Application

Postdoctoral Scholarship Program in Mathematics
for researchers from outside Sweden
Knut and Alice Wallenberg Foundation

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1 The research group

The topology group at the mathematics department at KTH consists currently of five members.

Senior faculty: Wojciech Chachólski (professor)
Tilman Bauer (associate professor)

Postdoc: Stephanie Ziegenhagen

Ph.D. students: Magnus Carlson (expected graduation fall 2017)
Sebastian Öberg (expected graduation spring 2016)

We collaborate closely with the topology group at the University of Stockholm, consisting of Gregory Arone and Alexander Berglund (associate professors), Kaj Börjeson, Felix Wierstra, and Bashar Saleh (Ph.D. students).

In addition, we currently supervise a number of master level students with the hope that one or two of them will join the group as Ph.D. students.

In the past two and a half years, topology in Stockholm has grown at a fast pace and Stockholm is in the process of establishing itself as one of the most active centers of topology in Europe. This has been in part due to the hiring of three permanent new faculty members (Tilman Bauer at KTH, Alexander Berglund and Gregory Arone at SU), but also due to our ability to attract excellent Ph.D. students and postdocs. We will strengthen our profile even more and broaden our research scope if this application for a postdoctoral researcher is successful. Based on what we have managed to build up in Stockholm, we are confident that we would be able to hire a young mathematician on the very top level.

2 Project and research area

The proposed project lies in the area of *algebraic K-theory*, which is one of the most active, most exciting, and also most successful areas of algebraic topology of recent decades. I will give a brief overview of the development and current status of this field.

Algebraic K-theory exists in many flavors, but the common starting point is the wish to construct an invariant of *rings* that captures arithmetic and geometric properties of the ring. The motivation for this came from two very distinct areas of mathematics. In the 1930s and 1940s, J.H.C. Whitehead [29] studied when a topological space Y which is homotopy equivalent to another topological spaces X can be built from X by a sequence of simple moves, which he called *simple homotopy equivalences*. He discovered that this is not always possible, and the obstruction to this lies in a group $Wh(\pi)$, the Whitehead group, which only depends on the fundamental group $\pi = \pi_1(X)$, or rather its group ring $\mathbf{Z}[\pi_1(X)]$.

A couple of decades later, A. Grothendieck defined group $K_0(X)$ of equivalence classes of algebraic vector bundles on a scheme X in his work on the Riemann-Roch theorem [8]. It was the work of many people, among them Bass, Milnor, and Swan [5], to see that the Whitehead group and K_0 were two sides of the same medal, and that the Whitehead group $Wh(R)$ had a close relative, $K_1(R)$, whose relation to $K_0(\operatorname{Spec} R)$ was analogous to the relationship between the first two ordinary homology groups.

Naturally, people now started looking for higher K-groups and their significance. In the course of the 1960s, Steinberg [24] and Milnor [21] defined K_2 for fields and related it to arithmetic invariants like the Brauer group, Galois cohomology, and quadratic forms. Milnor then extended this definition to define K_n^M, *Milnor K-theory*, for all $n \geq 0$. Famously, the *Milnor* and *Bloch-Kato conjectures*, proven in the late 1990 by Voevodsky and Rost [25, 26], state that the Milnor K-theory modulo a prime l of a field is isomorphic to its Galois cohomology.

D. Quillen entered the field in the late 1960s and used topological and category-theoretic methods to construct K_n in a unified and very general way [22]. Strikingly, he showed that there is a spectrum (cohomology theory) KR associated to any ring R, or in fact any category with exact sequences, whose homotopy groups are the algebraic K-groups of R. Because of their link to arithmetic algebraic geometry, stable homotopy theory, and manifold topology, there is immense interest in computing these groups. This is a notoriously difficult task, and in fact very few higher K-groups can be explicitly computed. Crucial tools for this are the related cohomology theories of Hochschild homology [17] via the Dennis trace (R. K. Dennis, unpublished) and cyclic homology.

In the mid 1980s, Waldhausen [28] generalized algebraic K-theory further to work in the topological setting of categories with cofibrations and weak equivalences, nowadays called Waldhausen categories. This, together with the development of new foundations of stable homotopy theory in the 1990s [15, 19], allowed for an extension of K-theory to ring spectra, along with corresponding extensions of Hochschild homology and cyclic homology [11]. Waldhausen conjectured [27] that analogously to the chromatic convergence theorem in stable homotopy theory by Hopkins et. al. [18, 23], the tower of K-theory spectra of the chromatic tower of the sphere converged to the sphere. This was proved in a special case by McClure and Staffeldt [20] but remains an important open problem.

It was soon observed that applying the K-theory functor to a ring spectrum somehow always produces a more complicated ring spectrum. This was made precise in Rognes's work on K-theory and in particular his famous redshift conjecture [3] and supported by numerous computations, e.g. [2, 1, 4]. This, together with Waldhausen's chromatic convergence conjecture, would imply that stable homotopy category can be faithfully approximated by iterated application of the K-theory functor.

A central theorem of K-theory goes by the name of *dévissage*. The following version was proved by Quillen [22]:

Theorem. Let C be an exact subcategory of an abelian category D, closed under subobjects and quotients. If every object of D has a finite filtration with filtration quotients in C, then $KC \simeq KD$.

This theorem is remarkable in that it allows the "untwisting" of extensions when computing K-theory. It is a long-standing open problem to find an analogous theorem for Waldhausen K-theory. A partial result was obtained by Blumberg and Mandell in [10], where they show that for connective associative ring spectra R, the K-theory of R-module spectra with finitely many, finitely generated homotopy groups is isomorphic to the K-theory of finitely generated modules over the coefficient ring $\pi_0 R$ of R.

The goal of the proposed project is to prove a generalized dévissage theorem in algebraic K-theory (and related theories) with methods that have not been previously applied to this question, and in which the applicants are experts. The concrete approach is outlined in the following section. With such a theorem at hand, one would have new attack vectors on Rognes's redshift and localization conjectures as well as Waldhausen's chromatic convergence conjecture.

3 The candidate and the planned project

As outlined in broad strokes in the previous section, the goal of the proposed project is to prove a generalized dévissage theorem in Waldhausen K-theory. The approach we want to take is based on the twisting construction for augmented functors:

Definition. A coaugmented functor on a topological category $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathcal{C}$ together with a natural transformation $\eta\colon \mathrm{id} \to F$. Given two coaugmented functors F, G, a new augmented functor $[F, G]$ is defined as the homotopy pullback

$$\begin{tikzcd} {[F,G](X)} \arrow[r] \arrow[d] & F(X) \arrow[d, "F(\eta_G)"] \\ G(X) \arrow[r, "\eta_{G(X)}"] & F(G(X)) \end{tikzcd}$$

This construction was first (implicitly) studied by Dror Farjoun [16] in the specific context where F is "homological", i. e. of the form $\Omega^\infty(E \wedge -)$ for some spectrum E, and the general construction was studied by Chachólski (unpublished) and (again, implicitly) applied for singular homology to cellularization questions in [14] where it is used to prove and generalize the so called Bousfield key lemma [12]. This construction, in a more algebraic setting, was also essential in [9] for understanding idempotent deformations of finite simple groups.

A striking property of the bracket construction is that for singular homology, the *injective objects* of $F^n = [[\dots[F, F], \dots, F]$ are precisely the n-stage Postnikov systems, or n-polyGEMs. These are precisely the kinds of spaces for which Blumberg and Mandell proved a dévissage theorem [10], begging the question whether this result holds more generally. The main objective of the proposed project is to attack the following questions:

Question. Let F be a coaugmented functor on some Waldhausen category $\mathcal{C}$. Assume the existence of reasonable Waldhausen categories $\mathcal{C}_{F^\infty}$ of F^n-injectives, for all finite n, and $\mathcal{C}_{F^n}$ of F^n-injectives, for a fixed n.

Under which conditions is there a dévissage isomorphism relating $K(\mathcal{C}_{F^\infty})$ and $K(\mathcal{C}_{F^n})$ with $K(C_{F^1})$?

While these questions can be asked for arbitrary coaugmented functors, experience (for instance with the Bousfield-Kan completion tower, [13]) tells us that it may be necessary to take more structure into account by assuming that the functors in question are actually monads, i. e. come with an associative multiplication $F(F(X)) \to F(X)$ for which the coaugmentation acts as a unit. Bauer and Libman [7, 6] have shown that completion with respect to a monad is again a monad up to higher homotopies, a so-called A_∞-monad. This makes it plausible to conjecture that the bracket construction will also be an A_∞-monad if the input functors F and G are, at least when some compatibility between F and G is given. This question is of interest beyond the current project as it would imply, for instance, that the category of polyGEMs is monadic, which has strong consequences for its structure.

Wojciech Chachólski and Tilman Bauer have great expertise in homotopy theory, in particular in questions about cellularity, operads, completions and pro-homotopy theory, which is crucial for this project. To make it succeed, we would like to hire a postdoctoral researcher who has expertise in homotopy theory and complements us with a strong background, and research track record, in algebraic K-theory. We do not have a particular candidate in mind, but here is a short list of recent Ph.D.s who would be exceptionally well-suited for this task, and who we would encourage to apply:

- Cary Malkiewich, postdoc at the University of Illinois at Urbana-Champaign, USA; student of Fred Cohen
- Marc Lange, Ph.D. student of Birgit Richter, Hamburg, Germany
- Eva Höning, Ph.D. student of Christian Ausoni, Paris XIII, France
- Kristian Jonsson Moi, postdoc at the University of Copenhagen; student of Ib Madsen

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