac{2\pi R^3}{3}(1+1) = \frac{4\pi R^3}{3}.$$

Integraler i $\mathbb{R}^n$.

Resultat: $\int_{B_R} h(|x|)\,dx = \int_0^R h(r)\cdot V'(r)\,dr$, där $V(r) = \mu(B_r)$ (Man kan visa att)

$$V(r) = \mu(B_r) = \int_{B_r} 1 \cdot dx = \{x = ry\} = \int_{B_1} r^n \, dy = r^n \mu(B_1).$$

$$V'(r) = n r^{n-1} \mu(B_1).$$

Ex: $\int_{B_1} \frac{1}{|x|^p}\,dx = \int_0^1 \frac{1}{r^p} \cdot n r^{n-1} \mu(B_1)\,dr = n\mu(B_1)\int_0^1 \frac{1}{r^{p-n+1}}\,dr$

konvergent om och endast om $p-n+1 < 1$, dvs

$\iff p < n$.

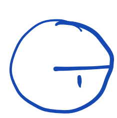
Monte-Carlo metoden: Beräkna volymen av $\mu(B_1)$ i $\mathbb{R}^n$

$n = 1, 2, 3, \dots$

n=1:

$\mu(B_1) = 2.$

n=2:

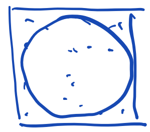


$\mu(B_1) = \pi$

n=3:

$\mu(B_1) = \frac{4\pi}{3}.$

$\underline{n=4}$: ?



Slumpa och se hur mycket som är utanför/innanför.