p-compact groups as framed manifolds

by

Tilman Bauer

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Author . . . . . .
Department of Mathematics
April 22, 2002

Certified by . . . . . .
Michael J. Hopkins
Professor
Thesis Supervisor

Accepted by . . . . . .
Tomasz Mrowka
Chairman, Department Committee on Graduate Students

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Abstract

We describe a natural way to associate to any p-compact group an element of the p-local stable stems, which, applied to the p-completion of a compact Lie group G, coincides with the element represented by the manifold G with its left-invariant framing. To this end, we construct a d-dimensional sphere S_G with a stable G-action for every d-dimensional p-compact group G, which generalizes the one-point compactification of the Lie algebra of a Lie group. The homotopy class represented by G is then constructed by means of a transfer map between the Thom spaces of spherical fibrations over BG associated with S_G.

Thesis Supervisor: Michael J. Hopkins
Title: Professor

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Contents

Chapter 1

Introduction

Let G be a compact Lie group with (real) Lie algebra $\mathfrak{g} = T_eG$. Left multiplication with an element $g \in G$ gives an isomorphism $\mathfrak{g} \cong T_gG$, and by choosing a basis for $\mathfrak{g}$, we thus obtain a framing L of the manifold G, called the left-invariant framing. The Pontryagin-Thom construction produces from this data an element in $\pi_d^s(\mathbf{S}^0)$, where $d = \dim G$. Computations of homotopy classes that arise in this way have been made by Smith [Smi74], Wood [Woo76], Knapp [Kna78], and others. The most extensive table of homotopy classes represented by Lie groups can be found in [Oss82].

This construction is intimately related to the transfer map for the universal bundle over the classifying space of the Lie group G. More generally, for every subgroup inclusion $H \overset{i}{\hookrightarrow} G$ of Lie groups, there is a transfer map in the stable homotopy category $\Sigma^\infty BG^{\mathfrak{g}} \overset{i_!}{\to} \Sigma^\infty BH^{\mathfrak{h}}$. Here, $BG^{\mathfrak{g}}$ stands for the Thom space of the bundle associated to the adjoint representation of G on $\mathfrak{g}$. This map is a twisted version of the well-known Umkehr map for the fibration $G/H \to BH \overset{i}{\to} BG$,

$$BG_+ \to BH^\nu, \tag{1.1}$$

where ν stands for the normal bundle along the fibers of p. Note that the tangent bundle along the fibers of p is $\mathfrak{g}/\mathfrak{h}$ and hence $\nu = \mathfrak{h} - \mathfrak{g}$ as virtual vector bundles. By taking Thom spaces with respect to the bundle $\mathfrak{g}$ resp. $p^*\mathfrak{g}$ on both

sides in (1.1), we obtain the desired map.

Lemma 1.2. *The homotopy class represented by the d-dimensional compact Lie group G is given by the following composite of maps:*

$$S^d \to \Sigma^\infty BG^{\mathfrak{g}} \xrightarrow{i_!} \Sigma^\infty EG_+ \simeq S^0.$$

Here the left hand map is the inclusion of the bottom cell into $BG^{\mathfrak{g}}$, and i is the inclusion of the trivial subgroup into G.

Note that we can factor this map through any $BH^{\mathfrak{h}}$, where $H < G$. For $H = T$ a maximal torus in a semisimple G, this leads to an explicit way of computing the corresponding element in π_d^s.

In this paper, we go one step further and show that the transfer functor $(-)_!$ can be extended to the class of all p*-compact groups*. A p-compact group ([DW94]) G is a $H_*(-;\mathbf{Z}/p)$-local space BG such that $G =_{\text{def}} \Omega BG$ has totally finite mod-p homology. Prominent examples are given by $H\mathbf{Z}/p$-localizations of compact Lie groups. Dwyer and Wilkerson have worked out an extensive Lie theory of p-compact groups [DW94]. It turns out that the classification of p-compact groups, at least at odd primes, boils down to the classical classification of complex reflection groups by Sheppard and Todd [ST54], refined by Clark and Ewing [CE74] to p-adice reflection groups. These groups occur as "Weyl groups" of p-compact groups, and the p-compact groups themselves have been constructed on a case-by-case basis; no general method to construct them from their Weyl groups is known so far.

The main results of this paper are

Theorem 1.3. 1. *For every p-compact group G of $\mathbf{F}_p$-homological dimension d, there is a $H\mathbf{Z}/p$-local d-dimensional sphere S_G with a stable G-action, which in the case of the localization of a compact Lie group is equivalent to the localization of the one-point compactification of the Lie algebra $\mathfrak{g}$ with the adjoint action.*

2. *For every monomorphism $H < G$ of p-compact groups, there is a map $S_G \to G\wedge_H S_H$ which is an isomorphism in $H_d(-;\mathbf{F}_p)$.*

Annoyingly, the morphism in (2) fails to be G-equivariant, but it does so in a well-behaved manner. In fact, there is an extension to $EG_+ \wedge S_G \to G \wedge_H S_H$ that is G-equivariant.

Theorem 1.4. *There is a contravariant functor t from the category of p-compact groups and monomorphisms to the stable homotopy category with the following properties:*

1. *The spectrum*

$$BG^g := t(G) = EG_+ \wedge_G S_G$$

is $\mathbf{Z}/p$-local and connective, and $H^*(BG^g; \mathbf{Z}_p)$ is a free module over $H^*(BG; \mathbf{Z}_p)$ on a Thom class in dimension d, the dimension of G.

2. *The functor t makes the following diagram commute:*

$$\begin{array}{ccc} S_G & \longrightarrow & G \wedge_H S_G \\ \downarrow & & \downarrow \\ BG^g & \longrightarrow & BH^h \end{array}$$

3. *The composition $t \circ L_p$, defined on the category of compact Lie groups and monomorphisms (where L_p is $H\mathbf{Z}/p$ localization), is equivalent to the functor $L_p \circ (-)_!$.*

A few explanations are in order. A *monomorphism* of p-compact groups is, by definition, a pointed map $BH \xrightarrow{f} BG$ whose homotopy fiber has finite mod p homology. Hence, even for Lie groups, we allow additional maps such as unstable Adams operations $L_pBU(n) \xrightarrow{\psi^k} L_pBU(n)$ ($k \in \mathbf{Z}_p^\times$). Of course, in that case the map is a homotopy equivalence and will not yield an interesting transfer map.

Theorem 1.4 enables us, by means of Lemma 1.2, to associate to any p-compact group an element in the stable stems, which one might provocatively call "the p-compact group in its invariant framing".

Table 1 shows p-compact groups with the homotopy classes they represent.

Notation. The symbol $\mathbf{Z}_p$ denotes the p-adic integers. All homology and cohomology theories in this paper are assumed to be reduced, and all spaces to be compactly generated weak Hausdorff.

Name	dim	rank	ST number	prime	homotopy class
$\mathbf{A_n}$	$n(n+2)$	n	1	any	ν, ...
$\mathbf{X}(m,q,n)$	*	n	2a	1 (m)	?
I_{2m}	$2m+2$	2	2b	±1 (m)	0
μ_m	$2m-1$	1	3	1 (m)	α_1 for $m = p-1$
	18	2	4	1 (3)	0
	34	2	5	1 (3)	0
	30	2	6	1 (12)	0
	46	2	7	1 (12)	0
	38	2	8	1 (4)	β_1 for $p = 5$
	62	2	9	1 (8)	0
	70	2	10	1 (12)	0
	94	2	11	1 (24)	0
$\mathbf{Za_2}$	26	2	12	1, 3 (8)	0†
	38	2	13	1 (8)	0
	58	2	14	1, 19 (24)	0
	70	2	15	1 (24)	0
	98	2	16	1 (5)	0
	158	2	17	1 (20)	0
	178	2	18	1 (15)	0
	238	2	19	1 (60)	0
	82	2	20	1, 4 (15)	0
	142	2	21	1, 49 (60)	0
	62	2	22	1, 9 (20)	0
	33	3	23	1, 4 (5)	0
$\mathbf{DW_3}$	45	3	24	1, 2, 4 (7)	w for $p = 2$?
	51	3	25	1 (3)	0
	69	3	26	1 (3)	0
	93	3	27	1, 4 (15)	0
$\mathbf{F_4}$	52	4	28	any	?
$\mathbf{Za_4}$	84	4	29	1 (4)	0
	124	4	30	1, 4 (5)	0
	124	4	31	1 (4)	$\beta_1\beta_2$ for $p = 5$?
	164	4	32	1 (3)	β_1^2 for $p = 7$?
	95	5	33	1 (3)	0
$\mathbf{Ag_6}$	258	6	34	1 (3)	0
$\mathbf{E_6}$	78	6	35	any	β_2^3 for $p = 3$? $p = 2$?
$\mathbf{E_7}$	133	7	36	any	?
$\mathbf{E_8}$	248	8	37	any	?

*$m(n^2 - n + 2\frac{n}{q}) - n$

†does not vanish for purely dimensional and filtration reasons.

Table 1.1.1: p-compact groups and the homotopy classes they represent.

Chapter 2

$H\mathbf{Z}/p$-local equivariant spectra

2.1 $H\mathbf{Z}/p$-localization and p-completion

In [Bou79], a localization functor $X \to X_E$ is constructed for every spectrum E with the property that $X \to X_E$ is the terminal E_*-equivalence out of X. We will need this for $E = H\mathbf{Z}/p$, the Eilenberg-MacLane spectrum with coefficients in $\mathbf{Z}/p$. If X is connective, this functor is very well-behaved:

Lemma 2.1 (Bousfield [Bou79]). *Let X be a connective spectrum. Then localization with respect to $H\mathbf{Z}/p$ is equivalent to localization with respect to $M(\mathbf{Z}/p)$, the Moore spectrum for $\mathbf{Z}/p$. This localization can be constructed explicitly as the p-completion of X, i.e.*

$$X_{M(\mathbf{Z}/p)} = X_p^\wedge = \underleftarrow{\text{holim}} \left\{ \cdots \to X \wedge M(\mathbf{Z}/p^3) \to X \wedge M(\mathbf{Z}/p^2) \to X \wedge M(\mathbf{Z}/p) \right\}.$$

I will denote the $H\mathbf{Z}/p$-localization functor by L_p.

For a finite spectrum X, smashing with X commutes with homotopy limits, and therefore

$$L_p X = X \wedge L_p \mathbf{S}^0.$$

Let $\mathcal{S}$ be the full subcategory of $H\mathbf{Z}/p$-local spectra, i.e. of spectra X such that $X \to L_p X$ is a weak equivalence. This category has all homotopy limits, homotopy

colimits, smash products and function spectra if we compose the usual construction with the functor L_p. (In fact, a homotopy limit of E-local spectra is already E-local.) The smash product is associative up to homotopy, with unit object $L_p\mathbf{S}^0$. When working in $\mathcal{S}$, I will omit any mention of L_p and also write $\mathbf{S}^0$ for the unit of the smash product.

2.2 G-spectra

To construct the transfer map t, we will need to work in a point-set category of equivariant spectra. For our purposes, it is enough to work in the category of so-called naive G-spectra. I will drop the word "naive" since it will make this work appear so puny. Let $G\mathcal{S}$ be the category whose objects are $H\mathbf{Z}/p$-local spectra E, together with a (left) G-action on every space E_n ($n \in \mathbf{Z}$), such that the structure maps $E_n \to \Omega E_{n+1}$ are G-equivariant homeomorphisms. Morphisms are defined as usual. This category has again all homotopy limits and colimits, smash products, and function spectra. The unit is given by $L_p\mathbf{S}^0$ with the trivial G-action. It may be worth pointing out that the G-action on a smash product is the diagonal one, whereas the G-action on $\mathrm{map}(X, Y)$ is given by conjugation.

There are at least two notions of equivariant equivalences in $G\mathcal{S}$, and it is important to distinguish between them.

Definition. I will call a G-equivariant map $f : X \to Y$ between G-spectra a **coarse G-equivalence** if it is a weak equivalence of underlying spectra. It is called a **G-homotopy equivalence** if there is an inverse map up to homotopies through G-equivariant maps.

For a Lie group G, a coarse equivalence f that also induces an equivalence on H-fixed points for every closed subgroup H is sometimes called a weak G-equivalence.

By the equivariant Whitehead theorem for spaces with for a Lie group action of G (cf. [Ada84], [LMSM86]), a weak G-equivalence between G-CW complexes

is a G-homotopy equivalence; this need not be true for coarse G-equivalences in general. For example, the obvious coarse G-equivalence $EG \to *$ does not have an equivariant inverse.

Define a free G-CW spectrum to be a G-spectrum which is built from cells of the form $\mathbf{S}^n \wedge G_+$.

Lemma 2.2. *If E is a free G-CW spectrum and $X \to Y$ is a coarse G-equivalence of G-spectra, then it induces weak equivalences*

$$\mathrm{map}^G(E, X) \tilde{\to} \mathrm{map}^G(E, Y) \quad \textit{and} \quad E \wedge_G X \tilde{\to} E \wedge_G Y.$$

Proof: Both equivalences are clear if E is a single cell $\mathbf{S}^n \wedge G_+$ because in that case,

$$\mathrm{map}^G(E, -) = \mathrm{map}(\mathbf{S}^n, -) \quad \text{and} \quad E \wedge_G X = \mathbf{S}^n \wedge X.$$

It follows for finite spectra by induction and the five-lemma, and in general by a direct limit argument. □

For a G-spectrum X, define

$$X_{hoG} = EG_+ \wedge_G X = (EG_+ \wedge X)/G \quad \text{and} \quad X^{hoG} = \mathrm{map}^G(EG_+, X)$$

where map^G denotes G-equivariant based maps.

The spectrum $\Sigma^\infty EG_+$ is a free G-CW spectrum. Therefore Lemma 2.2 implies in particular that a coarse G-equivalence $f : X \to Y$ induces weak equivalences $f^{hoH} : X^{hoH} \to Y^{hoH}$ and $f_{hoH} : X_{hoH} \to Y_{hoH}$ for any subgroup $H < G$. If H is normal in G then these maps are coarse G/H-equivalences.

2.3 Duality

For a nonequivariant spectrum X, let $DX =_{def} \mathrm{map}(X, \mathbf{S})$ be its dual. This spectrum DX will not have good duality properties in general. For instance, there is no

guarantee that $D(DX) \simeq X$. We call X *strongly dualizable* if there is a map $\mathbf{S} \xrightarrow{\eta} X \wedge DX$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\eta} & X \wedge DX \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tau} \\ \mathrm{map}(X, X) & \xleftarrow{\nu} & DX \wedge X \end{array} \tag{2.3}$$

Here, τ is the flip involution, ι is adjoint to the identity map $X \to X$, and ν is the map adjoint to

$$X \wedge DX \wedge X \xrightarrow{\mathrm{eval} \wedge \mathrm{id}_X} X.$$

The existence of such a map η is equivalent to ν being a homotopy equivalence. It implies that $D(DX) \simeq X$. Cf. [May96].

It turns out that the category $G\mathcal{S}$ contains very few strongly dualizable objects, i.e. objects for which in the above diagram, there is an equivariant map η, or equivalently, ν is a G-homotopy equivalence. This is mainly due to the fact that we are considering naive G-spectra. For example, if M is a compact G-manifold, we usually construct a duality morphism η by embedding M equivariantly into some G-representation V, use the Thom-Pontryagin-construction to get an equivariant map $\mathbf{S}^V \to M^\nu \wedge M_+$, and desuspend by $\mathbf{S}^V$. This last step is impossible in the category of naive G-spectra unless V is a trivial representation, i. e., unless M has a trivial G-action.

If G is a Lie group, and we work in the category of non-naive G-spectra, it is known that a G-CW spectrum is strongly dualizable if and only if it is a wedge summand of a finite G-CW spectrum. It seems plausible that if one succeeded to set up the "right" category of non-naive G-equivariant spectra for a p-compact group G, all the objects in this work that are nonequivariantly dualizable but do not appear to be strongly dualizable in $G\mathcal{S}$ would actually have a strong dual in that category. From a philosophical point of view, this would be desirable and make some cumbersome technical problems disappear. However, in my opinion,

the effort needed for setting up such a category is not warranted by the purposes of the present work.

Suppose that X is a G-spectrum that, as a nonequivariant spectrum, is strongly dualizable. Then the map $\nu : DX \wedge X \to \mathrm{map}(X, X)$, which is always G-equivariant by naturality, is a *coarse* G-equivalence, and η exists but is not necessarily G-equivariant. As should be expected, X will have about half of all the good properties of a strongly dualizable object. For instance, there is a weak equivalence

$$\mathrm{map}^G(A, B \wedge DX) \to \mathrm{map}^G(A \wedge X, B) \tag{2.4}$$

given by

$$A \wedge X \to B \wedge DX \wedge X \xrightarrow{\mathrm{id} \wedge \mathrm{eval}} B$$

but in general no such map

$$\mathrm{map}^G(A \wedge DX, B) \not\to \mathrm{map}^G(A, X \wedge B).$$

A spectrum or space X is called **p-finite** if $H_*(X; \mathbf{F}_p)$ is totally finite.

The following lemma has a rather long history of my advisor suggesting a proof using the Adams spectral sequence and me rejecting it and finding another (erroneous) proof without it. Eventually, I caved in. Here's his proof. Kudos for Mike.

Lemma 2.5. *For every connective, $H\mathbf{Z}/p$-local, p-finite spectrum X, there is a finite spectrum X' and p-equivalence $X' \to X$.*

Remark. This association is not claimed to be functorial.

Proof: Let $k \in \mathbf{Z}$ be minimal with $H_k(X; \mathbf{F}_p) \neq 0$. We proceed by induction on the size of $H_*(X; \mathbf{F}_p)$.

We will first show that there is a nontrivial map

$$f : \pi_k(X) \to H_k(X) \to H_k(X; \mathbf{F}_p).$$

This would be a simple application of the Hurewicz theorem relative to a Serre class if the class of groups that vanish when tensored with $\mathbf{F}_p$ were actually a Serre class, which it is not.

Since X is connective and $H\mathbf{Z}/p$-local, its $H\mathbf{Z}/p$-nilpotent completion and its $H\mathbf{Z}/p$-localization agree (Lemma 2.1) and are equal to X, hence the classical $H\mathbf{Z}/p$-based Adams spectral sequence converges to $\pi_*(X)$. Since $\mathrm{Ext}_{A^*}^{s,t}(H^*(X;\mathbf{F}_p),\mathbf{F}_p) = 0$ for $t-s<k$, the Hurewicz map $f:\pi_k(X)\to H_k(X;\mathbf{F}_p)$ has to be nonzero.

Let $\beta:\mathbf{S}^k\to X$ be a map such that $f([\beta])\neq 0$. Let F be the $H\mathbf{Z}/p$-localization of the homotopy fiber of β. F is p-finite, $H\mathbf{Z}/p$-local and the size of its $\mathbf{Z}/p$-homology is smaller than that of X, hence by induction, there is a finite spectrum F' and a p-equivalence $F'\to F$. Let X' be the cofiber of $F'\to F\to \mathbf{S}^k$; X' is a finite spectrum and comes with a map $X'\to X$ which is a p-equivalence. □

Corollary 2.6. *Let X be a connective, $H\mathbf{Z}/p$-local, p-finite spectrum. Then X has a strong dual in $\mathcal{S}$.*

Proof: By Lemma 2.5, there is a finite spectrum X' and a p-equivalence $X'\to X$. Hence there is a p-equivalence of $H\mathbf{Z}/p$-local spectra $L_pX'\to L_pX = X$, which therefore is a weak equivalence. It remains to show that $L_p(D(X'))$ is a strong dual of L_pX' for a finite spectrum X'.

We need to show that

$$L_p(\mathrm{map}(X',\mathbf{S})) = L_p\,\mathrm{map}(L_pX',L_p\mathbf{S}).$$

Indeed,

$$L_p\,\mathrm{map}(L_pX',L_p\mathbf{S}) \simeq \mathrm{map}(X',L_p\mathbf{S}) \simeq DX'\wedge L_p\mathbf{S} \simeq L_p(DX').$$

Now $\eta:\mathbf{S}\to X\wedge DX$ induces a duality map

$$L_p\eta : L_p\mathbf{S}\to L_p(X'\wedge DX') = L_pX'\wedge L_pDX' = L_pX'\wedge D(L_pX'),$$

which shows that $L_p(D(X'))$ is a strong dual. □

Chapter 3

p-compact groups

This chapter will provide some background about p-compact groups, a topic that has become very popular starting in the early nineties, largely due to some beautiful work of Dwyer and Wilkerson [DW94] and Dwyer, Miller, and Wilkerson [DMW92].

3.1 Definition and examples

Definition ([DW94]). A **p-compact group** is a triple (X, BX, e) where BX is a $H\mathbf{Z}/p$-local space, X is an $\mathbf{F}_p$-finite space, and $e : X \to \Omega BX$ is a homotopy equivalence.

As noted in the introduction, the $H\mathbf{Z}/p$-localization L_pG of a Lie group G gives rise to a p-compact group (L_pG, L_pe, L_pBG) for every prime p. Here $e : G \to \Omega BG$ is the canonical equivalence.

To illustrate how to obtain other p-compact groups, it is instructive to recall the connection between spaces with polynomial cohomology rings and finite loop spaces. If X is a space such that

$$H^*(X; \mathbf{F}_p) \cong \mathbf{F}_p[\sigma_1, \sigma_2, \ldots, \sigma_r] \quad \text{with} \quad \sigma_i \in H^{d_i}(X; \mathbf{F}_p),\ d_i \text{ even},$$

then by the Eilenberg-Moore spectral sequence,

$$H^*(\Omega X; \mathbf{F}_p) \cong \bigwedge\{\tau_1, \tau_2, \ldots, \tau_r\} \quad \text{with} \quad \tau_i \in H^{d_i - 1}(\Omega X; \mathbf{F}_p),$$

and τ_i is the image of σ_i under the transgression. In particular, $H^*(\Omega X; \mathbf{F}_p)$ is finite, and L_pX is a p-compact group. The reader should be warned that not all p-compact groups are polynomial in this sense.

A large class of p-compact groups, called the *non-modular* groups, can be constructed as follows:

First pick a finite group $W < GL_r(\mathbf{Z}_p)$ (a "Weyl" group for the p-compact group); W acts on $\mathbf{Z}_p^r$ and hence also on $K(\mathbf{Z}_p^r, 2) = L_p(\mathbf{C}P^\infty)^r$. Define a space

$$BG =_{\text{def}} L_p \left(K(\mathbf{Z}_p^r, 2)\right)_{\text{ho}\, W}.$$

We want to determine what restrictions on W we have to make to ensure that BG is a space with polynomial cohomology. There is a spectral sequence converging to $H^*(BG; \mathbf{F}_p)$ whose E_2 term is

$$E_2^{r,s} = H^r(BW; H^s(K(\mathbf{Z}_p, 2); \mathbf{F}_p)) = H^r(BW; \mathbf{F}_p[\![t_1, \ldots, t_r]\!])$$

If p does not divide $|W|$ then $E_2^{r,s} = 0$ for $r > 0$, and

$$E_2^{0,s} = \mathbf{F}_p[\![t_1, \ldots, t_r]\!]^W = H^s(BG; \mathbf{F}_p)$$

Theorem 3.1 (Sheppard-Todd, Clark-Ewing [ST54, CE74]).
Let $W < GL_r(\mathbf{F}_p)$ be finite.

If $\mathbf{F}_p[t_1, \ldots, t_r]^W$ is polynomial then W is a pseudo-reflection group*, i.e., it is generated by a finite set of finite order elements that fix a hyperplane in $\mathbf{F}_p^r$.*

The converse is true if (but not only if) p does not divide the order of W. □

Moreover, in the non-modular case, every representation of W over $\mathbf{F}_p$ can be lifted to a representation over $\mathbf{Z}_p$.

We can thus construct a p-compact group BG for every pseudo-reflection group defined as a subgroup of $GL_r(\mathbf{Z}_p)$ such that p does not divide the order of W. All such groups are classified [ST54, CE74], and Table 1 lists some statistics about them. In that table, all exotic groups of rank bigger than 1 that are given a name are non-modular.

Definition. A morphism $BH \to BG$ of p-compact groups is just a pointed map $Bf : BH \to BG$. It is a **monomorphism** if its homotopy fiber is $\mathbf{F}_p$-finite, and an **epimorphism** if its homotopy fiber is a p-compact group.

Two morphisms $BH \to BG$ are called **conjugate** if they are freely homotopic.

For Lie groups H and G, being conjugate in the p-compact sense is indeed the same as being conjugate as Lie group homomorphisms.

3.2 Maximal tori

In the non-modular case considered in the previous section, BG naturally comes with a map

$$BT := K(\mathbf{Z}_p^r, 2) \to L_p(K(\mathbf{Z}_p^r, 2)_{hoW}) = BG$$

given by the inclusion of the fiber of the bundle $BG \to BW$. Call a monomorphism of p-compact groups $BT \to BG$ a **maximal torus** if $BT = K(\mathbf{Z}_p^r, 2)$ for some r, and it does not factor through a larger torus. One of the main results of [DW94] is that such tori also exist in the non-modular case:

Theorem 3.2 (Dwyer-Wilkerson [DW94]).

1. *For every connected p-compact group BG, there is a maximal torus $BT \to BG$, unique up to conjugacy.*
2. *The monoid $\mathrm{map}_{BG}(BT, BT)$ of endomorphisms of BT over BG is homotopy equivalent to a finite group W acting as a group of pseudo-reflections on $H^2(BT; \mathbf{Z}_p) \cong \mathbf{Z}_p^r$.*
3. *$H^*_{\mathbf{Q}_p}(BG_+) \cong H^*_{\mathbf{Q}_p}(BT_+)^W$, and $H^*_{\mathbf{Q}_p}(BT_+)$ is a free module over $H^*_{\mathbf{Q}_p}(BG_+)$.*

□

Here, $H^*_{\mathbf{Q}_p}(X) =_{\mathrm{def}} H^*(X; \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. (Note that $H^*(X; \mathbf{Q}_p)$ would be an unreasonably large group; whereas $\mathrm{Hom}(\mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p$, we have

$$\mathrm{Hom}(\mathbf{Z}_p, \mathbf{Q}_p) = \mathrm{Hom}(\mathbf{Q}_p, \mathbf{Q}_p) = \mathbf{Q}_p^{\aleph_2}.)$$

Corollary 3.3. *The* p*-compact flag variety* $G/T = \mathrm{hofib}(BT \to BG)$ *has*

$$H^*_{\mathbf{Q}_p}(G/T_+) = H^*_{\mathbf{Q}_p}(BT_+) \,/\, (H^*_{\mathbf{Q}_p}(BT)^W).$$

Proof: There is an Eilenberg-Moore spectral sequence

$$E_2^{s,t} = \widehat{\mathrm{Tor}}^{s,t}_{H^*_{\mathbf{Q}_p}(BG_+)}(H^*_{\mathbf{Q}_p}(BT_+), \mathbf{Q}_p) \Longrightarrow H^*_{\mathbf{Q}_p}(G/T_+),$$

where $\widehat{\mathrm{Tor}}^s$ is the sth derived functor of the completed tensor product $\hat{\otimes}$. In this spectral sequence, $E_2^{s,t} = 0$ for $s > 0$ because $H^*_{\mathbf{Q}_p}(BT_+)$ is free, hence flat, over $H^*_{\mathbf{Q}_p}(BG_+)$, and

$$E_2^{0,t} = H^*_{\mathbf{Q}_p}(BT_+) \hat{\otimes}_{H^*_{\mathbf{Q}_p}(BG_+)} \mathbf{Q}_p = H^*_{\mathbf{Q}_p}(BT_+) \,/\, (H^*_{\mathbf{Q}_p}(BT)^W). \quad \square$$

It will become important in calculations to know exactly what the degree of the map

$$c : H^*(BT_+; \mathbf{Z}_p) \,/\, (H^*(BG; \mathbf{Z}_p)) \hookrightarrow H^*(G/T_+; \mathbf{Z}_p)$$

is in the top dimension.

Lemma 3.4. *Let* $p > 2$ *or* G *of Lie type or* $G = DW_3$*. Then the cohomology ring* $H^*(G/T; \mathbf{Z}_p)$ *is concentrated in even dimensions and torsion free.*

Proof: This is a result that follows from Schubert calculus in the case where G is a Lie group. For polynomial p-compact groups, we have

$$H^*(G/T_+; \mathbf{Z}_p) \cong H^*(BT_+; \mathbf{Z}_p) \,/\, (H^*(BG; \mathbf{Z}_p))$$

by the same argument as in Corollary 3.3, and the assertion holds. Now the only non-polynomial p-compact groups for odd p are [KM97, Not99b, Not99a]:

- Type A_n with a fundamental group that is a p-group;
- types F_4, E_6, E_7, E_8 for $p = 3$; and
- type E_8 for $p = 5$.

In particular, they are all Lie groups. □

Remark 1. Since the classification of 2-compact groups in not finished at this time, we cannot claim Lemma 3.4 holds for $p = 2$. However, the only *known* non-Lie 2-compact group is DW_3, which is polynomial [DW93]. It is conjectured that it actually is the only one.

Remark 2. It would be much more satisfying to find a proof that does not rely on the accidental fact that all non-polynomial p-compact groups are of Lie type. For example, it would be exciting to produce a Schubert calculus for p-compact flag varieties.

I am grateful to Nitu Kitchloo for pointing out to me the implication (i) ⇒ (iii) in the following proposition:

Proposition 3.5. *Let* G, p *be as in Lemma 3.4 and* G *be simply connected. Then the following are equivalent:*

(i) c *is an isomorphism in the top dimension;*

(ii) c *is an isomorphism in all dimensions;*

(iii) $H^*(G/T_+; \mathbf{Z}_p)$ *is generated by degree 2 classes;*

(iv) $H^*(BG_+; \mathbf{Z}_p)$ *has no torsion;*

(v) $H^*(BG_+; \mathbf{Z}_p)$ *is a polynomial algebra.*

Proof: If G is simply connected it follows from the Serre spectral sequence associated to $G/T \to BT \to BG$ that

$$H^2(BT_+; \mathbf{Z}_p) \xrightarrow{\cong} H^2(G/T_+; \mathbf{Z}_p).$$

This shows (iii) ⇔ (ii) ⇒ (i). For (iv) ⇒ (ii), assume c fails to be an isomorphism in dimension k. Then in the above Serre spectral sequence, a class x in $H^k(G/T_+; \mathbf{Z}_p)$ has to support a nontrivial differential d^i. Since rationally, $H^*_{\mathbf{Q}_p}(G/T_+)$ is always generated by degree 2 classes, $d^i(x)$ has to be a torsion class in

$$H^i(BG_+; H^{k+1-i}(G/T_+; \mathbf{Z}_p)).$$

By Lemma 3.4, the latter group is isomorphic to $H^i(BG_+;\mathbf{Z}_p)\hat{\otimes}H^{k+1-i}(G/T_+;\mathbf{Z}_p)$. Since by the same lemma, $H^*(G/T_+;\mathbf{Z}_p)$ is torsion free, there must be a torsion class in $H^i(BG_+;\mathbf{Z}_p)$.

For (i) $\Rightarrow$ (iv), assume $y \in H^j(BG_+;\mathbf{Z}_p)$ is torsion with j minimal. By the multiplicativity and Lemma 3.4, this implies that

$$y = d^j(x) \quad \text{for some} \quad x \in H^{j-1}(G/T_+;\mathbf{Z}_p).$$

Pick a generator $g \in H^{top}(G/T;\mathbf{Z}_p)$. Now

$$0 = d^j(gx) = d^j(g)x \pm gd^j(x).$$

Since $d^j(x)g = yg \neq 0$, $d^i(g)$ cannot be trivial, hence g is not a permanent cycle, and c is not an isomorphism in the top dimension.

For (iv) $\Leftrightarrow$ (v), note that if $H^*(BG_+;\mathbf{Z}_p)$ has no torsion, it has to be concentrated in even degrees since it injects into $H^*_{\mathbf{Q}_p}(BG_+)$. Hence $H^*(G_+;\mathbf{Z}_p)$ is a degreewise free Hopf algebra on odd-dimensional generators, which implies that it is an exterior algebra. Hence, by the Serre spectral sequence for the path-loop-fibration on BG, $H^*(BG_+;\mathbf{Z}_p)$ is a polynomial algebra. □

3.3 A comment on rigidity

In the definition of a p-compact group (X, BX, e), the data X and e are redundant and probably only classically included to provide some justification for speaking of "a p-compact group X" and not the more accurate "BX". On the other hand, it is always possible to choose a model for the loop space $X := \Omega BX$ such that X is actually a topological group and not just an H-space. A possible construction is the geometric realization of Kan's loop group functor G as described in [Kan56].

Let $\mathbf{S}$ denote the category of simplicial sets and $\mathbf{S}_0$ the full subcategory of *reduced* simplicial sets, i.e., simplicial sets X such that $X_0 = pt$. Equip $\mathbf{S}_0$ with the *projective* model structure, i.e. weak equivalences and *cofibrations* are shared with

$\mathbf{S}$. It turns out ([GJ99]) that a map $X \to Y$ between fibrant reduced simplicial sets is a fibration if and only if it induces a surjection on fundamental groups.

Let $s\,\mathrm{Gr}$ denote the category of simplicial groups, carrying the injective model structure (sharing weak equivalences and *fibrations* with the underlying simplicial sets).

Proposition 3.6 (Kan). *There is a Quillen equivalence*

$$\overline{W} : \ s\,\mathrm{Gr} \leftrightarrow \mathbf{S}_0 : \ G$$

Furthermore, there is a Quillen equivalence between the category $\mathbf{S}_0$ and the category of connected, pointed simplicial sets, $\mathbf{S}_c$, where the functor $F : \mathbf{S}_c \to \mathbf{S}_0$ is given by

$$F(X)_n = \{x \in X_n \mid i^*(x) = * \text{ for every } i : [0] \to [n]\}.$$

Passing to topological spaces, we also have Quillen equivalences

$$\mathbf{S}_c \leftrightarrow \{\mathtt{pointedconnectedtopologicalspaces}\}$$

and

$$s\,\mathrm{Gr} \leftrightarrow \{\mathtt{topologicalgroups}\}$$

This suggests the following alternative definition of a p-compact group:

Definition (alternative). The category of p-compact groups is the full subcategory of all $H\mathbf{Z}/p$-local topological groups (compactly generated, weak Hausdorff) whose objects are fibrant, cofibrant, $\mathbf{F}_p$-finite, and such that $\pi_0(G)$ is a finite p-group.

The condition on the group of components is necessary to ensure that BG is still $H\mathbf{Z}/p$-local. By the above Quillen equivalences, every map $BH \to BG$ is, up to homotopy, induced by a group homomorphism $H \to G$ if H and G are p-compact groups in this sense.

Moreover, a monomorphism $BH \to BG$ in the sense of the original definition is always, up to homotopy, induced by an injective group homomorphism $H \to G$.

In fact, we can functorially replace $BH \to BG$ by a cofibration, and Kan's functor G preserves cofibrations. Cofibrations of simplicial groups are injective.

We will therefore work in the category of p-compact groups according to the above alternative definition, and define monomorphisms as actual subgroup inclusions.

Chapter 4

Adjoint representations

Although much of Lie theory carries over to the more general setting of p-compact groups, the representation theory, and in particular the adjoint representation, does not seem to have a direct analogue for p-compact groups. We do not know how to construct a vector bundle on a p-compact group BG that plays that role, but we can manufacture something that, in the Lie cases, looks like its Thom spectrum.

Definition. For any connected p-compact group G, define

$$S_G = (\Sigma^\infty G_+)^{hoG^{op}}.$$

Note that G acts on $\Sigma^\infty G_+$ by both left and right multiplication. We agree to use the right action for the formation of this homotopy fixed point spectrum, leaving us a left G-action on S_G.

The **adjoint Thom spectrum** of G is the spectrum

$$BG^{\mathfrak{g}} =_{def} (S_G)_{hoG} = EG_+ \wedge_G S_G.$$

Klein [Kle01] has shown that this construction for G a (non-localized) connected compact Lie group indeed gives rise to the Thom spectrum of the adjoint bundle. It is therefore reasonable to mimick this construction for a p-compact group G. The main point of this chapter is to show that S_G, defined as above for a connected p-compact group G, is homotopy equivalent to a sphere.

We will need two classical lemmas on finite-dimensional Hopf algebras. All cohomology and homology groups are with coefficients in $\mathbf{F}_p$.

Lemma 4.1. *If* G *is a topological group such that* $H_*(G)$ *is totally finite, then* $H^*(\Sigma^\infty G_+)$ *is a free* $H_*(G)$*-module on a generator in dimension* $\dim G$.

Proof: Note that $A = H_*(G)$ is a Hopf algebra, and

$$H^*(\Sigma^\infty G_+) \cong A^*.$$

by universal coefficients. The dual algebra A^* is a Hopf algebra with antipode c coming from inversion in the group G, and A is a right Hopf module over A^*: the module structure is given by

$$A \otimes A^* \to A, \quad \text{the adjoint map of the coproduct } \psi : A \to A \otimes A,$$

and the comodule structure by

$$A \to A \otimes A^*, \quad \text{the adjoint map of the product on } A.$$

Let $P(A)$ denote the $\mathbf{F}_p$-vector space of primitives of A as an A^*-comodule, i.e.

$$P(A) = \left\{ a \in A \;\middle|\; ax = a\epsilon(x) \text{ for all } x \in A \right\},$$

where ϵ is the augmentation $H_*(G_+) \to H_*(\mathbf{S}^0)$.

Then (cf [Par71]), we have a splitting

$$A \xrightarrow{\cong} P(A) \otimes A^*$$

as right A^*-Hopf modules, given by

$$\begin{array}{ccccccccc} A & \xrightarrow{\psi} & A \otimes A^* & \xrightarrow{\mathrm{id}\otimes\psi} & A \otimes A^* \otimes A^* & \xrightarrow{\mathrm{id}\otimes c\otimes \mathrm{id}} & A \otimes A^* \otimes A^* & \xrightarrow{\mu\otimes\mathrm{id}} & A \otimes A^* \\ & & & & & & & & \uparrow \\ & & & & & & & & P(A) \otimes A^* \end{array} \tag{4.2}$$

Since A is finite dimensional, it follows that $\dim P(A) = 1$. The assertion of Lemma 4.1 follows. □

We will show later (Proposition 5.8) that for G a p-compact group, this map is realizable as a map of spectra.

An algebra like $H_*(G;\mathbf{F}_p)$, which, as a module, is isomorphic to a suspension of its dual, is called a Frobenius algebra.

Lemma 4.3 (Moore-Peterson [MP73]). *If A is a finite-dimensional Frobenius algebra over a field, then the class of its projective modules coincides with the class of its injective modules.* □

Proposition 4.4. *S_G is homotopy equivalent to a $H\mathbf{Z}/p$-local sphere of dimension d.*

Proof: It is enough to know that S_G has the mod p homology of a sphere because the proof of Lemma 2.5 produces a p-equivalence $\mathbf{S}^d \to S_G$ in that case.

To see that S_G has the correct homology, we will use a spectral sequence associated to the cosimplicial spectrum

$$(\Sigma^\infty G_+)^{h\circ G^{op}} = \mathrm{map}^G(EG_+, \Sigma^\infty G_+),$$

where $EG = \mathrm{map}(-, G) = \Delta^1 \otimes G$ is the usual simplicial space with G^{n+1} in dimension n. The E^2-term is given by

$$E^2_{p,q} = H_p(\mathrm{map}^G(G^{(q+1)}, \Sigma^\infty G_+); \mathbf{F}_p),$$

and by the Lemma below, this spectral sequence collapses at the E^2-term with

$$E_2^{p,q} = \begin{cases} 0; & p \neq 0 \text{ or } q \neq d \\ \mathbf{F}_p; & \text{otherwise.} \end{cases}$$

and converges strongly. Therefore and $H_*(S_G) = H_*(\mathbf{S}^d)$. □

This proves the first part of Theorem 1.3.

Lemma 4.5. *Let $k \in \mathbf{N}_0 \cup \{\infty\}$, and let $EG_+^{(k)}$ be the G-equivariant k-skeleton of the simplicial space EG_+. Then*

$$H_n(\mathrm{map}^G(EG_+^{(k)}, \Sigma^\infty G_+)) \cong \begin{cases} \mathbf{F}_p; & n = d < k \\ 0; & n < k \textit{ and } n \neq d \end{cases}$$

No statement is made about the homology groups beyond degree k. By G-equivariant k-skeleton we mean the truncation of the simplicial space EG_+ at the kth stage.

From now on, until the end of this chapter, all homology groups are taken with coefficients in $\mathbf{F}_p$.

Proof: Since $EG_n = G^{n+1}$, we have that

$$(\mathrm{map}^G(EG_+, \Sigma^\infty G_+))^n = \mathrm{map}^G(G_+^{n+1}, \Sigma^\infty G_+).$$

The evaluation map

$$\mathrm{map}^G(G_+^{n+1}, \Sigma^\infty G_+) \wedge G_+^{n+1} \to \Sigma^\infty G_+$$

induces a natural map

$$H_n(\mathrm{map}^G(G_+^{n+1}, \Sigma^\infty G_+) \to \mathrm{Hom}^n_{H_*(G_+)}(H_*(G_+)^{\otimes(n+1)}, H_*(\Sigma^\infty G_+)),$$

where Hom^n stands for module homomorphisms that raise degree by n. This map is an isomorphism because the following diagram commutes:

$$\begin{array}{ccc}
H_n(\mathrm{map}^G(G_+^{n+1}, \Sigma^\infty G_+)) & \longrightarrow & \mathrm{Hom}^n_{H_*(G_+)}(H_*(G_+)^{\otimes(n+1)}, H_*(\Sigma^\infty G_+)) \\
\downarrow \sim & & \downarrow \sim \\
H_n(\mathrm{map}(G_+^n, \Sigma^\infty G_+)) & & \mathrm{Hom}^n(H_*(G_+)^{\otimes n}, H_*(\Sigma^\infty G_+)) \\
\downarrow \sim & & \downarrow \sim \\
H_n((DG_+)^{\wedge n} \wedge \Sigma^\infty G_+) & \xrightarrow{\sim} & (H_*(DG_+)^{\otimes n} \otimes H_*(\Sigma^\infty G_+))_n .
\end{array}$$

The coboundary operators are induced by the simplicial operators on $H_*(G_+)^{\otimes \bullet}$ from the bar construction on $H_*(G_+)$. Hence $H_n(\mathrm{map}^G(EG_+^{(k)}, \Sigma^\infty G_+))$ is the group of homomorphisms from a truncated projective resolution of k over $H_*(G)$ to $H_*(G)$.

Associated to the tower

$$\mathrm{map}^G(EG_+^{(k)}, \Sigma^\infty G_+) \to \cdots \to \mathrm{map}^G(EG_+^{(j)}, \Sigma^\infty G_+) \to$$
$$\to \cdots \to \mathrm{map}^G(EG_+^{(0)}, \Sigma^\infty G_+) = \Sigma^\infty G_+,$$

we therefore obtain a spectral sequence with E^2-term

$$E^2_{p,q} = \begin{cases} \mathrm{Ext}^{p,q}_{H_*(G_+)}(\mathbf{F}_p, H_*(G_+)); & q < k \\ 0; & q > k \end{cases} \Longrightarrow H_{p+q}(\mathrm{map}^G(EG^{(k)}_+, \Sigma^\infty G_+).$$

Because of Lemmas 4.1 and 4.3, $H_*(G_+)$ is injective as a module over itself, and hence $E^2_{p,q} = 0$ for $q > 0$ and $q \neq k$. On the other hand,

$$E^2_{p,0} = E^\infty_{p,0} = \mathrm{Hom}^q_{H_*(G_+)}(\mathbf{F}_p, H_*(G_+)) = P(H_*(G_+))$$

with the notation of Lemma 4.1, and hence

$$H_n(\mathrm{map}^G(EG^{(k)}_+, \Sigma^\infty G_+)) \cong \begin{cases} \mathbf{F}_p; & n = d \\ 0; & n < k \text{ and } n \neq d \end{cases}$$

□

Chapter 5

Self-duality for p-compact groups

5.1 Two Lemmas on restricted homotopy fixed points

Lemma 5.1. *For a sub-p-compact group $H < G$, there is a coarse G-equivalence*

$$G \wedge_H S_H \xrightarrow{\sim} (\Sigma^\infty G_+)^{h_o H^{op}}.$$

Proof: First note that as $(G \times H^{op})$-spectra,

$$\Sigma^\infty G_+ \simeq G \wedge_H \Sigma^\infty H_+,$$

where on the right hand side, H acts on the right factor from the right and G acts on the left factor from the left.

We therefore have a map

$$G \wedge_H S_H = G \wedge_H \mathrm{map}^{H^{op}}(EH_+, \Sigma^\infty H_+)$$
$$\longrightarrow \mathrm{map}^{H^{op}}(EH_+, G \wedge_H \Sigma^\infty H_+) = (\Sigma^\infty G_+)^{h_o H^{op}}.$$

This map is clearly G-equivariant, and it is a weak equivalence because G and H are nonequivariantly dualizable. □

If $X \in (G \times G^{op})\mathcal{S}$ and $Y \in H^{op}\mathcal{S}$, we have G-equivariant homotopy equivalences (given by shearing maps)

$$G \wedge_H X \simeq G/H_+ \wedge X$$

and

$$\mathrm{map}^{H^{op}}(Y \wedge G_+, X) \simeq \mathrm{map}(Y \wedge_H G, X)$$

In particular, if $Y \in (G \times G^{op})\mathcal{S}$, we have

$$\mathrm{map}^{H^{op}}(Y \wedge G_+, X) \simeq \mathrm{map}(Y \wedge G/H_+, X). \tag{5.2}$$

Lemma 5.3. *Let $H < G$ be as above. Then there is a coarse G-equivalence*

$$(DG_+)^{h_o H^{op}} \xrightarrow{\sim} D(G/H_+),$$

natural on subgroups of G.

Proof: The map is the following composite of coarse G-equivalences, all of which are natural:

$$\begin{aligned}
(DG_+)^{h_o H^{op}} &= \mathrm{map}^H(EH_+, DG_+) \\
&\simeq \mathrm{map}^H(EG_+, DG_+) \\
&\longrightarrow \mathrm{map}^H(EG_+ \wedge G_+, \mathbf{S}^0) \\
&\xrightarrow{f} \mathrm{map}(EG_+ \wedge G/H_+, \mathbf{S}^0) \\
&\xrightarrow{g} D(G/H_+).
\end{aligned}$$

For the first homotopy equivalence, we use that EG is a valid model for EH. f is a G-equivariant homotopy equivalence by (5.2). Since EG_+ has the usual right action and a trivial left action, the map $\mathbf{S}^0 \to EG_+$ is a left G-homotopy equivalence, and hence so is g. □

5.2 Absolute Poincaré duality

Denote by G_c the suspension spectrum of G with G acting by conjugation. For G a Lie group, S_G can be identified with the one-point compactification of a neighborhood of the identity in G; this identification is G-equivariant if we equip G with the conjugation action. The following lemma shows that such a "logarithm" also exists for p-compact groups, at least up to a coarse G-equivalence.

Lemma 5.4. *For every* p*-compact group* G*, there is a* G*-spectrum* $E(G)$*, a natural coarse* G*-equivalence* $E(G) \to (G_c)_+$*, and a* G*-equivariant retraction* $E(G) \to S_G$*.*

Remark: An equivariant retraction $X \to Y$ means two equivariant maps

$$Y \to X \to Y$$

such that the composite is a *coarse* G-equivalence.

Proof: The auxiliary spectrum $E(G)$ is defined as

$$E(G) = \mathrm{map}^{G^{op}}(EG_+, \Sigma^\infty G_+ \wedge DG_+).$$

Consider $\Sigma^\infty G_+$ as a $(G \times G^{op})$-spectrum by left and right multiplication. Then the diagonal map

$$\Sigma^\infty G_+ \to \Sigma^\infty G_+ \wedge \Sigma^\infty G_+$$

is $(G \times G^{op})$-equivariant and has an equivariant adjoint

$$\Sigma^\infty G_+ \wedge DG_+ \to \Sigma^\infty G_+. \tag{5.5}$$

Similarly, the $(G \times G^{op})$-equivariant projection map to the first factor

$$\Sigma^\infty G_+ \wedge \Sigma^\infty G_+ \to \Sigma^\infty G_+$$

has an equivariant adjoint

$$\Sigma^\infty G_+ \to DG_+ \wedge \Sigma^\infty G_+.$$

The composite

$$\Sigma^\infty G_+ \to \Sigma^\infty G_+ \wedge DG_+ \to \Sigma^\infty G_+ \tag{5.6}$$

is a weak equivalence.

Taking homotopy fixed points with respect to $G^{op} = 1 \times G^{op} \subseteq G \times G^{op}$ on the left hand side of (5.5) yields

$$\begin{aligned} E(G) = \mathrm{map}^{G^{op}}(EG_+, \Sigma^\infty G_+ \wedge DG_+) &\xrightarrow[(2.4)]{\sim} \mathrm{map}^{G^{op}}(EG_+ \wedge G_+, \Sigma^\infty G_+) \\ &\xrightarrow{\sim} \mathrm{map}(EG_+, \Sigma^\infty G_+) \\ &\xrightarrow{\sim} \mathrm{map}(\mathbf{S}^0, \Sigma^\infty G_+) \simeq \Sigma^\infty G_+ \end{aligned} \tag{5.7}$$

As in Lemma 5.3, the map induced by $\mathbf{S}^0 \to EG_+$ is indeed a G-homotopy equivalence because the left G-action on EG_+ is trivial. In fact, all maps but the first one are G-homotopy equivalences.

We have to check that the G-action on $E(G)$ corresponds to the conjugate action on $\Sigma^\infty G_+$.

The action of G on $M = \mathrm{map}^{G^{op}}(EG_+ \wedge G_+, \Sigma^\infty G_+)$ is given by

$$(g.f)(x \wedge \gamma) = gf(x \wedge g^{-1}\gamma) \quad (g \in G,\ f \in M,\ x \in EG_+,\ \gamma \in G_+).$$

The induced action of G on $\mathrm{map}(EG_+, \Sigma^\infty G_+)$ is

$$(g.f)(x) = gf(xg)g^{-1} \quad (g \in G,\ f \in \mathrm{map}(EG_+, \Sigma^\infty G_+),\ x \in EG_+)$$

since

$$\begin{array}{rcl} \mathrm{map}^{G^{op}}(EG_+ \wedge G_+, \Sigma^\infty G_+) & \to & \mathrm{map}(EG_+, \Sigma^\infty G_+) \\ f & \mapsto & x \mapsto f(x, 1) \\ g.f & \mapsto & x \mapsto gf(x, g^{-1}) = gf(xg, 1)g^{-1}. \end{array}$$

The restricted G-action on $\mathrm{map}(\mathbf{S}^0, \Sigma^\infty G_+)$ becomes

$$(g.f)(x) = gf(x)g^{-1}$$

since $\mathbf{S}^0$ has the trivial G-action, and the G-action on $\Sigma^\infty G_+$ is indeed by conjugation.

Applying $(-)^{h\mathrm{o}\, G^{op}}$ to (5.6) yields the desired retraction $E(G) \to S_G$. □

Proposition 5.8. *Regard the* G*-spectrum* S_G *as a* $(G \times G^{op})$*-spectrum with trivial* G^{op}*-action. Then there is a coarse* $G \times G^{op}$*-equivalences*

$$S_G \wedge DG_+ \xrightarrow{\simeq} \Sigma^\infty G_+$$

On G^{op}*-homotopy fixed points, these maps make the following diagram commute:*

$$\begin{array}{ccccc} S_G = (\Sigma^\infty G_+)^{h\mathrm{o}\, G^{op}} & \xleftarrow{\simeq} & (S_G \wedge DG_+)^{h\mathrm{o}\, G^{op}} & \xleftarrow{\simeq} & S_G \wedge (DG_+)^{h\mathrm{o}\, G^{op}} \\ & \searrow & & & \simeq \downarrow \text{Lemma 5.3} \\ & & & & S_G \wedge \mathbf{S}^0 \end{array}$$

Proof: We will have to deal with spectra with three G-actions, and for ease of notation, for a $(G \times G^{op})$-spectrum X, I will denote by ${}^{a}X^{b}_{c}$ the spectrum X with the left action a and the two right actions b and c, where a, b, c, are one of the following:

- '$\circ$' denotes a trivial action
- 'l' denotes the action from the left — if this symbol appears on the right then G acts by inverses from the left
- 'r' denotes the action from the right — if this symbol appears on the left then G acts by inverses from the right

The main ingredient is a shearing map

$$ {}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{l}(DG_{+})^{r}_{\circ} \xrightarrow{sh} {}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{\circ}(DG_{+})^{l}_{r}. \tag{5.9} $$

which is adjoint to

$$ \begin{array}{ccc} {}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{\circ}(\Sigma^{\infty}G_{+})^{l}_{r} & \longrightarrow & {}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{l}(\Sigma^{\infty}G_{+})^{r}_{\circ} \\ g \wedge h & \mapsto & g \wedge hg. \end{array} $$

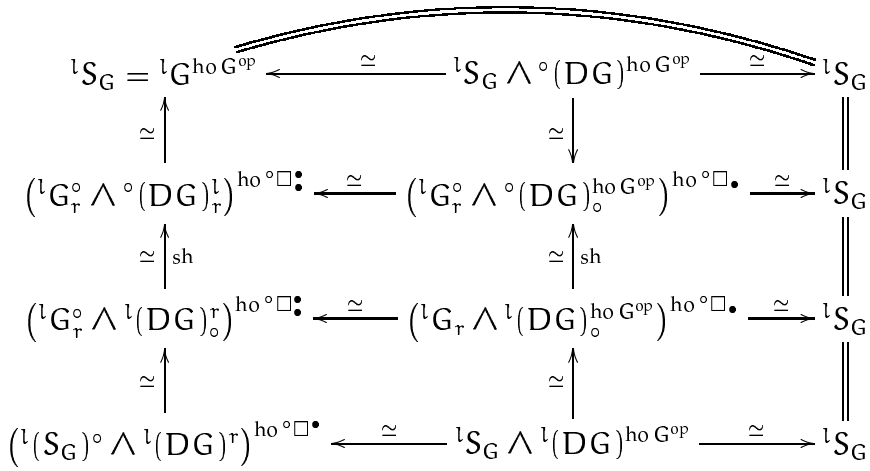
This map is clearly a weak equivalence, and it is straightforward to check that it is $(G \times G^{op} \times G^{op})$-equivariant as claimed.

By passing to homotopy orbits with respect to the ${}^{\circ}\square^{\circ}_{\bullet}$ action of G^{op} in (5.9), we obtain a $(G \times G^{op})$-equivariant homotopy equivalence

$$ \begin{array}{ccc} \left({}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{l}(DG_{+})^{r}_{\circ}\right)^{hoG^{op}} & \longrightarrow & \left({}^{l}(\Sigma^{\infty}G_{+})^{\circ}_{r} \wedge {}^{\circ}(DG_{+})^{l}_{r}\right)^{hoG^{op}} =: E'(G) \\ \simeq \uparrow & & \simeq \downarrow \\ {}^{l}(S_{G})^{\circ} \wedge {}^{l}(DG_{+})^{r} & & {}^{l}(\Sigma^{\infty}G_{+})^{r} \end{array} $$

The underlying spectrum of $E'(G)$ is the spectrum $E(G)$ of Lemma 5.4. It is easy to see that with the remaining operations, the map $E'(G) \to \Sigma^{\infty}G_{+}$, described in (5.7), is $(G \times G^{op})$-equivariant.

For the assertion about G^{op}-homotopy orbits, observe that by changing the order of taking G^{op}-homotopy orbits, we have a large commutative diagram



For space reasons, the disjoint basepoint for $\Sigma^\infty G$ and DG have been omitted as well as the suspension functor Σ^∞ for G.

The important, if trivial, observation is that the shear map becomes homotopic to the identity when passing to $^\circ\square^\bullet_\circ$-homotopy orbits on the DG_+ factor. The diagram claimed to be commutative in the proposition is the "boundary" of the diagram above. □

5.3 Relative Poincaré duality

Corollary 5.10. *For any sub-*p*-compact group* H *of* G*, there is a zigzag of coarse* G*-equivalences*

$$G \wedge_H S_H \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} D(G/H_+) \wedge S_G$$

This zigzag is natural in the following sense: for any chain of p*-compact groups* $K < H < G$*, the following diagram commutes:*

$$\begin{array}{ccccc} (\Sigma^\infty G_+)^{h\,H^{op}} & \xleftarrow{\simeq} & G \wedge_H S_H & \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} & D(G/H_+) \wedge S_G \\ \downarrow{\scriptstyle res} & & & & \downarrow{\scriptstyle D(proj)\wedge id} \\ (\Sigma^\infty G_+)^{h\,K^{op}} & \xleftarrow{\simeq} & G \wedge_K S_K & \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} & D(G/K_+) \wedge S_G \end{array}$$

Proof: From Proposition 5.8, we have a coarse $(G \times H^{op})$-equivalence

$$DG_+ \wedge S_G \to \Sigma^\infty G_+.$$

Applying H^{op}-homotopy orbits turns coarse $(G \times H^{op})$-equivalences into coarse G-equivalences, and since the right actions on S_G and S_H are trivial, we obtain

$$\begin{array}{ccccc} (DG_+ \wedge S_G)^{hoH^{op}} & \xleftarrow{\simeq} & DG_+^{hoH^{op}} \wedge S_G & \xrightarrow{\simeq} & D(G/H_+) \wedge S_G \\ \downarrow \simeq & & & & \\ (\Sigma^\infty G_+)^{hoH^{op}} & \xleftarrow[\simeq]{\text{Lemma 5.1}} & G \wedge_H S_H & & \end{array}$$

For naturality, consider the following diagram:

$$\begin{array}{ccccccc} D(G/H_+) \wedge S_G & \longleftarrow & DG_+^{hoH^{op}} \wedge S_G & \longrightarrow & (DG_+ \wedge S_G)^{hoH^{op}} & \longrightarrow & (\Sigma^\infty G_+)^{hoH^{op}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D(G/K_+) \wedge S_G & \longleftarrow & DG_+^{hoK^{op}} \wedge S_G & \longrightarrow & (DG_+ \wedge S_G)^{hoK^{op}} & \longrightarrow & (\Sigma^\infty G_+)^{hoK^{op}} \end{array}$$

The left hand square commutes by Lemma 5.3, the other two for trivial reasons. □

5.4 Definition of the transfer

Proof of Theorem 1.3: S_G was constructed in Chapter 4. We obtain a (nonequivariant) map

$$\overline{t} : S_G = (\Sigma^\infty G_+)^{hoG^{op}} \longrightarrow (\Sigma^\infty G_+)^{hoH^{op}} \xrightarrow[f]{\simeq} G \wedge_H S_H$$

coming from restricting from G- to H-homotopy fixed points. Here, f is the non-equivariant homotopy inverse of the coarse G-equivalence given by Lemma 5.1.

By Lemma 2.2, there is also a G-equivariant map

$$\tilde{t} : EG_+ \wedge S_G \longrightarrow G \wedge_H S_H$$

such that the composite

$$S_G \to EG_+ \wedge S_G \to G \wedge_H S_H$$

is homotopic to $\tilde{t}$, and $\tilde{t}$ is unique up to homotopy with this property.

To finish the proof of Theorem 1.3, we need to show that

$$\tilde{t}_* : H_d(S_G; \mathbf{F}_p) \to H_d(G \wedge_H S_H; \mathbf{F}_p)$$

is an isomorphism for $d = \dim G$. This now follows easily from Corollary 5.10: The map $\tilde{t}$ is by construction the composite

$$S_G \to D(G/H_+) \wedge S_G \xrightarrow{\sim} G \wedge_H S_H.$$

Since the first map is an isomorphism in $H_d(-; \mathbf{F}_p)$, so is the composite. □

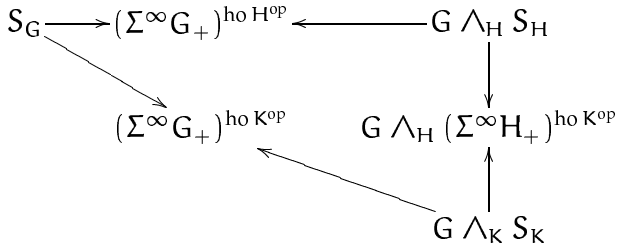
The first part of Theorem 1.4 claims that $H^*(BG^\wedge_p; \mathbf{F}_p)$ is a Thom module over $H^*(BG_+; \mathbf{F}_p)$. This follows from the spectral sequence

$$E_2 = H^*(BG_+; H^*(S_G; \mathbf{Z}_p)) \Longrightarrow H^*(EG_+ \wedge_G S_G; \mathbf{F}_p) = H^*(BG^\wedge_p; \mathbf{Z}_p).$$

□

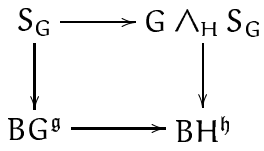
Definition. For a monomorphism $H < G$ of p-compact groups, the transfer map $t_{G,H}$ is given by applying G-homotopy orbits to the G-equivariant map $\tilde{t}$.

The domain of $\tilde{t}$ is $EG_+ \wedge_G (EG_+ \wedge S_G)$, which by Lemma 2.2 is homotopy equivalent to $BG^\wedge_p$. For the functoriality, it is sufficient to notice that the following diagram of G-equivariant maps commutes:



That is a less than remarkable statement since no two maps are composable. But all of the maps going left or up or both are coarse G-equivalences, and the diagram stays (non-equivariantly) homotopy commutative if we invert them.

By its definition, the commutativity of



as claimed in Theorem 1.4 is immediate.

The next chapter will be devoted to identifying the transfer map on the category of $H\mathbb{Z}/p$-localizations of compact Lie groups and monomorphisms.

Chapter 6

Identification of the transfer map

Using the construction of t for $\{1\} < G$, we have a commutative diagram coming from the natural transformation $\mathrm{id} \to (-)_{hoG}$:

$$\begin{array}{ccccc} S_G & \longrightarrow & \Sigma^\infty G_+ & & \\ \downarrow & & \downarrow & & \\ BG^g & \xrightarrow{t} & B\{1\}_+ & = & \mathbf{S}^0 \end{array} \tag{6.1}$$

We will now identify t with the transfer map from the introduction in the case where G is of Lie type.

First note that in (6.1), the composite map $\mathbf{S}^d \to \mathbf{S}^0$ is indeed the same as the Thom-Pontryagin construction on G if G is a Lie group: The Thom-Pontryagin construction on G is given by the composition of maps

$$\mathbf{S}^0 \to DG_+ \simeq S^{-d} \wedge G_+ \to \mathbf{S}^{-d},$$

where the first map is a desuspension of the map from the embedding sphere to the Thom space of the normal bundle of G, which is DG; by Proposition 5.8 or since the tangent bundle of G is trivial, this is equivalent to a desuspension of $\Sigma^\infty G_+$; and the second map is the projection of G_+ to $\mathbf{S}^0$, the map classifying the (trivial) stable normal bundle of G.

Using the G-equivariant isomorphism from Proposition 5.8, we have

$$\begin{array}{ccccc} S_G & \longrightarrow & \Sigma^\infty G_+ & \xrightarrow{(-)_{\mathrm{ho}\,G}} & \mathbf{S}^0 \\ & \searrow & \downarrow \wr & & \\ & & S_G \wedge DG_+ & & \end{array}$$

The bottom composition is the Thom-Pontryagin construction, the upper one the map from (6.1).

6.1 An alternative construction of the transfer map

To show that t agrees with the Umkehr map not only on the bottom cell, we will compare its definition to another, equally general construction, reminiscent of Dwyer's construction of the Becker-Gottlieb transfer in [Dwy96]. This will be equivalent to the classical construction of the Umkehr map in the Lie case.

Let $H < G$ be p-compact groups. The quotient $\Sigma^\infty G/H_+$ is dualizable, and the projection $D(G/H_+) \to \mathbf{S}^0$ onto the top cell is equivariant and has a section α; however, α is not G-equivariant unless $H = G$. But we do get an equivariant map if we "free up" the G-action on $\mathbf{S}^0$: consider the following diagram of G-equivariant maps:

$$\begin{array}{ccccc} \Sigma^\infty EG_+ & \xrightarrow{\sim} & \mathbf{S}^0 & \longrightarrow & \mathrm{map}(\Sigma^\infty G/H_+, \Sigma^\infty G/H_+) \\ & & & & \uparrow{\scriptstyle \eta}\,{\scriptstyle \simeq} \\ & & & & D(G/H)_+ \wedge \Sigma^\infty G/H_+ \xrightarrow{1\wedge\pi} D(G/H)_+ \wedge \mathbf{S}^0. \end{array}$$

The map π is the projection $\Sigma^\infty G/H_+ = \Sigma^\infty G/H \vee \mathbf{S}^0 \to \mathbf{S}^0$, and η is a coarse G-equivalence. Therefore, by Lemma 2.2, there exists an equivariant lifting

$$\Sigma^\infty EG_+ \to D(G/H_+) \tag{6.2}$$

which is nonequivariantly homotopic to the map

$$\Sigma^\infty EG_+ \to \mathbf{S}^0 \xrightarrow{\alpha} D(G/H_+).$$

In the case where H and G are (localizations of) Lie groups, the homotopy orbit space of $D(G/H_+)$ under this G-action is the (localization of the) Thom space of ν, the normal bundle along the fibers of $BH \to BG$. This follows from the observation that $BH = EG \times_G G/H$, and that stably, the normal bundle along the fiber is the fiberwise Spanier-Whitehead dual, i.e. $\nu = EG \times_G D(G/H)$. Hence its Thom spectrum is $EG_+ \wedge_G D(G/H_+)$, as claimed.

By passage to G-homotopy orbits in (6.2), we therefore obtain a map

$$\Sigma^\infty BG_+ \simeq EG_+ \wedge_G \Sigma^\infty EG_+ \to BH^{-\mathfrak{g}/\mathfrak{h}},$$

where $BH^{-\mathfrak{g}/\mathfrak{h}}$ denotes the Thom spectrum of the virtual inverse of the adjoint bundle of G, pulled back to BH, modulo the adjoint bundle of H. This is the Lie theoretic model of the normal bundle along the fibers.

Returning to the case of a general p-compact group, we now introduce a "twisting" by smashing source and target of the map with S_G:

$$EG_+ \wedge S_G \longrightarrow \mathrm{map}(\Sigma^\infty G/H_+, \mathbf{S}^0) \wedge S_G \xrightarrow{\simeq} D(G/H_+) \wedge S_G \underset{\text{Cor. 5.10}}{\overset{\simeq}{\dashleftarrow\!\dashrightarrow}} G \wedge_H S_H$$

By Lemma 2.2 and since $EG_+ \wedge S_G$ is a free G-spectrum, we obtain a G-map (unique up to homotopy)

$$\tilde{t}' : EG_+ \wedge S_G \to G \wedge_H S_H,$$

and passing to G-homotopy orbits, we obtain:

$$t' : BG^{\mathfrak{g}} = EG_+ \wedge_G S_G \to EH_+ \wedge_H S_H = BH^{\mathfrak{h}}.$$

Lemma 6.3. *Let $H < G$ be p-compact groups. Then*

$$\tilde{t} \simeq \tilde{t}' : \ EG_+ \wedge S_G \longrightarrow G \wedge_H S_H$$

Proof: We have to show that the following G-equivariant diagram commutes:

$$\begin{array}{ccc} EG_+ \wedge S_G & \xrightarrow{(6.2)} & D(G/H_+) \wedge S_G \\ \downarrow{\scriptstyle \tilde{t}} & & \uparrow{\scriptstyle \sim} \\ G \wedge_H S_H & \overset{\sim}{\dashleftarrow\!\dashrightarrow} & (DG_+ \wedge S_G)^{\mathrm{ho}\, H^{\mathrm{op}}} \end{array}$$

Since $EG_+ \wedge S_G$ is a free G-spectrum, there is by Lemma 2.2 a G-equivariant map going diagonally

$$EG_+ \wedge S_G \to (DG_+ \wedge S_G)^{h_o H^{op}}$$

and making the upper right triangle of the diagram commute. The commutativity of the lower left triangle then follows from the observation that in the commutative diagram

$$\begin{array}{ccccc} S_G & \xleftarrow{\sim} & (DG_+ \wedge S_G)^{h_o G^{op}} & \longrightarrow & (DG_+ \wedge S_G)^{h_o H^{op}} \\ \downarrow & & \downarrow & & \downarrow \\ S_G & = & (\Sigma^\infty G_+)^{h_o G^{op}} & \longrightarrow & (\Sigma^\infty G_+)^{h_o H^{op}} \end{array},$$

the left hand map is the identity by Proposition 5.8. □

Conclusion of the proof of Theorem 1.4: The previous lemma implies the third part of the theorem (namely, that $t \circ L_p \simeq L_p \circ (-)_!$ on compact Lie groups and monomorphisms). Indeed, by applying G-homotopy orbits to the diagram in Lemma 6.3, the map induced by $\tilde{t}'$ is homotopic to t, and the preceding discussion shows that the former map is the classical Umkehr map in the Lie case. □

Chapter 7

Computational methods

In this chapter, I will describe a general method for computing the homotopy class represented by a p-compact group by constructing a representing cycle in the Adams spectral sequence for a complex oriented cohomology theory E. Let G be a simply connected d-dimensional p-compact group of rank r with maximal torus T. We want to identify the maps the following diagram induces in the E_2-term for the E-cohomology ASS:

$$\mathbf{S}^d \to BG^g \to BT^t \to \mathbf{S}^0$$

7.1 The S^1-transfer

The right-hand map is a suspension of the r-fold smash product of the $\mathbf{S}^1$-transfer map

$$\tau : \mathbf{C}P^\infty_+ \to \mathbf{S}^{-1}.$$

It is well-known that the homotopy fiber of this transfer map is the spectrum $\mathbf{C}P^\infty_{-1}$, the Thom spectrum of the inverse of the universal line bundle on $\mathbf{C}P^\infty$, the fiber inclusion $\mathbf{C}P^\infty_{-1} \to \mathbf{C}P^\infty_+$ being the obvious projection map onto the 0-coskeleton.

For a complex oriented cohomology theory E and a finite spectrum X, there is an Adams-Novikov spectral sequence

$$E_2 = \mathrm{Ext}(E^*(X)) \Longrightarrow [X, L_E\mathbf{S}],$$

where, for an (E_*, E_*E)-comodule A, $\mathrm{Ext}(A)$ is a shorthand for $\mathrm{Ext}_{E_*E}(E_*, A)$.

We will now first restrict to the case of finite dimensional projective spaces and study the map $\mathbf{C}P^m_+ \to \mathbf{S}^{-1}$ as a map of E_2-terms of this ANSS

$$\begin{array}{ccc} \mathrm{Ext}(E^*(\mathbf{S}^{-1})) & \Longrightarrow & \pi_*(L_E\mathbf{S}^1) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(E^*(\mathbf{C}P^m_+)) & \Longrightarrow & [\mathbf{C}P^m_+, L_E\mathbf{S}]. \end{array}$$

By a change of rings isomorphism, this spectral sequence is isomorphic to the one associated to the Hopf algebroid

$$(A_m, \Gamma_m) = (E^*(\mathbf{C}P^m_n), (E \wedge E)^*(\mathbf{C}P^m_n)).$$

Note that A_m represents the following functor:

$$R \mapsto \left\{ (\alpha, f) \;\middle|\; \begin{array}{l} E^* \xrightarrow{\alpha} R, \\ f \text{ is a function modulo degree } m+n+1 \text{ on the formal group} \\ \text{law on } R \text{ given by the image of the universal formal group law} \\ \text{under } MU^* \to E^* \to R \text{ such that } f \text{ vanishes to the } n\text{th order at} \\ \text{the identity.} \end{array} \right\}$$

Similarly, Γ_m represents isomorphisms of such data. Hence, for $E = MU$, (A_m, Γ_m) classifies formal groups with an $m+n+1$-truncated function on it that vanishes to the nth order at the identity. This interpretation makes it easy to compute the structure maps of (A_m, Γ_m).

First assume that $n = 0$. Pick coordinates z such that

$$E^*(\mathbf{C}P^m_0) = E^*[\![z]\!]/(z^{m+1}) \quad \text{and} \quad (E \wedge E)^*(\mathbf{C}P^\infty_0) = (E \wedge E)^*[\![z]\!]/(z^{m+1}).$$

Since there is a map of Hopf algebroids $(E_*, E_*E) \to (A_m, \Gamma_m)$, we only need to determine $\eta_L(z)$ and $\eta_R(z)$. We can make $\eta_L(z) = z$ by choice of coordinates; then, $\eta_R(z)$ will be the image of the universal isomorphism

$$\sum_{i=0}^{m-1} b_i z^{i+1} \in (MU \wedge MU)^*(\mathbf{C}P^m_0)$$

in Γ_m.

As usual, $MU_* = \mathbf{Z}[m_i]$ and $MU_*MU = MU_*[b_i]$.

If $n \neq 0$, $E^*(\mathbf{C}P^m_n)$ is still a free E^*-module, generated by $\{z^n, z^{n+1}, \ldots, z^{n+m}\}$, and the above formula for η_R is correct when interpreted as $\eta_R(z^i) = \eta_R(z)^i$.

For our purposes, it would be easier to use BP-theory instead of MU since we are working in a p-local setting anyway. However, it only affects the complexity of the computations, not the method.

Assembling all spectral sequences for varying $m \geq 0$, we obtain towers

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Ext}_{\Gamma_{m+1}}(A_{m+1}, A_{m+1}) & \Longrightarrow & [\mathbf{C}P^{m+1}_n, L_E\mathbf{S}] \\
\downarrow & & \downarrow \\
\mathrm{Ext}_{\Gamma_m}(A_m, A_m) & \Longrightarrow & [\mathbf{C}P^m_n, L_E\mathbf{S}] \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

The inverse limit of the left tower is not quite the Ext term associated to the Hopf algebroid $(A, \Gamma) = (E^*(\mathbf{C}P^\infty), (E \wedge E)^*(\mathbf{C}P^\infty))$. This is due to the fact that

$$(E \wedge E \wedge E)^*(\mathbf{C}P^\infty) \overset{\approx}{\leftarrow} \Gamma \otimes_A \Gamma$$

(the left hand side is a completion of the right hand side).

Similarly, the inverse limit of the tower on the right hand side is not quite $[\mathbf{C}P^\infty_n, L_E\mathbf{S}]$. It does not include the phantom maps.

Coming back to the problem of determining the induced map of the $\mathbf{S}^1$-transfer on E_2-terms, we look at the cobar construction functor

$$B^n(M) = M \otimes_{E_*} (E_*E)^{\otimes_{E_*} n}$$

for an (E_*, E_*E)-comodule M.

Since E_*, E_*E, and $H^*(\mathbf{C}P^\infty)$ are concentrated in even dimensions, and since B is an exact functor on flat E_*-modules, we have a short exact sequence of $B(E_*, E_*E)$-modules

$$0 \leftarrow B(E^*\Sigma^{-1}\mathbf{S}^{-1})) \leftarrow B(E^*(\mathbf{C}P^\infty_{-1})) \leftarrow B(E^*(\mathbf{C}P^\infty_0)) \leftarrow 0.$$

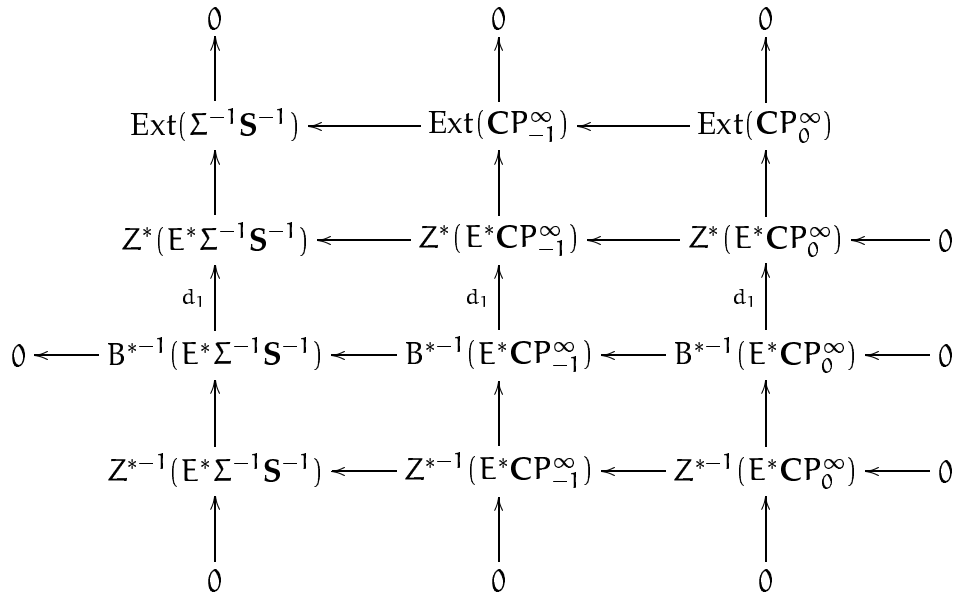


Figure 7.7-1: This diagram admits a snake map.

If we denote by $Z^*(X) \subseteq B^*(X)$ the cycles under the cosimplicial differential d_1, we have a diagram as shown in Figure 7.7-1. It follows from the snake lemma that the kernel of the top right map is the image under the snake map

$$Z^{*-1}(E^*\mathbf{S}^{-2}) \to \mathrm{Ext}(\mathbf{C}P^{\infty}_{0}),$$

which is, by following through the diagram, the image of $d_1|_{E^*\{z^{-1}\}}$.

Now if T is a p-compact torus, the transfer map in cohomotopy is simply the r-fold smash product of the map represented at the E_2-level by $d|_{E^*\{z^{-1}\}}$.

7.2 The map $S^d \to BG^g \to \Sigma^r BT$

We will first study the effect of this map on rational cohomology. By Theorem 3.2, $H^*_{\mathbb{Q}_p}(BG) = H^*_{\mathbb{Q}_p}(BT)^{W(G)}$ is always a polynomial algebra.

Proposition 7.1. *For a p-compact groups G with maximal torus T, the following diagram*

commutes:

$$\begin{array}{ccccc} H^*_{\mathbf{Q}_p}(BT_+) & \xrightarrow{proj} & H^*_{\mathbf{Q}_p}(BT_+)\ /\ (H^*_{\mathbf{Q}_p}(BG)) & \underset{Cor.\ 3.3}{=\!=} & H^*_{\mathbf{Q}_p}(G/T_+) \\ \downarrow{\scriptstyle \Sigma^{-r}t^*} & & & & \downarrow \\ H^*_{\mathbf{Q}_p}(\Sigma^{-r}BG^g) & \xrightarrow[\cong]{} & H^*_{\mathbf{Q}_p}(BG_+)\{\tau\} & \xrightarrow[\tau\mapsto\iota]{} & H^*_{\mathbf{Q}_p}(\mathbf{S}^{d-r}), \end{array}$$

where τ *is the Thom class of* BG^g*, and* ι *is the generator in* $H^*(\mathbf{S}^{d-r})$.

This allows us to compute the effect of the map

$$\mathbf{S}^d \to BG^g \to \Sigma^r BT\mathrm{fiber}$$

in cohomology (this is the bottom composition of maps in the diagram) by simply evaluating at the image in $H^*_{\mathbf{Q}_p}(BT_+)\ /\ (H^*_{\mathbf{Q}_p}(BG))$ of the fundamental class of G/T.

Proof of the proposition:

By the construction of the transfer map, we have a commutative diagram

$$\begin{array}{ccc} G \wedge_T \mathbf{S}^r & \longrightarrow & \Sigma^r BT_+ \\ \uparrow & & \uparrow \\ \mathbf{S}^d & \longrightarrow & BG^g \end{array},$$

and by Theorem 1.3(2), the left hand map $\mathbf{S}^d \to \Sigma^r G/T_+$ is an isomorphism in the top homology group. Desuspending r times and applying $H^*_{\mathbf{Q}_p}$ yields the commutativity of the diagram of the proposition. □

Now let E be a $H\mathbf{Z}/p$-local complex oriented torsion free cohomology theory. Denote by $E_{\mathbf{Q}_p}$ the cohomology theory $X \mapsto E^*(X) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We have $E^*(\mathbf{C}P^\infty) = E^*[\![z]\!] \hookrightarrow E^*_{\mathbf{Q}_p}(\mathbf{C}P^\infty)$, and the same is true for $E \wedge E$. Hence for computing a cobar representative, we can work with rational coefficients and always hope that in the end of our computations, everythings turns out integral. To compute this, we can use the Chern characters

$$\exp : E^*_{\mathbf{Q}_p}(X) \to H^*_{\mathbf{Q}_p}(X) \hat{\otimes} E^*$$

and

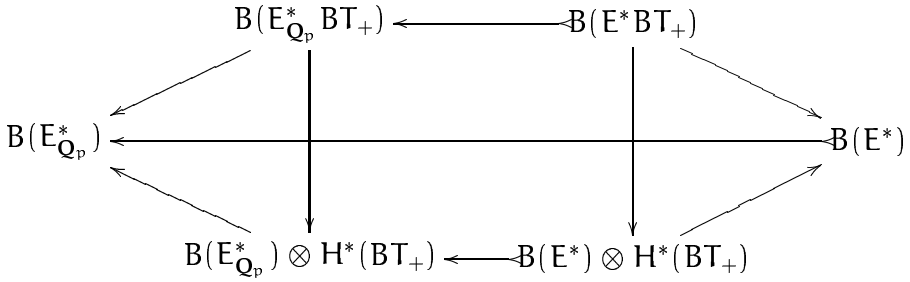
$$\exp : (E \wedge E)^*_{\mathbf{Q}_p}(X) \to H^*_{\mathbf{Q}_p}(X) \hat{\otimes} \pi_*(E \wedge E)$$

which, for $X = \mathbf{C}P^\infty_+$, is the exponential map for the formal group law associated with E and an isomorphism, and for $X = BT_+$, a tensor power thereof. The smash product is formed in the $H\mathbf{Z}/p$-local category, as always.

This induces an isomorphism of $(E_{Q_p *}, (E \wedge E)_{Q_p *})$-modules

$$B(\exp) : B(E^*_{Q_p} BT_+) \to B(E^*) \hat{\otimes} H^*_{Q_p}(BT_+).$$

We have a commutative diagram



So, to evaluate the class in $B(E^*(BT))$ computed in the first part, we apply $B(\exp)$ to it and obtain a class in $B(E^*) \otimes H^*_{Q_p}(BT)$, which we then evaluate at the image of the fundamental class $[G/H]$ in $(H_{Q_p})_{d-r}(G/H_+) \to (H_{Q_p})_{d-r}(BT_+)$. This class then must actually be integral.

Chapter 8

The family no. 3 of groups μ_m

The p-compact group μ_m, for $p \cong 1 \ (m)$, has rank 1 and Weyl group $W \leq \mathbf{Z}_p^\times$ a cyclic subgroup of order m of the p-adic units. It is a nonmodular group and can therefore be constructed as

$$B\mu_m = L_p\,(K(\mathbf{Z}_p, 2) \times_W EW).$$

Therefore, $H^*(B\mu_m; \mathbf{Z}_p) = \mathbf{Z}_p[\![z]\!]^W$, where a W acts on z by multiplication. This shows that

$$H^*(B\mu_m; \mathbf{Z}_p) = \mathbf{Z}_p[\![z^m]\!] \hookrightarrow \mathbf{Z}_p[\![z]\!].$$

The fundamental class

$$[\mu_m/T] \in H^*_{\mathbf{Q}_p}(BT)/(H^*_{\mathbf{Q}_p}(B\mu_m)) = \mathbf{Q}_p[z]/(z^m)$$

is z^{m-1}, and we conclude that μ_m has dimension $1 + 2(m-1)$. It is straightforward to see that for $m < p-1$, μ_m cannot represent a nontrivial homotopy class in the p-stems because $(\pi^s_n)_{(p)} = 0$ for $0 < n < 2p-3$. But $(\pi^s_{2p-3})_{(p)} = \mathbf{Z}/p\{\alpha_1\}$, and we will see that μ_{p-1} represents this class.

In the p-completed BP-spectral sequence for $\pi^*(\mathbf{CP}^\infty_{-1})$,

$$\eta_R(z) = z + t_1 z^p + O(z^{p+1}), \quad \text{hence } \eta_R(z^{-1}) = z^{-1} - t_1 z^{p-2} + O(z^{p-1}).$$

Applying the Chern character to this power series does not change it up to $O(z^{p-1})$, and hence $[\mu_{p-1}]$ is the coefficient of z^{p-2} of this series, which is t_1. Lying in filtration 1, t_1 represents the homotopy class α_1.

Chapter 9

Some exceptional cases

9.1 The 5-compact group no. 8

The pseudo-reflection group G which is no. 8 in Shephard and Todd's list has order 96 and is generated, as a complex reflection group, by the two reflections $\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{pmatrix}$. The ring of invariants $\mathbf{Z}_5[x_1, x_2]^G$ is polynomial because 5 does not divide the order of G; a straightforward calculation shows that it is generated by the polynomials

$$\mu = x_1^8 + 14x_1^4x_2^4 + x_2^8$$

and

$$\nu = x_1^{12} - 33x_1^8x_2^4 - 33x_1^4x_2^8 + x_2^{12}.$$

Hence $H^*(BG; \mathbf{Z}_5) = \mathbf{Z}_5[\mu, \nu]$, and by Proposition 3.5 the cohomology of G/T is given by

$$H^*(G/T_+; \mathbf{Z}_5) = H^*(BT_+; \mathbf{Z}_5) \;/\; (H^*(BG; \mathbf{Z}_5)).$$

A Gröbner basis calculation shows that the top class in $H^{36}(G/T; \mathbf{Z}_5)$ is

$$x_1^7x_2^{11} = -x_1^{11}x_2^7 = -\frac{1}{13}x_1^3x_2^{15} = \frac{1}{13}x_1^{15}x_2^3. \tag{9.1}$$

We will use the 5-primary BP-spectral sequence for $\pi^*(\mathbf{C}P^\infty_{-1})$ to determine the homotopy class G represents. In this spectral sequence,

$$\begin{aligned}\eta_R(z) = z - t_1 z^5 + \left(5 {t_1}^2 + t_1 v_1\right) z^9 + \left(-35 {t_1}^3 - 12 {t_1}^2 v_1 - t_1 {v_1}^2\right) z^{13} \\ + \left(285 {t_1}^4 + 137 {t_1}^3 v_1 + 21 {t_1}^2 {v_1}^2 + t_1 {v_1}^3\right) z^{17} \\ + O(z^{21})\end{aligned}$$

and hence

$$\begin{aligned}d_1(z^{-1}) = -\left(t_1 z^3\right) + \left(4 {t_1}^2 + t_1 v_1\right) z^7 + \left(-26 {t_1}^3 - 10 {t_1}^2 v_1 - t_1 {v_1}^2\right) z^{11} \\ + \left(204 {t_1}^4 + 106 {t_1}^3 v_1 + 18 {t_1}^2 {v_1}^2 + t_1 {v_1}^3\right) z^{15} \\ + O(z^{19}).\end{aligned}$$

Applying the Chern character to this class yields

$$\begin{aligned}f(z) = -t_1 z^3 + \left(4 {t_1}^2 + \frac{8 t_1 v_1}{5}\right) z^7 + \left(-26 {t_1}^3 - \frac{78 {t_1}^2 v_1}{5} - \frac{78 t_1 {v_1}^2}{25}\right) z^{11} \\ + \left(204 {t_1}^4 + \frac{816 {t_1}^3 v_1}{5} + \frac{1224 {t_1}^2 {v_1}^2}{25} + \frac{816 t_1 {v_1}^3}{125}\right) z^{15} \\ + O(z^{19}).\end{aligned}$$

We need to evaluate the class $f(z) \otimes f(z)$ at the classes given in (9.1) and add them up. This yields:

$$\begin{aligned}[G] \quad &= -204 t_1 \otimes t_1^4 - 808 t_1^2 \otimes t_1^3 - 1208 t_1^3 \otimes t_1^2 - 604 t_1^4 \otimes t_1 \\ &\quad -160 v_1 t_1 \otimes t_1^3 - 480 v_1 t_1^2 \otimes t_1^2 - 320 v_1 t_1^3 \otimes t_1 \\ &\quad -48 v_1^2 t_1 \otimes t_1^2 - 48 v_1^2 t_1^2 \otimes t_1.\end{aligned}$$

By adding a suitable boundary, namely

$$d_1(4 t_1^5 + 45 v_1 t_1^4 + 34 v_1^2 t_1^3 + 10 v_1^3 t_1^2 + v_1^4 t_1),$$

we see that this class is homologous to

$$t_1 \otimes t_1^4 + t_1^4 \otimes t_1 + 2(t_1^2 \otimes t_1^3 + t_1^3 \otimes t_1^2),$$

which is the representative of β_1 in the ANSS.

9.2 The 3-compact group Za_2 (no. 12)

The Weyl group W of the modular group Za_2 constructed by Zabrodsky [Zab84] is generated by the two matrices

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2}+\frac{i}{2} & \frac{1}{2}+\frac{i}{2} \\ -\frac{1}{2}+\frac{i}{2} & \frac{1}{2}-\frac{i}{2} \end{pmatrix}.$$

Although $3 \mid \#W$, the ring of invariants $\mathbf{Z}_3[x_1, x_2]^W$ is polynomial, generated by the polynomials

$$\mu = x_1^8 + 14x_1^4x_2^4 + x_2^8$$

and

$$\nu = x_1^5x_2 - x_1x_2^5.$$

We find that the top class in $H^{36}(G/T_+; \mathbf{Z}_3)$ is

$$x_1^4x_2^8 = x_1^8x_2^4 = -\frac{1}{15}x_1^{12} = \frac{1}{15}x_2^{12}. \tag{9.2}$$

In the 3-primary BP-ANSS, logarithm, exponential, and universal isomorphism are all odd power series; hence, evaluation at the class above yields 0 without further computations.

This means that $[Za_2]$ is of filtration at least 3; however, the only 38-dimensional class in the Adams-Novikov E_2-term is $\beta_{3/2}$ in filtration 2.

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