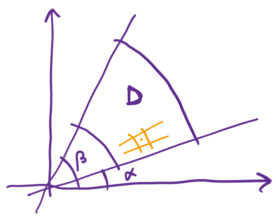
Riemannsummor:

Sats: Om f är kontinuerlig på $D \subset \mathbb{R}^2$ så gäller att
(D kompakt)

$$\sum_k f(\alpha_k, \beta_k)\,\mu(D_k) \longrightarrow \iint_D f(x,y)\,dx\,dy \quad \text{då}$$

$$\max_k \mu(D_k) \longrightarrow 0.$$

Ex:



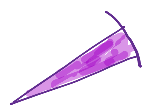
Vi vill uttrycka $\iint_D f(x,y)\,dx\,dy$ i polära koordinater

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

Vi delar in D i små sektorer $D_{ij} = \{(r,\varphi) : r_{i-1} \le r < r_i,\ \varphi_{j-1} \le \varphi < \varphi_j\}$

I D_{ij} väljer vi punkten $\rho_i = \frac{r_{i-1}+r_i}{2}$ och $\theta_j = \frac{\varphi_{j-1}+\varphi_j}{2}$.

Notera att $\mu(D_{ij}) = \dfrac{r_i^2(\varphi_j - \varphi_{j-1}) - r_{i-1}^2(\varphi_j - \varphi_{j-1})}{2}$



$$A = \pi \cdot r^2 \cdot \frac{(\beta - \alpha)}{2\pi}$$

$$= \frac{1}{2}(r_i^2 - r_{j-1}^2)(\varphi_j - \varphi_{j-1}) = \rho_i (r_i - r_{i-1})(\varphi_j - \varphi_{j-1})$$

Riemannsumman blir nu $\sum_{i,j} f(\xi_{ij}, \eta_{ij})\,\mu(D_{ij}) =$

$$= \sum_{i,j} \underbrace{f(\rho_i \cos\theta_j, \rho_i \sin\theta_j) \cdot \rho_i}_{} (r_i - r_{i-1})(\varphi_j - \varphi_{j-1})$$

Notera att detta kan uppfattas som en Riemannsumma

till $(r, \varphi) \longmapsto f(r\cos\varphi, r\sin\varphi) \cdot r$ över $E = \{(r,\varphi) : a \leq r \leq b, \alpha \leq \varphi \leq \beta\}$

(rektangel)

När indelningen går mot noll får vi

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(r\cos\varphi, r\sin\varphi)\, r\, dr\, d\varphi.$$

Variabelbyte då $d=1$:

Låt $g: [\alpha, \beta] \longrightarrow [a,b]$ vara monoton och surjektiv.

Då gäller

(g växande) $\displaystyle\int_a^b f(x)\,dx = \left\{\begin{matrix} x = g(t) \\ dx = g'(t)dt \end{matrix}\right\} = \int_\alpha^\beta f(g(t))\, g'(t)\,dt = \int_\alpha^\beta f(g(t))\cdot |g'(t)|\,dt$

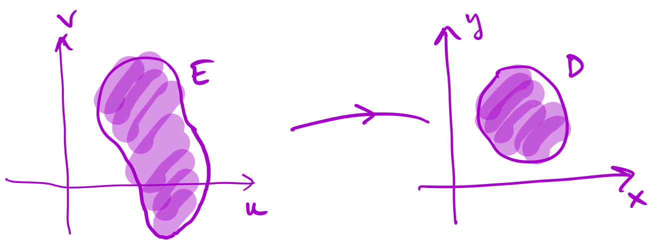
(g avtagande) $\displaystyle\int_a^b f(x)\,dx = \left\{\begin{matrix} x = g(t) \\ dx = g'(t)dt \end{matrix}\right\} = \int_\beta^\alpha f(g(t))\, g'(t)\,dt$

$\displaystyle = \int_\alpha^\beta f(g(t))\,(-g'(t))\,dt = \int_\alpha^\beta f(g(t))\,|g'(t)|\,dt$

(eftersom $|g'(t)| = -g'(t)$)

$d=2$: Ett variabelbyte i planet ges av en bijektiv avbildning $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ $G(u,v) = (x,y)$

från ett område E i uv-planet till D i xy-planet.



Dela in $D = \bigcup D_k$, disjunkta delar. Låt E_k vara sådan att $G(E_k) = D_k$.

Area förstoringen i en punkt ges (från linjärisering) av funktionaldeterminanten i den punkten.

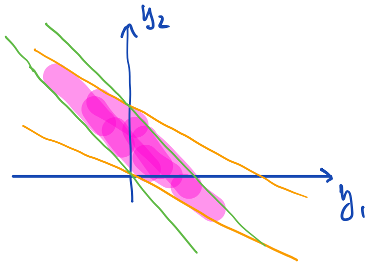
$$J(u,v) = \frac{d(x,y)}{d(u,v)} = \frac{d(g,h)}{d(u,v)} \qquad \text{Alltså} \qquad \mu(D_k) \approx |J(u_k,v_k)|\,\mu(E_k)$$

$$\sum_k f(x_k,y_k)\,\mu(D_k) \approx \sum_k f(g(u_k,v_k),h(u_k,v_k))\,|J(u_k,v_k)|\,\mu(E_k).$$

↓ Riemannsumma ↓ Riemannsumma

$$\iint_D f(x,y)\,dx\,dy = \iint_D f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv.$$

Ex: Beräkna $\iint_D (y_1^2 + y_2)\,dy_1\,dy_2$ om $D = \{(y_1,y_2) : 0 < y_1 + 2y_2 < 1,\ 0 < 3y_1 + 4y_2 < 2\}$



$0 < y_1 + 2y_2 < 1$: $\quad y_2 = -\frac{y_1}{2}$, $\quad y_2 = \frac{-y_1+1}{2}$

$0 < 3y_1 + 4y_2 < 2$: $\quad y_2 = -\frac{3y_1}{4}$, $\quad y_2 = \frac{-3y_1+2}{4}$.

Nya variabler
$$\begin{cases} x_1 = y_1 + 2y_2 \\ x_2 = 3y_1 + 4y_2 \end{cases} \iff \begin{cases} y_1 = -2x_1 + x_2 \\ y_2 = \frac{3x_1}{2} - \frac{x_2}{2} \end{cases}$$

$$I = \iint_D f(y)\,dy_1\,dy_2 = \iint_R f\left(-2x_1 + x_2, \frac{3x_1}{2} - \frac{x_2}{2}\right) \frac{d(y_1,y_2)}{d(x_1,x_2)} \cdot dx_1\,dx_2 .$$

$$\frac{d(y_1,y_2)}{d(x_1,x_2)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{vmatrix} = 1 - \frac{3}{2} = -\frac{1}{2}.$$

$$I = \iint_R f\left(-2x_1 + x_2, \frac{3x_1}{2} - \frac{x_2}{2}\right) \left|-\frac{1}{2}\right| dx_1\,dx_2 = \int_0^2\int_0^1 \left((-2x_1 + x_2)^2 + \frac{3x_1}{2} - \frac{x_2}{2}\right) \frac{dx_1\,dx_2}{2}$$

$$= \int_0^2\int_0^1 \left(4x_1^2 - 4x_1x_2 + x_2^2 + \frac{3x_1}{2} - \frac{x_2}{2}\right)\frac{dx_1\,dx_2}{2} = \int_0^2 \frac{1}{2}\left[\frac{4x_1^3}{3} - 2x_1^2x_2 + x_1x_2^2 + \frac{3x_1^2}{4} - \frac{x_1x_2}{2}\right]_0^1 dx_2$$

$$= \int_0^2 \frac{1}{2}\left(\frac{4}{3} - 2x_2 + x_2^2 + \frac{3}{4} - \frac{x_2}{2}\right)dx_2 = \frac{1}{2}\left[\frac{4x_2}{3} - x_2^2 + \frac{x_2^3}{3} + \frac{3x_2}{4} - \frac{x_2^2}{4}\right]_0^2 =$$

$$= \frac{1}{2}\left(\frac{8}{3} - 4 + \frac{8}{3} + \frac{6}{4} - 1\right) = \frac{1}{2}\left(\frac{16}{3} - 5 + \frac{3}{2}\right) = \frac{1}{2}\cdot\frac{1}{6}\left(32 - 30 + 9\right) = \underline{\underline{\frac{11}{12}}}.$$

Ex: Beräkna $I = \iint_E y^2\,dx\,dy$ över $E = \left\{(x,y) : \frac{x^2}{9} + y^2 < 1\right\}$

Variabelbyte: $\begin{cases} \frac{x}{3} = r\cos\theta & 0 < r < 1 \\ y = r\sin\theta & 0 < \theta < 2\pi \end{cases}$

$$\left|\frac{d(x,y)}{d(r,\theta)}\right| = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 3r$$

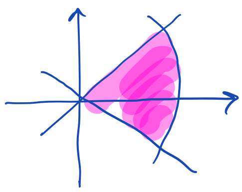
$$I = \int_0^1\int_0^{2\pi} r^2\sin^2\theta \cdot 3r\,d\theta\,dr = 3\int_0^1 r^3\,dr \cdot \int_0^{2\pi}\sin^2\theta\,d\theta = 3\left[\frac{r^4}{4}\right]_0^1 \cdot \int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= \frac{3}{4}\int_0^{2\pi}\sin^2\theta\,d\theta = \left\{\text{symmetri}\right\} = \frac{3}{4}\cdot\frac{1}{2}\int_0^{2\pi}\left(\sin^2\theta + \cos^2\theta\right)d\theta$$

$$= \frac{3}{8}\cdot 2\pi = \frac{3\pi}{4}.$$

Ex: Beräkna $I = \iint_\Omega (x^2 - y^2)e^{2xy}\,dx\,dy$ om Ω ges av olikheterna $x^2 + y^2 < 1$, $-x < y < x$, $x > 0$.

$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$, $\quad 0 < r < 1$, $\quad |\theta| < \frac{\pi}{4}$.



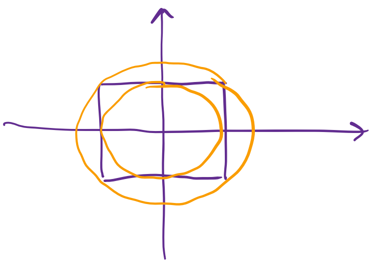
$$I = \int_0^1\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(r^2\cos^2\theta - r^2\sin^2\theta\right)e^{2r^2\cos\theta\sin\theta}\,r\,dr\,d\theta$$

$$= \int_0^1\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos 2\theta\, e^{r^2\sin 2\theta}\,dr\,d\theta = \int_0^1\left[\frac{r}{2}e^{r^2\sin 2\theta}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}dr = \frac{1}{2}\int_0^1 r\left(e^{r^2} - e^{-r^2}\right)dr$$

endast θ ☺

$$= \frac{1}{2}\left[\frac{e^{r^2}}{2} + \frac{e^{-r^2}}{2}\right]_0^1 = \underline{\frac{1}{4}\left(e + e^{-1} - 2\right)}.$$

Ex: Beräkna $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$.



$S_R = \{(x,y) : |x| < R,\ |y| < R\}$.

$C_R = \{(x,y) : x^2 + y^2 < R\}$.

Låt $f(x,y) = e^{-x^2-y^2}$.

$$\iint_{C_R} f(x,y)\,dx\,dy \le \iint_{S_R} f(x,y)\,dx\,dy \le \iint_{C_{\sqrt{2}R}} f(x,y)\,dx\,dy.$$

$$\iint_{C_R} f(x,y)\,dx\,dy = \int_0^R \int_0^{2\pi} e^{-r^2} r\,dr\,d\theta = 2\pi\left[\frac{e^{-r^2}}{-2}\right]_0^R$$

$$= -\pi\left(e^{-R^2} - 1\right) = \pi\left(1 - e^{-R^2}\right).$$

$$\iint_{S_R} e^{-x^2} \cdot e^{-y^2}\,dx\,dy = \int_{-R}^{R} e^{-x^2}\,dx \cdot \int_{-R}^{R} e^{-y^2}\,dy = \left(\int_{-R}^{R} e^{-x^2}\,dx\right)^2.$$

Alltså

$$\begin{array}{ccccc} \pi\left(1 - e^{-R^2}\right) & \le & \left(\displaystyle\int_{-R}^{R} e^{-x^2}\,dx\right)^2 & \le & \pi\left(1 - e^{-(\sqrt{2}R)^2}\right) \\ \downarrow & & \downarrow & & \downarrow \\ \pi & \le & I^2 & \le & \pi. \end{array}$$

Alltså gäller att $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.