THE REALIZABILITY OF LOCAL LOOP SPACES AS MANIFOLDS

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Abstract. As an extension of earlier work, we show that every P-local loop space, where P is a set of primes, is homotopy equivalent to the P-localization of a compact, smooth, parallelizable manifold. A similar result is also proved for P-complete loop spaces.

1. Introduction

In [BKNP04], it was shown that every quasifinite loop space is homotopy equivalent to a parallelizable, compact, smooth manifold. In this paper, we prove a stronger local version of this result.

Recall from [Bou75] that for any homology theory E, a space X is called E-*local* if for every E_*-isomorphism $f\colon Y_1 \to Y_2$ the induced map of homotopy classes $[Y_2, X] \to [Y_1, X]$ is a bijection. There is a *localization functor* $L_E\colon \mathrm{Top} \to \mathrm{Top}$ with a natural transformation $\eta\colon X \to L_E X$ which is both the terminal E_*-isomorphism out of X and the initial morphism into an E-local space.

In this paper, we will consider the case of $E_*(X) = H_*(X; R)$ for a commutative ring R. For $R = \mathbf{Q}$, we call the localization functor *rationalization* and abbreviate it by $L_{\mathbf{Q}} = L_{H_*(-;\mathbf{Q})}$. For a set of primes P, we call the localization functor $L_{H_*(-;\mathbf{Z}_{(P)})}$ *P-localization* and abbreviate if by $L_{(P)}$; similarly, we call $L_{H_*(-;\mathbf{Z}/P)}$ *P-completion* and abbreviate it by L_P, where by convention, $\mathbf{Z}/P = \prod_{p\in P} \mathbf{Z}/p$. Thus a nilpotent space X is P-complete if and only if $X \simeq \prod_{p\in P} L_p X$.

We call a space X *R-finite* if $H_*(X; R)$ is totally finitely generated as an R-module.

A *loop structure* on a space X is an equivalence class of pairs (Y, e), where $e\colon X \to \Omega Y$ is a homotopy equivalence. Two loop structures (Y, e), (Y', e') are identified if there is a map $\phi\colon Y \to Y'$ such that $e' \simeq (\Omega\phi) \circ e$. A *$p$-compact group* [DW94] is a $\mathbf{Z}/p$-finite, p-complete space X with a loop structure (BX, e) such that BX is also p-complete.

We can now formulate the main result of this paper, which can be cast in a local and in a complete setting:

Theorem 1.1. *Let P be a collection of primes, and let X be a P-local, $\mathbf{Z}_{(P)}$-finite loop space. Then X is homotopy equivalent to the P-localization of a compact, smooth, parallelizable manifold.*

Theorem 1.2. *Let P be a collection of primes, and let X be a P-complete loop space. Assume that there exist integers $(d_1, \ldots, d_r)$ such that for all primes $p \in P$,*

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$H^*(X;\mathbf{Z}_p)\otimes\mathbf{Q}$ is an exterior algebra over $\mathbf{Q}_p$ with generators in degrees d_i. Then X is homotopy equivalent to the P-completion of a compact, smooth, parallelizable manifold.

Setting $P=$ the set of all primes, we recover the main result of [BKNP04] from Thm. 1.1: every quasifinite (i. e. $\mathbf{Z}$-finite) loop space is homotopy equivalent to a compact, smooth, parallelizable manifold.

On the other extreme, we have as a special case of Thm. 1.2:

Corollary 1.3. *Every p-compact group is the p-completion of a compact, smooth, parallelizable manifold.*

A natural question to ask is whether these theorems can be reduced to the global setting of [BKNP04] by showing that every space satisfying the conditions can be represented as the P-localization (resp. P-completion) of a quasifinite loop space. This is in fact not the case, as seen by the following theorem [Cla63].

Theorem 1.4 (Clark). *Every nontrivial connected quasifinite loop space X with finite fundamental group has $H^3(X;\mathbf{Q})\neq 0$.*

On the other hand, the structure theory of p-compact groups initiated in [DW94] exhibits a plethora of p-compact groups with trivial third cohomology.

2. Loop structures on local spaces

In this section we assemble some well-known results about the interaction of loop structures with localization.

Conventions 2.1. Let P be a set of primes. We define $\mathbf{Z}_{(P)}$ to be the ring of integers localized at P and $\mathbf{Z}_P$ to be the product of p-adic number rings $\prod_{p\in P}\mathbf{Z}_p$. If X is a $\mathbf{Z}/P$-local nilpotent space then its singular chain complex is a complex of $\mathbf{Z}_P$-modules; in particular, it carries a natural topology. If A is a topological abelian group (e.g., $\mathbf{Z}_P$ or $\mathbf{Q}_p$, the p-adic rationals), we denote by $H_*(X;A)$ the homology of the chain complex $C_*(X)\hat{\otimes}A$ and by $H^*(X;A)$ the cohomology of the cochain complex $\mathrm{Hom}^{\mathrm{cont.}}(C_*(X);A)$. We abbreviate $H_*(X)=H_*(X;\mathbf{Z}_P)$ and $H^*(X)=H^*(X;\mathbf{Z}_P)$.

Definition 2.2. If X is a P-local, $\mathbf{Z}_{(P)}$-finite loop space, $H^*(X;\mathbf{Q})$ is a finite dimensional Hopf algebra over $\mathbf{Q}$ and thus isomorphic to an exterior algebra over $\mathbf{Q}$ on generators in odd dimensions $(2d_1-1,\dots,2d_r-1)$. We call these numbers d_i the *degrees* of X, r the *rank* of X and $\min\{d_i\}$ the *level* of X. Similarly, if X is a p-compact group, $H^*(X;\mathbf{Q}_p)$ is a finite dimensional Hopf algebra over $\mathbf{Q}_p$, and we have the corresponding notions of degrees, rank, and level as well. However, if P is a set of more than one prime, and X is a P-complete $\mathbf{Z}/P$-finite loop space, these notions are not well-defined since $\mathbf{Z}_P\otimes\mathbf{Q}$ is not a field. In fact, $X\simeq\prod_{p\in P}L_pX$, and the p-compact groups L_pX are completely independent. We say that X has *uniform degrees* $(d_1,\dots,d_r)$ if L_pX has these degrees for each $p\in P$.

Theorem 2.3 (Serre [Ser53], Kumpel [Kum65], Wilkerson [Wil73], [ABGP04]). *Let R be either $\mathbf{Z}_{(P)}$ or $\mathbf{Z}/P$ for a set of primes P, and let X be a connected, R-local, R-finite loop space with uniform degrees $d_1\leq\cdots\leq d_r$. Then for each $p>d_r$,*

$$L_pX\simeq\left(\prod_{i=1}^{r}L_p\mathbf{S}^{2d_i-1}\right).$$

The interesting primes here are only the $p \in P$, otherwise the statement of the theorem is vacuous.

This decomposition does of course not respect the loop structure (although the factor spheres may be p-compact groups). The rational situation is simpler:

DEFINITION 2.4. For a set of primes P, the *ring of P-local finite adeles* $\mathbf{A}_P$ is defined as

$$\mathbf{A}_P = \mathbf{Z}_P \otimes \mathbf{Q}$$

(If P is finite, this is of course just $\prod_{p\in P} \mathbf{Q}_p$.)

LEMMA 2.5. *Let X be either a connected rational $\mathbf{Q}$-finite loop space, or, for a set of primes P, the rationalization of a connected, $\mathbf{Z}/P$-local loop space with uniform degrees which is p-finite for every $p \in P$.*

1. *As a space, X is homotopy equivalent to a product of rationalized (resp. rationalized $\mathbf{Z}/P$-localized) spheres.*
2. *For any loop structure $\xrightarrow{\simeq} \Omega Y$, Y is homotopy equivalent to a product of $K(\mathbf{Q}, 2d_i)$ (resp. $K(\mathbf{A}_P, 2d_i)$). In particular, every loop structure is abelian.*
3. *The functor*

$$\left\{\begin{array}{l}\textit{loop spaces } X \textit{ of}\\ \textit{the given form}\end{array}\right\} \xrightarrow{H_*} \left\{\begin{array}{l}\textit{connected, finite dimensional, primitively gen-}\\ \textit{erated, graded commutative } \mathbf{Q}\text{– } (\mathbf{A}_P\text{–}) \textit{ Hopf}\\ \textit{algebras whose underlying modules are free}\end{array}\right\}$$

is an equivalence.

Proof. The assertion (1) for $\mathbf{Q}$-finite loop spaces, or when P consists of a single prime, is classical and can be found in [Sch85].

We show that (1) implies (2). Let $X \xrightarrow{\simeq} \Omega Y$ be a loop structure. Then since X is an Eilenberg-Mac Lane space on some graded vector space which is trivial in even dimensions, Y has nontrivial homotopy groups only in even dimensions and must thus be a product of Eilenberg-Mac Lane spaces as well, as can be seen from the Postnikov tower. For (3), (2) implies that the homology functor actually takes values in graded commutative algebras. The inverse functor is given by $A \to K(P(A))$, where $P(A)$ is the graded vector space of primitive elements, and K is the Eilenberg-Mac Lane space functor.

It thus remains to show that (1) holds for infinite sets of primes P. Let $\{d_i\}$ be the uniform degrees of X. By Theorem 2.3, we have for $h = \max_i\{d_i\}$:

$$\begin{aligned} L_{\mathbf{Q}}X &\simeq \prod_{p\le h} L_{\mathbf{Q}}X_p \times L_{\mathbf{Q}}\Big(\prod_{p>h}\prod_{i=1}^{r} L_p\mathbf{S}^{2d_i-1}\Big) \\ &\underset{\text{Lemma 2.5}}{\simeq} \prod_{p\le h} L_{\mathbf{Q}}L_p\prod_{i=1}^{r}\mathbf{S}^{2d_i-1} \times \prod_{i=1}^{r} L_{\mathbf{Q}}\prod_{p>h} L_p\mathbf{S}^{2d_i-1} \\ &\simeq \prod_{i=1}^{r} L_{\mathbf{Q}}L_P\mathbf{S}^{2d_i-1} \simeq \prod_{i=1}^{r} K(\mathbf{A}_P, 2d_i-1) \end{aligned} \qquad \square$$

In particular, the equivalence classes of loop structures on a connected, rational, $\mathbf{Q}$-finite loop space X are in one-to-one correspondence with the choices of Hopf algebra structures extending the cohomology algebra $H^*(X)$, or equivalently, to the choice of a sub-vector space $P < H^*(X)$ of primitive elements which generate the algebra.

3. Lifts of loop structures

Lemma 3.1. *Let X be a connected, P-complete loop space with a finite set of uniform degrees (Def. 2.2). Let $BY \to BX$ be a $H_*(-;\mathbf{Z}/P)$-equivalence out of a P-local, $\mathbf{Q}$-finite loop space Y. Then Y is $\mathbf{Z}_{(P)}$-finite.*

Proof. Consider the pullback diagram

$$\begin{tikzcd} BY \arrow[r, "\eta_P"] \arrow[d, "\eta_0"] & L_P BY \arrow[r, "\simeq"] \arrow[d, "\eta_0"] & BX \arrow[d, "\eta_0"] \\ L_{\mathbf{Q}} BY \arrow[r, "L_{\mathbf{Q}}\eta_P"] & L_{\mathbf{Q}} L_P BY \arrow[r, "\simeq"] & L_{\mathbf{Q}} BX. \end{tikzcd}$$

The associated Mayer-Vietoris sequence for homotopy groups splits because A_P is generated by $\mathbf{Q}$ and $\mathbf{Z}_P$. Thus we have

$$\pi_n(BY) \cong \pi_n(L_{\mathbf{Q}}BY) \times_{\pi_n(L_{\mathbf{Q}}BX)} \pi_n(BX)$$

Since Y is $\mathbf{Q}$-finite, $L_{\mathbf{Q}}BY \simeq \prod_{i=1}^r K(\mathbf{Q}, 2d_i - 1)$. These are also the degrees of X because by assumption, $L_{\mathbf{Q}}BY \to L_{\mathbf{Q}}BX$ is a $H_*(-;\mathbf{Q}_P)$-equivalence. Thus

$$\pi_n(BY) \cong \mathbf{Z}_{(P)}^{\#\{i \mid 2d_i = n\}} \oplus \bigoplus_{p \in P} \operatorname{Tor}(\mathbf{Z}/p^\infty, \pi_n(L_p BY)).$$

In particular, $\pi_n(Y)$ is a finitely generated $\mathbf{Z}_{(P)}$-module, and thus so is $H^n(Y;\mathbf{Z}_{(P)})$.

By Theorem 2.3, there exists a d such that $H^i(Y;\mathbf{Z}_p) = H^i(X;\mathbf{Z}_p) = 0$ for all $p \in P$ and $i \geq d$. Since $H^i(Y;\mathbf{Z}_p) \cong H^i(Y;\mathbf{Z}_{(p)}) \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p$ and $H^i(Y;\mathbf{Z}_{(p)})$ is finitely generated, it follows that $H^i(Y;\mathbf{Z}_{(p)}) = 0$ for those i and all $p \in P$. Thus $H^i(Y;\mathbf{Z}_{(P)}) = 0$ for those i as well. □

Theorem 1.2 reduces to Theorem 1.1 by the implication (2)⇒(3) in the following essentially well-known lemma:

Lemma 3.2. *Let X be a connected, P-complete loop space with finite mod-p homology for every $p \in P$. Then the following are equivalent:*

1. *X is the P-completion of a $\mathbf{Z}$-finite CW complex;*
2. *There exist positive integers $\{d_1, \dots, d_r\}$ such that X has these as uniform degrees;*
3. *X is the P-completion of a $\mathbf{Z}_{(P)}$-finite, P-local loop space.*

Proof. (1)⇒(2): Let $X = L_P F$ for a finite CW-complex F.

$$H^*(L_pX;\mathbf{Z}_p) \otimes \mathbf{Q} \cong H^*(F;\mathbf{Z}_p) \otimes \mathbf{Q} \cong H^*(F;\mathbf{Q}) \otimes \mathbf{Q}_p.$$

This shows that $H^*(L_pX;\mathbf{Z}_p) \otimes \mathbf{Q}$ is an exterior algebra, and the degrees are independent of p.

(2)⇒(3): By Lemma (2.5),

$$L_{\mathbf{Q}}BX \simeq \prod_{i=1}^r K(\mathbf{A}_P, 2d_i).$$

Define

$$BK = \prod_{i=1}^{r} K(\mathbf{Q}, 2d_i),$$

and let $BK \to L_{\mathbf{Q}}BX$ be the product of the maps induced by the unit ring map $\mathbf{Q} \to A_P$. Let BF be the homotopy pullback of $BK \to L_{\mathbf{Q}}BX \leftarrow BX$. By construction, $F = \Omega BF$ is $\mathbf{Q}$-finite and $\mathbf{Z}_{(P)}$-local, thus Lemma 3.1 implies that F is $\mathbf{Z}_{(P)}$-finite.

(3)⇒(1): Let F_P be such a P-local, $\mathbf{Z}_{(P)}$-finite space, and let Q be the complementary set of primes. Choose F_Q to be a Q-local, $\mathbf{Z}_{(Q)}$-finite space such that $L_{\mathbf{Q}}F_Q \cong L_{\mathbf{Q}}F_P$, and let F be the pullback of $F_Q \to L_{\mathbf{Q}}F_P \leftarrow F_P$. Since F is $\mathbf{Z}_{(P)}$-finite and $\mathbf{Z}_{(Q)}$-finite, it is $\mathbf{Z}$-finite. □

Note that the previous lemma does not claim that given a $\mathbf{Z}_{(P)}$-finite P-local space Y and a loop structure on L_PY, there is a loop structure on Y such that the completion map is a loop map. This is in fact false: we thank the referee for pointing out the mistake in a proof of this claim in an earlier version of the present paper.

4. Double 1-tori

In [BKNP04], the pivotal tool for studying the finiteness obstruction and the surgery obstruction for a loop space X as well as for mixing homotopy types is

Definition 4.1. Let R be a ring and X be an R-finite, R-local, nilpotent, connected space. A 1-*torus* in X is a fibration of nilpotent, R-local spaces $L_R\mathbf{S}^1 \to X \xrightarrow{p} Y$ such that

(1) $X \to Y$ is orientable;
(2) $\pi_1 p$ is an isomorphism; and
(3) Y is R-finite and *stably reducible*, i. e. there is a map $\mathbf{S}^d \to Y$ which is an $H_*(-;R)$-isomorphism in the top degree.

A *double* 1-*torus* in X is a 1-torus $X \to Y$ together with a fibration of nilpotent, R-local spaces $Y \xrightarrow{j} Z \to L_RB\mathbf{Z}/2$ such that j is an orientable 2-sheeted covering such that $\pi_1(j)$ has a retraction.

Such an object was called rationally splitting in [BKNP04] if it rationally splits off a Hopf fibration. This condition needs to be relaxed for the present purposes.

Definition 4.2. A 1-torus $L_R\mathbf{S}^1 \to X \xrightarrow{p} Y$ is called *rationally splitting of level* l if p rationally has a retract of the form $L_{\mathbf{Q}}L_R\mathbf{S}^1 \to L_{\mathbf{Q}}L_R\mathbf{S}^{2l+1} \to L_{\mathbf{Q}}L_R\mathbf{C}P^l$.

If $g \in H^{2l+1}(X)\otimes\mathbf{Q}$ is a given nonzero class, a rational splitting is called *adapted to* g if there is a commutative diagram

$$\begin{tikzcd}
L_R\mathbf{S}^1 \arrow[r, "\eta"] \arrow[d] & L_{\mathbf{Q}}L_R\mathbf{S}^1 \arrow[d] \\
X \arrow[r, "g"] \arrow[d, "p"] & L_{\mathbf{Q}}L_R\mathbf{S}^{2l+1} \arrow[d] \\
Y \arrow[r, "\overline{g}"] & L_{\mathbf{Q}}L_R\mathbf{C}P^l,
\end{tikzcd}$$

where the right hand column is the standard fibration, g is the map representing the cohomology class of the same name, and all horizontal maps are supposed to have a splitting rationally.

Thus, a rational splitting in the sense of [BKNP04] is a rational splitting of level 1.

We will only consider rational splittings whose level equals the level of the loop space under consideration (Def. 2.2).

The reason for introducing these concepts is that they guarantee the vanishing of all surgery obstructions:

THEOREM 4.3 ([BKNP04, Prop. 3.3]). *If X is a Poincaré duality space of dimension at least 5 admitting a double 1-torus, then X is homotopy equivalent to a stably parallelizable, smooth, closed manifold.*

PROPOSITION 4.4. *Given a sequence of positive integers $(n_0, n_1, \ldots, n_k)$ where $k \geq 1$. Then there exists a bundle of manifolds $S^1 \to M \to N$ satisfying the following conditions:*

(1) *N is stably parallelizable*
(2) *M is parallelizable*
(3) *Rationally, the bundle is a product of the standard bundle*
$$L_{\mathbf{Q}}\mathbf{S}^1 \to L_{\mathbf{Q}}\mathbf{S}^{2n_0+1} \to L_{\mathbf{Q}}\mathbf{C}P^{n_0}$$
and trivial bundles $* \to L_{\mathbf{Q}}\mathbf{S}^{2n_i+1} \xrightarrow{=} L_{\mathbf{Q}}\mathbf{S}^{2n_i+1}$ for $1 \leq i \leq k$.
(4) *There exists an $N_0 \in \mathbf{N}$, depending only on the integers n_i, such that*
$$L_p M \simeq \prod_{i=0}^{k} L_p \mathbf{S}^{2n_i+1} \quad \text{for all } p \geq N_0.$$

Proof. We first consider the manifold $X = \mathbf{C}P^{n_0} \times \prod_{i=1}^{k} \mathbf{S}^{2n_i+1}$. This could be a candidate for N, but it is not, in general, stably parallelizable. We thus apply surgery to alter it without changing the rational type.

As for any Poincaré duality space, the Spivak normal fibration of X is rationally trivial, so there is a map $\mathbf{S}^m \to \Sigma^d X$ of nonzero degree for sufficiently large d. Making this transverse to X establishes a rational surgery problem

$$\begin{tikzcd} \nu_N \arrow[r] \arrow[d] & X \times \mathbf{R}^d \arrow[d] \\ N \arrow[r, "f"] & X \end{tikzcd}$$

with f of some nonzero degree on the top homology. Since X is an integral Poincaré duality space, the only possible surgery obstruction to turning f into a rational homotopy equivalence occurs when $m = \dim X$ is divisible by 4 [TW79]. This is the signature obstruction, the difference of the signatures of X and of N. However, the signature of X is zero since $k \geq 1$, and Hirzebruch's signature theorem implies that the signature of N is also zero since its normal bundle is trivial (condition (1)). Thus we may assume that N is a stably parallelizable manifold rationally equivalent to X.

Since $H^2(N, \mathbf{Q}) = \mathbf{Q}$, there is a nonzero integral class represented by a map $N \to B\mathbf{S}^1$, classifying a bundle $S^1 \to M \to N$. The Serre spectral sequence for this fibration immediately shows that requirement (3) is satisfied. To see (2), that M is parallelizable, we proceed as in [BKNP04], using the criterion of [Dup70, Sut76]. If the dimension of M is even, the parallelizability of a stably parallelizable manifold is determined by the vanishing of the Euler characteristic of M, which is clearly 0 in this case. If the dimension is 1, 3, or 7, M is automatically parallelizable; for all other odd dimensions, M is parallelizable if and only if its *Kervaire semi-characteristic* $\kappa(M)$ vanishes. It is defined as

$$\kappa(M) = \frac{1}{2}\left(\sum_{i\geq 0} \dim H_i(M;\mathbf{F}_2)\right) \pmod 2.$$

In this case $\dim N = m = 2l$ is even, and an easy rational computation shows that the Euler characteristic of N is 0. Thus the total dimension of $H_*(N;\mathbf{F}_2)$ is even. We may assume the Serre spectral sequence of $S^1 \to M \to N$ with $\mathbf{F}_2$-coefficients collapses, since we otherwise could have composed $N \to B\mathbf{S}^1$ with the degree two map $B\mathbf{S}^1 \to B\mathbf{S}^1$ without affecting the rational types. Thus

$$\dim_{\mathbf{F}_2} H_*(M;\mathbf{F}_2) = \left(\dim_{\mathbf{F}_2} H_*(\mathbf{S}^1;\mathbf{F}_2)\right)\left(\dim_{\mathbf{F}_2} H_*(N;\mathbf{F}_2)\right) = 2\dim_{\mathbf{F}_2} H_*(N;\mathbf{F}_2)$$

is divisible by four, and hence the Kervaire semi-characteristic is zero.

To prove (4), let $\{e_i \mid 0 \leq i \leq k\}$ denote a basis of the indecomposables of $H^*(M;\mathbf{Z})$ modulo torsion and consider the lifting problem

$$\begin{tikzcd}
 & \prod_{i=0}^{k} \mathbf{S}^{2n_i+1} \arrow[d] \\
M \arrow[ur, dotted] \arrow[r, "\prod e_i"'] & \prod_{i=0}^{k} K(\mathbf{Z}, 2n_i+1)
\end{tikzcd}$$

where the vertical map is the Hurewicz map, whose fiber will be denoted by F. The obstructions to this lifting problem lie in $H^{i+1}(M;\pi_i(F))$. The homotopy groups $\pi_i(F)$ are all finite since the dimensions $2n_i+1$ are odd. If

$$N_0 = l.c.m.\{\#\pi_i(F) \mid 1 \leq i \leq \dim M\}$$

then a lift exists after inverting N_0. Being parallelizable, M is in particular orientable and hence $\dim M = n_0 n_1 \cdots n_k$; thus N_0 only depends on the integers n_i. The lift is an $H\mathbf{F}_p$-isomorphism for $p > N_0$, and thus induces an equivalence $L_p M \to \prod L_p \mathbf{S}^{2n_i+1}$. □

Proposition 4.5. *Let G be a p-compact group of level l with finite fundamental group, and let $g \in H^{2l+1}(G;\mathbf{Q}_p)$ be a nonzero class. Assume*

($\star$) $\quad$ *$p \neq 2$ or $G/Z(G) \not\simeq L_2\,\mathrm{SO}(3)^l \times L_2\,\mathrm{SO}(5)^\epsilon$ for $l \geq 0$ and $\epsilon = 0,\ 1$.*

Then there exists an $n \geq 0$ such that for all $\gamma \in \mathbf{Z}_p^\times$, G possesses a double 1-torus adapted to $\gamma p^n g$.

Proof. The proof proceeds along the lines of [BKNP04, Prop. 5.3]. Let

$$T = T' \times T'' < \tilde{G}$$

be a maximal torus in the universal cover of G such that T'' is the smallest subtorus containing the p-compact center $Z(\tilde{G})$. By [BKNP04, Lemma 5.2], the rank r' of

T' is at least 1. Denote a basis for $H^1(T;\mathbf{Z}_p)$ by $\{t_1,\dots,t_r\}$ such that $\{t_1,\dots,t_{r'}\}$ is a basis of $H^1(T';\mathbf{Z}_p)$ and $\{t_{r'+1},\dots,t_r\}$ is a basis of $H^1(T'';\mathbf{Z}_p)$.

Now let $f\in\mathbf{Q}_p[t_1,\dots,t_r]=H^*(BT;\mathbf{Q}_p)$ be the image of g under the composition

$$H^{2l+1}(G;\mathbf{Q}_p)\xrightarrow{\sim}H^{2l+2}(BG;\mathbf{Q}_p)\to H^{2l+2}(BT;\mathbf{Q}_p).$$

If $\alpha=(\alpha_1,\dots,\alpha_r)\in\mathbf{Z}_p^r$ then α defines a 1-dimensional free submodule in $H^1(T;\mathbf{Z}_p)$ by using the basis $\{t_i\}$ and thus a torus $BS_\alpha\to BT$. This map is an inclusion into T (and thus into $\tilde G$) if and only if the α_i are relatively prime; they form a sub-circle group of G if also the $\{\alpha_i\mid 1\le i\le r'\}$ are relatively prime. We can always arrange this by choosing $\alpha_1=1$.

We now want to choose α in such a way that it forms a 1-torus in G adapted to $\gamma p^n g$. For the 1-torus to be splitting of level l, it is necessary and sufficient that $f(\alpha)\neq 0$. Since f is homogeneous of degree $l+1$, $\tilde f=f(1,\alpha_2,\dots,\alpha_r)$ is a nonzero polynomial and thus $\alpha_2,\dots,\alpha_r$ can be chosen such that $(1,\alpha_2,\dots,\alpha_r)$ is not a zero of f.

We have thus constructed a splitting 1-torus in G, but it is not yet adapted. To study this situation, consider the lifting problem

$$\begin{tikzcd}
G/S_\alpha \arrow[d] & L_{\mathbf{Q}}L_p\mathbf{C}P^l \arrow[d] \\
BS_\alpha \arrow[d] \arrow[r, dotted] & K(\mathbf{Q}_p,2) \arrow[d] \\
BG \arrow[r, "\gamma p^n g"] & K(\mathbf{Q}_p,2l+2)
\end{tikzcd}$$

The lift exists if and only if $\gamma p^n f(\alpha)$ is an $(l+1)$st power. In this case we have found a splitting 1-torus adapted to $\gamma p^n g$ which is even a sub-p-compact group. Choose n such that the sum of the p-adic valuation of $f(\alpha)$ and n is a multiple of $l+1$. We thus can write $\gamma p^n f(\alpha)=y^{l+1}k$, where k is a p-adic unit and y can be taken to be a power of p. The kth power map $G\to G$ (which is not a p-compact group homomorphism) is a homotopy equivalence because k is a p-adic unit; thus we can construct a diagram

$$\begin{tikzcd}
G \arrow[r, "\gamma p^n g"] \arrow[d, "k"] & K(\mathbf{Q}_p,2l+1) \arrow[dd] \\
G \arrow[d] & \\
G/S \arrow[r] & L_{\mathbf{Q}}L_p\mathbf{C}P^{2l}
\end{tikzcd}$$

which yields a 1-torus adapted to $\gamma p^n g$. $\square$

Proposition 4.6. *Let $P=P_1\cup P_2$ be a partition of a set of primes, and let G be a P-local space with finite fundamental group such that $G_{(P_1)}$ is a P_1-local finite loop space of level l and rank bigger than one, and $G_{(P_2)}$ is the P_2-localization of the total space of a bundle $\mathbf{S}^1\to M\to N$ satisfying the properties of Prop. 4.4 and* *such that $L_{\mathbf{Q}}M \simeq L_{\mathbf{Q}}G$. Assume that for each $p \in P_1$, L_pG satisfies $(\star)$. Then G admits a rationally splitting double 1-torus of level l.*

Remark. The 1-torus $\mathbf{S}^1 \to G$ constructed here is in general not a map of loop spaces. It would be interesting to know whether it can always be arranged to be one.

Proof. By Theorem 2.3 (for $p \in P_1$) and Prop. 4.4(4) (for $p \in P_2$), there exists an $N \gg 0$ such that

$$L_pG \simeq \prod_{i=1}^{r} L_p\mathbf{S}^{2d_i-1} \quad \text{as spaces, for all } p \geq N,\ p \in P.$$

Thus L_pG is homotopy equivalent to L_pM for all but finitely many $p \in P_1$. Hence we may assume without loss of generality that P_1 is finite.

Let $g\colon G \to L_{(P)}\mathbf{S}^{2l+1}$ be a map inducing a splitting for large $p \in P$. Rationalizing, we obtain a class $g \in H^{2l+1}(L_pG; \mathbf{Q}_p)$ for every $p \in P_1$. Proposition 4.5 yields splitting 1-tori $L_p\mathbf{S}^1 \to L_pG \to Y_p$ adapted to $\gamma p^{n_p} g$ for every $\gamma \in \mathbf{Z}_p^\times$. If we set $N = \prod_{p\in P_1} p^{n_p}$, these 1-tori are all adapted to Ng since $\frac{N}{p^{n_p}} \in \mathbf{Z}_p^\times$. Replacing g by Ng, we may therefore assume that all 1-tori are adapted to g.

As in the proof of [BKNP04, Prop. 4.3], we can construct a map h extending this to diagrams

$$\begin{tikzcd}
L_pG \arrow[r] \arrow[d] & L_{\mathbf{Q}}L_pG \arrow[r, "\sim", "L_{\mathbf{Q}}L_p(g\times h)"'] \arrow[d] & L_{\mathbf{Q}}L_p\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}L_pK \arrow[d] \\
Y_p \arrow[r] & L_{\mathbf{Q}}Y_p \arrow[r, "\sim", "\overline{g}_p\times\overline{h}_p"'] & L_{\mathbf{Q}}L_p\mathbf{C}P^l \times L_{\mathbf{Q}}L_pK.
\end{tikzcd}$$

for $p \in P_1$. For $p \in P_2$, the same diagram exists if we set $Y_p = L_pN$; by Prop. 4.4, a splitting exists and it is adapted to g by construction.

This whole diagram can be lifted to the $L_{\mathbf{Q}}L_P$-local category; the top row map $L_{\mathbf{Q}}L_p(g \times h)$ lifts by definition to $L_{\mathbf{Q}}L_PG \xrightarrow{\sim} L_{\mathbf{Q}}L_P\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}L_PK$; likewise, for $p \in P_2$, $\overline{g}_p \times \overline{h}_p$ is the image of a rationally defined map. This together with the fact that P_1 is finite implies that the diagram above can be lifted to a diagram

$$\begin{tikzcd}
L_PG \arrow[r] \arrow[d] & L_{\mathbf{Q}}L_PG \arrow[r, "\sim"] \arrow[d] & L_{\mathbf{Q}}L_P\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}L_PK \arrow[d] \\
Y_P \arrow[r] & L_{\mathbf{Q}}Y_P \arrow[r, "\sim"] & L_{\mathbf{Q}}L_P\mathbf{C}P^l \times L_{\mathbf{Q}}L_PK.
\end{tikzcd}$$

where $Y_P = \prod_{p\in P} Y_p$.

We now address the problem of lifting this to G.

Consider the diagram

$$\begin{tikzcd}
G \arrow[rr] \arrow[dd] & & L_{\mathbf{Q}}G \arrow[rr, "\sim"] \arrow[dd] \arrow[dr, dotted] & & L_{\mathbf{Q}}\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}K \arrow[dr] \arrow[dd] & \\
& & & Y_0 \arrow[rr, "\sim"] \arrow[dd] & & L_{\mathbf{Q}}\mathbf{C}P^l \times L_{\mathbf{Q}}K \arrow[dd] \\
L_PG \arrow[rr] \arrow[dr] & & L_{\mathbf{Q}}L_PG \arrow[rr, "\sim"] \arrow[dr] & & L_{\mathbf{Q}}L_P\mathbf{S}^{2l+1} \times L_{\mathbf{Q}}L_PK \arrow[dr] & \\
& Y_P \arrow[rr] & & L_{\mathbf{Q}}(Y_P) \arrow[rr, "\sim"] & & L_{\mathbf{Q}}L_P\mathbf{C}P^l \times L_{\mathbf{Q}}L_PK
\end{tikzcd}$$

where the bottom two rows are the diagram constructed before, and Y_0 is defined as the pullback. Hence we can also fill in a map $L_{\mathbf{Q}}G \to Y_0$. Let $\mathbf{S}^1 \to G \to Y$ be the pullback of this diagram. The $\mathbf{Z}_{(P)}$-finiteness of Y follows from Lemma 3.1. Therefore this constitutes a rationally splitting 1-torus of level l. ☐

Corollary 4.7. *Let P be any set of primes, and let G be an n-dimensional P-local, $\mathbf{Z}_{(P)}$-finite loop space of level l and rank bigger than one satisfying $(\star)$. Then there exists a finite n-dimensional CW-complex X whose P-localization (resp. P-completion) is homotopy equivalent to G and such that X admits a rationally splitting double* 1*-torus of level l.*

Proof. Clearly we may assume that G is connected. It was shown in [BKNP04] how to reduce to the case where G has finite fundamental group. For the reader's convenience, we recall the argument. Choose a map $BG \to L_{(P)}(B\mathbf{S}^1)^k$ which is an isomorphism on $H^2(-;\mathbf{Z}_{(P)})$, and denote its homotopy fiber by BG'. Then G' has finite fundamental group. Moreover, the map given by

$$L_{(P)}(\mathbf{S}^1)^k \to G^k \xrightarrow{\text{mult.}} G,$$

representing a basis of $\pi_1(G)$ modulo torsion, is easily seen to be a section of $G \to L_{(P)}(\mathbf{S}^1)^k$, thus $G \simeq G' \times L_{(P)}(\mathbf{S}^1)^k$, although not as loop spaces. Now if $X \xrightarrow{p} Y$ is a rationally splitting double 1-torus for G', then $X \times (\mathbf{S}^1)^k \xrightarrow{p\times\mathrm{Id}} Y \times (\mathbf{S}^1)^k$ is such for G.

Thus assume that the fundamental group of X is finite. Let X be constructed by mixing G with the manifold constructed in Proposition 4.4 at the complementary set of primes. By Prop. 4.6, X has a rationally splitting double 1-torus of level l. The stable reducibility condition is satisfied since that is a p-local condition and it is satisfied at every prime. ☐

Proof of Thm. 1.1. Let X be as in Theorem 1.1. If X has rank 1, the classification of rank 1 p-compact groups says that either $2 \in P$ and $X \in \{L_{(P)}SO(3), L_{(P)}\mathbf{S}^3\}$ or $2 \notin P$ and $X \simeq L_{(P)}\mathbf{S}^{2l-1}$ for some l. In the former case, the claim is obvious. In the latter case, the obvious choice for the manifold model M would be the sphere $\mathbf{S}^{2l-1}$, but if parallelizability is needed, as claimed in the theorem, then the manifold M, constructed by a surgery on twice the $l-1$-dimensional class in $S^{l-1} \times S^l$, will do, since its Kervaire semi-characteristic (cf. proof of Proposition 4.4) is obviously zero.

Now let X be as in Theorem 1.1 of rank ≥ 2, and let Z be constructed by mixing as in the corollary above. By the above corollary Z admits a double 1-torus. By Theorem 4.3, Z is homotopy equivalent to a compact, stably parallelizable, smooth manifold. This manifold is in fact parallelizable on the nose: in even dimensions, this follows from the vanishing of the Euler characteristic of Z; in odd dimensions, it suffices to show that the Kervaire semi-characteristic vanishes. If $2 \notin P$, this follows from Prop. 4.4, and otherwise from the fact that the $\mathbf{F}_2$-cohomology of the 2-compact group $L_2 Z$ is a tensor product of truncated polynomial algebras $\mathbf{F}_2[z]/(z^{2^k})$, as in [BKNP04, Proof of the Main Theorem]. □

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