GRADED p-POLAR RINGS AND THE HOMOLOGY OF $\Omega^n\Sigma^n X$

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Abstract. As an extension of previous ungraded work, we define a graded p-polar ring to be an analog of a graded commutative ring where multiplication is only allowed on p-tuples (instead of pairs) of elements of equal degree. We show that the free affine p-adic group scheme functor, as well as the free formal group functor, defined on k-algebras for a perfect field k of characteristic p, factors through p-polar k-algebras. It follows that the same is true for any affine p-adic or formal group functor, in particular for the functor of p-typical Witt vectors. As an application, we show that the homology of the free E_n-algebra $H^*(\Omega^n\Sigma^n X;\mathbf{F}_p)$, as a Hopf algebra, only depends on the p-polar structure of $H^*(X;\mathbf{F}_p)$ in a functorial way.

1. Introduction

In [Bau21], I introduced the notion of a p-polar k-algebra, which, roughly speaking, is a k-module with a p-fold associative and commutative multiplication defined on it. Here p is a prime and k is any commutative ring. If k is a perfect field of characteristic p, I showed that the free affine abelian p-adic group functor on $\operatorname{Spec} R$ for a k-algebra R factors through the category of p-polar k-algebras, and as a result, so does the functor of points for *every* p-adic group defined over k.

In this sequel, I prove the corresponding results for graded commutative k-algebras, where k is an (ungraded) perfect field of characteristic p. Both the definition of a graded p-polar k-algebra and the proofs are quite distinct, but not independent, from the ungraded case, and the results are more striking in the presence of a grading. This is demonstrated by applications concerning the mod-p homology of $\Omega^n\Sigma^n X$.

In this paper, all graded objects are understood to be nonnegatively graded.

Definition. Let k be an ungraded ring and Alg_k the category of graded-commutative k-algebras. Let M_k denote the category of graded k-modules A together with a graded symmetric k-multilinear map $\mu\colon A_j^{\otimes_k p} \to A_{jp}$, and let $\mathrm{pol}_p\colon \mathrm{Alg}_k \to M_k$ denote the forgetful functor from graded commutative k-algebras to M_k, where μ is given by p-fold multiplication.

A *graded p-polar k-algebra* is an object $A \in M_k$ which is a subobject of $\mathrm{pol}_p(B)$ for some algebra $B \in \mathrm{Alg}_k$.

This definition agrees with the one given in [Bau21] when A is concentrated in degree 0 (Lemma 2.8).

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We denote the category of graded p-polar k-algebra by $\mathrm{Pol}_p(k)$. We also let alg_k and $\mathrm{pol}_p(k)$ denote the full subcategories of objects that are finite-dimensional as k-vector spaces in Alg_k and $\mathrm{Pol}_p(k)$, respectively.

The restriction functors $\mathrm{pol}_p\colon \mathrm{Alg}_k \to \mathrm{Pol}_p(k)$ and $\mathrm{pol}_p\colon \mathrm{alg}_k \to \mathrm{pol}_p(k)$ are called *polarization*.

Our algebraic main results parallel those in [Bau21]. Let AbSch_k denote the category of representable, abelian-group-valued functors on Alg_k, and let AbSch_k^p denote the full subcategory of functors taking values in abelian pro-p-groups. Let Fgps_k be the category of ind-representable functors on alg_k taking values in abelian groups. We will refer to objects of AbSch_k and Fgps_k as affine and formal groups, respectively.

Theorem 1.1. *Let k be a perfect field of characteristic p. Then the forgetful functors $\mathrm{AbSch}_k^p \to \mathrm{Alg}_k^{\mathrm{op}}$ resp. $\mathrm{Fgps}_k \to (\mathrm{Pro}-\mathrm{alg}_k)^{\mathrm{op}}$ have left adjoints Fr and these factor through pol:*

$$\begin{array}{ccc} \mathrm{Alg}_k^{\mathrm{op}} & \xrightarrow{\mathrm{Fr}} & \mathrm{AbSch}_k^p; \\ & \searrow^{\mathrm{pol}} & \uparrow{\scriptstyle \tilde{\mathrm{Fr}}} \\ & & \mathrm{Pol}_p(k)^{\mathrm{op}} \end{array} \qquad \begin{array}{ccc} \mathrm{alg}_k^{\mathrm{op}} & \xrightarrow{\mathrm{Fr}} & \mathrm{Fgps}_k \\ & \searrow^{\mathrm{pol}} & \uparrow{\scriptstyle \tilde{\mathrm{Fr}}} \\ & & \mathrm{pol}_p(k). \end{array}$$

Corollary 1.2. *Let k be a perfect field of characteristic p and $M \in \mathrm{AbSch}_k^p$ or $M \in \mathrm{Fgps}_k$. Then M factors naturally through pol:*

$$\begin{array}{ccc} \mathrm{Alg}_k & \xrightarrow{M} & \{\textit{abelian pro-}p\textit{-groups}\}; \\ & \searrow^{\mathrm{pol}} & \uparrow{\scriptstyle \tilde{M}} \\ & & \mathrm{Pol}_p(k) \end{array} \qquad \begin{array}{ccc} \mathrm{alg}_k & \xrightarrow{M} & \mathrm{Ab} \\ & \searrow^{\mathrm{pol}} & \uparrow{\scriptstyle \tilde{M}} \\ & & \mathrm{pol}_p(k). \end{array}$$

Proof. Given any $M \in \mathrm{AbSch}_k$ or $M \in \mathrm{Fgps}_k$, we have that

$$M(R) = \mathrm{Hom}(\operatorname{Spec} R, M) = \mathrm{Hom}(\mathrm{Fr}(R), M),$$

where the last Hom group is of objects of AbSch_k^p or Fgps_k, respectively. The Corollary now follows from Thm. 1.1. □

In complete analogy with the ungraded case, we define:

Definition. An affine group G is called *unipotent* if its representing Hopf algebra $\mathcal{O}_G$ is unipotent (conilpotent). The latter means that the reduced comultiplication $\tilde{\mathcal{O}_G} \to \tilde{\mathcal{O}_G} \otimes \tilde{\mathcal{O}_G}$ is nilpotent, where $\tilde{\mathcal{O}_G}$ is the cokernel of the unit map. The group G is called *of multiplicative type* if its representing Hopf algebra is isomorphic to a group algebra, possibly after a finite field extension of k.

The Cartier dual G^* of a formal group is the affine group represented by the Hopf algebra $\mathrm{Hom}(\mathcal{O}_G, k)$. Cartier duality is an anti-equivalence between Fgps_k and AbSch_k.

A formal group G is called *connected* if is G^* unipotent, and *étale* if G^* is of multiplicative type.

Affine and formal graded groups over perfect fields split naturally:

Proposition 1.3. *Let k be a perfect field of characteristic $p > 2$. Then there are equivalences of categories*

$$\mathrm{AbSch}_k \simeq \mathrm{AbSch}_k^{\mathrm{odd}} \times \mathrm{AbSch}_k^{u} \times \mathrm{AbSch}_k^{m},$$
$$\mathrm{AbSch}_k^{p} \simeq \mathrm{AbSch}_k^{\mathrm{odd}} \times \mathrm{AbSch}_k^{u} \times \mathrm{AbSch}_k^{m,p},$$

where

- $\mathrm{AbSch}_k^{\mathrm{odd}}$ *denotes groups represented by Hopf algebras isomorphic to primitively generated exterior algebras on elements of odd degree,*
- AbSch_k^{u} *denotes evenly graded, unipotent groups, and*
- AbSch_k^{m} *denotes ungraded groups of multiplicative type and* $\mathrm{AbSch}_k^{m,p}$ *the full subcategory of p-adic groups.*

Dually, there is an equivalence of categories

$$\mathrm{Fgps}_k \simeq \mathrm{Fgps}_k^{\mathrm{odd}} \times \mathrm{Fgps}_k^{c} \times \mathrm{Fgps}_k^{e},$$

where

- $\mathrm{Fgps}_k^{\mathrm{odd}}$ *denotes groups represented by complete Hopf algebras isomorphic to primitively generated exterior pro-algebras on elements of odd degree,*
- Fgps_k^{c} *denotes evenly graded, connected formal groups, and*
- Fgps_k^{e} *denotes ungraded, étale formal groups.*

The free functors in Thm. 1.1 split into components $\mathrm{Fr}^{\mathrm{odd}}$, Fr^{u} etc. accordingly.

Proposition 1.4. *Let k be a field of characteristic $p > 2$. Then the forgetful functors $\mathrm{AbSch}_k^{\mathrm{odd}} \to \mathrm{Alg}_k^{\mathrm{op}}$ and $\mathrm{Fgps}_k^{\mathrm{odd}} \to (\mathrm{Pro}-\mathrm{alg}_k)^{\mathrm{op}}$ have left adjoints $\mathrm{Fr}^{\mathrm{odd}}$, and these factor as*

$$\begin{array}{ccc} \mathrm{Alg}_k^{\mathrm{op}} & \xrightarrow{\mathrm{Fr}^{\mathrm{odd}}} & \mathrm{AbSch}_k^{\mathrm{odd}} \\ & {\scriptstyle (-)^{\mathrm{odd}}}\searrow & \uparrow{\scriptstyle \hat{\mathrm{Fr}}} \\ & & (\mathrm{Mod}_k^{\mathrm{odd}})^{\mathrm{op}}, \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{alg}_k^{\mathrm{op}} & \xrightarrow{\mathrm{Fr}^{\mathrm{odd}}} & \mathrm{Fgps}_k^{\mathrm{odd}} \\ & {\scriptstyle (-)^{\mathrm{odd}}}\searrow & \uparrow{\scriptstyle \hat{\mathrm{Fr}}} \\ & & (\mathrm{mod}_k^{\mathrm{odd}})^{\mathrm{op}}, \end{array}$$

where the diagonal map assigns to A the odd part of the underlying k-module A and $\mathcal{O}_{\tilde{\mathrm{Fr}}(M)} = \bigwedge(M)$.

There are Dieudonné equivalences

$$D\colon (\mathrm{AbSch}_k^{u})^{\mathrm{op}} \to \mathrm{Dmod}_k^{V,\mathrm{nil}}$$

and

$$\mathbb{D}^{\mathrm{f}}\colon (\mathrm{Fgps}_k^{c})^{\mathrm{op}} \to \mathbb{D}\,\mathrm{mod}_k^{F,c}$$

with certain categories of modules over the rings $W(k)$ of p-typical Witt vectors over k with certain Frobenius and Verschiebung operations. More generally, for a graded ring R, let $W_n(R)$ denote the ring of p-typical Witt vectors of length $n \leq \infty$ and

$$CW^u(R) = \mathrm{colim}(W_1(R) \xrightarrow{V} W_2(R) \xrightarrow{V} \cdots)$$

the graded abelian group of unipotent co-Witt vectors. Moreover, for a finite-dimensional graded k-algebra R with radical $\tilde{R}$, let

$$CW^c(R)_i = \begin{cases} CW^c(R_0); & i = 0 \\ CW^{c,u}(R)_i; & i \neq 0 \end{cases}$$

be the graded abelian group of connected co-Witt vectors, where the ungraded $CW^c(R_0)$ is the classical group of connected co-Witt vectors [Fon77, §2.4] and $CW^{c,u}(R) < CW^u(R)$ consists of those unipotent co-Witt vectors all of whose entries lie in $\tilde{R}$.

Lemma 1.5. *The functors W_n and CW^u from Alg_k to $\mathrm{Dmod}_k^{V,\mathrm{nil}}$ factor naturally through $\mathrm{Pol}_p(k)$, and the functor $CW^c\colon \mathrm{alg}_k \to \mathbb{D}\,\mathrm{mod}_k^{F,c}$ factors naturally through $\mathrm{pol}_p(k)$.*

We prove:

Theorem 1.6. *For $R \in \mathrm{Alg}_k$, there are natural isomorphisms*

$$\mathbb{D}^{\mathrm{f}}(\mathrm{Fr}^c(R)) \cong CW^c(R) \quad \textit{for } R \in \mathrm{alg}_k$$

and

$$D(\mathrm{Fr}^u(R)) \cong CW^u(R) \quad \textit{for } R \in \mathrm{Alg}_k.$$

Proof of Thm. 1.1. Thm 1.6 together with Lemma 1.5 implies that Fr^c (resp. Fr^u) factor naturally through alg_k and Alg_k, respectively. The factorization of $\mathrm{Fr}^{\mathrm{odd}}$ is guaranteed by Prop. 1.4. Finally, the factorization of the ungraded Fr^e and Fr^m was proved in [Bau21, Thm. 1.3]. □

As an algebraic application, we show:

Theorem 1.7. *The affine group scheme of p-typical Witt vectors is the free unipotent abelian group scheme on the p-polar affine line, i.e. on the free p-polar algebra on a single generator.*

This is an analog of the fact that the Hopf algebra representing the big Witt vectors, the algebra Λ of symmetric functions, is cofree on a polynomial ring in one generator [Haz03]. The corresponding Hopf algebra Λ_p for p-typical Witt vectors is definitely not cofree on a (non-polar) algebra.

We conclude with a topological application. Let X be a connected space. We show:

Theorem 1.8. *Let p be a prime and $n \geq 1$. Then the functor $D_n\colon \mathrm{Top} \to \mathrm{Hopf}_{\mathbf{F}_p}$ defined by $D_n(X) = H^*(\Omega^{n+1}\Sigma^{n+1}X; \mathbf{F}_p)$ factors through the forgetful functor $\mathrm{Top} \xrightarrow{H^*(-;\mathbf{F}_p)} \mathrm{Alg}_{\mathbf{F}_p} \xrightarrow{\mathrm{pol}} \mathrm{Pol}_p(\mathbf{F}_p)$.*

In particular, the Hopf algebra $H^*(\Omega^n\Sigma^n X; \mathbf{F}_p)$ depends only on the p-polar structure of $H^*(X; \mathbf{F}_p)$. Under a finite-type condition, using work of Kuhn [Kuh20], we even have:

Corollary 1.9. *For $n \geq 0$ and spaces X, Y of finite type, $D_n(X) \cong D_n(Y)$ as Hopf algebras if and only if $H^*(X; \mathbf{F}_p) \cong H^*(Y; \mathbf{F}_p)$ as vector spaces with pth power maps. In particular, $D_n(X)$ only depends on the stable homotopy type of X, up to (noncanonical) isomorphism.*

2. Graded p-polar algebras

We begin the study of p-polar k-algebras with some observations and examples.

Remark 2.1. A graded p-polar k-algebra A does not supply a map $A^{\otimes_k p} \to A$ – only elements of the same degree can be multiplied together. In particular, a graded p-polar k-algebra is not a p-polar k-algebra when one forgets the grading.

Remark 2.2. The embeddability $i\colon A \to B$ into a graded commutative algebra can be thought of as saying that for any elements $x_1, \dots, x_n \in A$ and scalars $\lambda_1, \dots, \lambda_m \in k$, there is at most one way of multiplying them together using μ (up to sign); namely, the element $\lambda_1 \cdots \lambda_k i(x_1) \cdots i(x_n) \in B$, which is either in the image of i or it isn't.

Example 2.3. The submodule $k\langle x^{p^i} \mid i \geq 0\rangle \subset k[x]$ is a sub-p-polar algebra of $\mathrm{pol}(k[x])$, where $|x| > 0$. It is the free p-polar algebra on a generator x. This shows that in contrast to the ungraded case, even for $p = 2$, p-polar algebras are much weaker structure than actual algebras.

Remark 2.4. Let Mod_F denote the category of *F-modules*, i. e. graded k-modules M with a linear map $F\colon M_i \to M_{pi}$ satisfying $F(\alpha x) = \alpha^p F(x)$ for $\alpha \in k$, $x \in M_i$. ($F = 0$ if p and i are odd.) The forgetful functor

$$U_F\colon \mathrm{Pol}_p(k) \to \mathrm{Mod}_F; \quad M \to (M, F) \text{ with } F(x) = \mu(x, \dots, x)$$

is an equivalence when restricted to evenly graded, torsion-free M by what is classically called polarization. E.g., for $p = 2$, $\mu(x, y) = \frac{1}{2}\mu(x+y, x+y) - \mu(x, x) - \mu(y, y)$. In the situation of most interest to us, where k is a field of characteristic p, U_F is not an equivalence, but it has a left adjoint.

Remark 2.5. For any p-polar algebra A, the submodule $A_{(j)} = \bigoplus_{i\geq 0} A_{jp^i}$ is a polar subalgebra and direct factor, and thus

$$A \cong A_0 \times \prod_{p\nmid j} A_{(j)}.$$

We call a p-polar algebra of this form $A_{(j)}$ a *p-typical* polar algebra of degree j. The inclusion of such p-polar algebras into all p-polar algebras is biadjoint to the functor $A \mapsto A_{(j)}$. We write $\mathrm{pol}_{(j)}(A) := (\mathrm{pol}(A))_{(j)}$. This is a sub-$p$-polar algebra, but not a subalgebra of A.

In particular, if $p > 2$, we see that every graded p-polar k-algebra splits as a product $A = A_{\mathrm{odd}} \times A_{\mathrm{ev}}$, where $A_{\mathrm{odd}} = \bigoplus_n A_{2n+1}$ and A_{ev} the obvious complement.

Example 2.6. Consider the stable splitting $P = \Sigma^\infty(\mathbf{C}P^\infty)^\wedge_p \simeq P_1 \vee \cdots \vee P_{p-1}$ of the p-completion of complex projective space [McG81] with

$$H^*(P_i) = \langle x^j \mid j \equiv i \pmod{p-1}\rangle < \mathbf{F}_p[x] = H^*(\mathbf{C}P^\infty, \mathbf{F}_p).$$

By [Sul74], P_{p-1} is the suspension spectrum of a space (the classifying space of the Sullivan sphere), but all other P_i are not. However, the maps

$$P_i \hookrightarrow P = \Sigma^\infty(\mathbf{C}P^\infty)^\wedge_p \xrightarrow{\Sigma^\infty\Delta} \Sigma^\infty((\mathbf{C}P^\infty)^\wedge_p)^p \simeq P \wedge \cdots \wedge P \twoheadrightarrow P_i \wedge \cdots \wedge P_i$$

induces a p-polar algebra structure on $\tilde{H}^*(P_i)$, and the splitting $P \simeq P_1 \vee \cdots \vee P_{p-1}$ induces a splitting of p-polar algebras in cohomology. In fact,

$$H^*(P_i; \mathbf{F}_p) \cong \bigoplus_{\substack{j \equiv i \ (p-1) \\ p\nmid j}} \mathrm{pol}_{(j)} H^*(P; \mathbf{F}_p)$$

So while the P_i are not spaces for $i \neq p-1$, they do retain some likeness to spaces in that their cohomologies are p-polar algebras. This raises the question whether there is a reasonable notion of a "p-polar space" somewhere between connective spectra and spaces.

In a way, the definition of a p-polar k-algebra is wrong in the same way the definition of a manifold as a submanifold of $\mathbf{R}^n$ is wrong; it mentions an enveloping object which is not part of the data. The following proposition remedies this to a certain extent:

Proposition 2.7. *The functor* $\mathrm{pol}\colon \mathrm{Alg}_k \to M_k$ *has a left adjoint given for* $A \in M_k$ *by*

$$A \mapsto \mathrm{hull}(A) = \mathrm{Sym}(A)/(x_1 \cdots x_p - \mu(x_1, \dots, x_p) \mid x_1, \dots, x_p \in A_i).$$

An object $A \in M_k$ *is a* p*-polar* k*-algebra iff the unit map of this adjunction,* $u\colon A \to \mathrm{pol}(\mathrm{hull}(A))$*, is injective.*

Proof. The existence and structure of the left adjoint, hull, is obvious.

If u is injective, A is a p-polar algebra by definition. Conversely, if A is p-polar, say $i\colon A \hookrightarrow \mathrm{pol}(B)$ for some $B \in \mathrm{Alg}_k$, then by the universal property of the left adjoint, there is a factorization

$$\begin{array}{ccc} A & \xrightarrow{u} & \mathrm{pol}(\mathrm{hull}(A)) \\ & {\scriptstyle i}\searrow & \downarrow \\ & & \mathrm{pol}(B). \end{array}$$

Since i is injective, so is u. ∎

For the sake of explicitness, we will now give a list of axioms for objects of M_k to be a p-polar algebra.

Lemma 2.8. *If* $A = A_0$ *(i.e. in the ungraded case), the definition of a* p*-polar* k*-algebra agrees with the one given in* [Bau21]*; i.e.,* $A \in M_k$ *is* p*-polar iff*

(ASSOC): *For the symmetric group* Σ_{2p-1} *permuting the elements* $x_1, \dots, x_p, y_2, \dots, y_p \in A$,

$$\mu(\mu(x_1, \dots, x_p), y_2, \dots, y_p)$$

is Σ_{2p-1}*-invariant.*

Proof. Clearly, axiom (ASSOC) holds if A is p-polar. Conversely, suppose that (ASSOC) holds, and let $i\colon A \to \mathrm{hull}(A)$ be the adjunction unit. In [Bau21], it was shown that (ASSOC) implies that for any $i \geq 0$ and any set of $1 + i(p-1)$ elements $x_1, \dots, x_{1+i(p-1)}$, there is exactly one way of multiplying the x_i together using μ, and any other number of elements cannot be multiplied together. Write $\mu(x_1, \dots, x_{1+i(p-1)})$ for this unique product. Let $j\colon \bigoplus_{i=0}^{\infty} \mathrm{Sym}^{1+i(p-1)}(A) \hookrightarrow \mathrm{Sym}(A)$ be the inclusion and

$$\begin{aligned} \mathrm{hull}(A) \supseteq B &= \left(\bigoplus_{i=0}^{\infty} \mathrm{Sym}^{1+i(p-1)}(A)/j^{-1}((x_1 \cdots x_p - \mu(x_1, \dots, x_p))) \right) \\ &\cong \left(\bigoplus_{i=0}^{\infty} \mathrm{Sym}^{1+i(p-1)}(A) \right) / \langle x_1 \cdots x_{1+i(p-1)} - \mu(x_1, \dots, x_{1+i(p-1)}) \rangle \\ &\cong A \end{aligned}$$

We have thus exhibited A as a subobject of $\mathrm{hull}(A)$. ∎

Now we consider the general, graded case.

Lemma 2.9. *Let* A *be an object in* M_k*. Then* A *is a* p*-polar* k*-algebra iff*

(1) A_0 *is a p-polar k-algebra, and*
(2) *For the symmetric group* Σ_{2p} *permuting the elements* $x_1, \dots, x_{2p} \in A_j$ *and elements* $y_3, \dots, y_p \in A_{pj}$,

$$\mu(\mu(x_1, \dots, x_p), \mu(x_{p+1}, \dots, x_{2p}), y_3, \dots, y_p)$$

is Σ_{2p}*-invariant (up to multiplication with the sign of the permutation if* j *is odd).*

Proof. The implication A p-polar $\Rightarrow$ (1), (2) is straightforward. For the converse, we may assume by Remark 2.5 without loss of generality that either $A = A_0$ or $A = A_{(j)}$ is p-typical. Lemma 2.8 takes care of the first case, so assume A is p-typical and (2) holds.

For any graded p-typical k-module $M = M_{(j)}$, the free object T_M in M_k on M is given inductively by

$$(T_M)_n = M_n \oplus \mathrm{Sym}^p((T_M)_{\frac{n}{p}}),$$

where $(T_M)_n = 0$ if $n \notin \mathbf{Z}$. We call an element of $(T_M)_n$ a monomial if it is either an element of M_n or a monomial $\{x_1, \dots, x_p\} \in \mathrm{Sym}^p$ on monomial elements in $x_i \in (T_M)_{\frac{n}{p}}$, using curly braces for equivalence classes of tensors in Sym^p. Clearly, by linearity, any element of T_M is a linear combination of monomials. One could describe these monomial elements as some kinds of labelled trees. While this is a good picture to have in mind, I will not use that language.

Define an equivalence relation $\sim$ on monomials in T_M (and hence, by linear extension, on all of T_M) generated by

$$\begin{aligned}&\{\{x_1, \dots, x_p\}, \{x_{p+1}, \cdots, x_{2p}\}, y_3, \cdots, y_p\}\\ &\qquad\sim (-1)^{|x_1|}\{\{x'_1, \dots, x'_{p-1}, x'_{p+1}\}, \{x'_p, x'_{p+2}, \cdots, x'_{2p}\}, y'_3, \cdots, y'_p\}\end{aligned}$$

iff $y_j \sim y'_j$ for $3 \le h \le p$, $x_i \sim x'_i$ for $1 \le i \le 2p$. Then $T_M/\sim$ is the free object in M_k satisfying (2). (Obviously, the Σ_{2p}-invariance is equivalent to the invariance under interchanging x_p and x_{p+1}, given the guaranteed $\Sigma_p \times \Sigma_p$-invariance.)

There is a linear map $f\colon T_M \to \mathrm{Sym}(M)_{(j)}$ given on monomials by $f(m) = m$ for $m \in M$ and $f(\{x_1, \dots, x_p\}) = \{f(x_1), \dots, f(x_p)\}$ for $\{x_1, \dots, x_p\} \in \mathrm{Sym}^p(T_M)$. We claim that this map induces an injective map on $T_M/\sim$ with image $\mathrm{Sym}(A)_{(j)}$.

To see this is the image, let $X = \{x_1, \dots, x_n\} \in \mathrm{Sym}(M)_{jp^N}$. If $n = 1$, $\{x_1\} = f(x_1)$ and we are done. Otherwise, because M is p-typical, there has to be a partition of $\{1, \dots, n\}$ into p parts $I_1, \dots, I_p$ such that for $X_i = \pm\{x_j \mid j \in I_i\}$, $|X_i| = jp^{N-1}$. Inductively, all X_i are in the image of f, hence so is $X = f(X_1, \dots, X_p)$.

We proceed to show injectivity.

Let $x \in T_M$ be a monomial. We say that $y \in T_M$ occurs at depth d in x if either $d = 0$ and $y = x$ or $x = \{x_1, \dots, x_p\}$ and y occurs at depth $d-1$ in x_i for some i.

Now suppose that y_1 and y_2 occur at a common depth $d \ge 1$ in $x = \{x_1, \dots, x_p\}$, and let $x' \in T_M$ be the element obtained by interchanging y_1 and y_2. Then I claim that $x \sim \pm x'$. To see this, we proceed by induction. If $d = 1$ then the claim is true by symmetry. Suppose that $d > 1$. Then y_1 occurs at depth $d-1$ in some x_i and y_2 occurs at depth $d-1$ in some x_j. If $i = j$, we are done by induction. Otherwise, suppose without loss of generality that $i = 1$ and $j = 2$. Let $x_1 = \{x_{11}, \dots, x_{1p}\}$ and $x_2 = \{x_{21}, \dots, x_{2p}\}$. Without loss of generality, suppose that y_i occurs at depth $d-2$ in x_{i1} for $i=1,2$. Then

$$\begin{aligned} x \sim \pm x^{(1)} &= \{x_1^{(1)}, \dots, x_p^{(1)}\} \\ &= \{\{x_{11}, x_{21}, x_{13}, \dots, x_{1p}\}, \{x_{12}, x_{22}, \dots, x_{2p}\}, x_3, \dots, x_p\}. \end{aligned}$$

Then y_1 and y_2 occur at depth $d-1$ in $x_1^{(1)}$ and by induction, $x_1^{(1)} \sim (x^{(1)})'_1 = \{x'_{11}, x'_{21}, x_{13}, \dots, x_{1p}\}$, the element obtained from $x_1^{(1)}$ by interchanging y_1 and y_2. But then

$$\begin{aligned} x &\sim \pm\{\{x'_{11}, x'_{21}, x_{13}, \dots, x_{1p}\}, \{x_{12}, x_{22}, \dots, x_{2p}\}, x_3, \dots, x_p\} \\ &\sim \{\{x'_{11}, x_{12}, \dots, x_{1p}\}, \{x'_{21}, x_{22}, \dots, x_{2p}\}, x_3, \dots, x_p\} = x'. \end{aligned}$$

We conclude that if x, $x' \in T_M$ with $f(x) = f(x')$ (i.e. they contain the same leaf elements at any given level), then $x \sim x'$.

Now let $A = A_{(j)}$ be an object of M_k such that (2) holds and F_A the quotient of $(T_A/\sim) \cong \operatorname{Sym}(A)_{(j)}$ by the intersection of the ideal $(x_1 \dots x_p - \mu(x_1, \dots, x_p)) \triangleleft \operatorname{hull}(A)$ with $\operatorname{Sym}(A)_{(j)}$. Then the map $A \mapsto F_A$ is an isomorphism, showing that $A \hookrightarrow \operatorname{hull}(A)$. □

Corollary 2.10. *Let $A = A_{(j)}$ be a p-typical polar k-algebra and $x_1, \dots, x_N \in A$. Then there is a unique product $x_1 \dots x_N$ in A if $|x_1| + \dots + |x_N| = jp^k$ for some k, and no possible way to multiply these elements in A otherwise.* □

3. Graded Witt vectors

In this section, we will consider commutative graded rings A instead of graded-commutative rings, i.e. graded rings A whose underlying ungraded ring is commutative. We will apply the results of this section to evenly graded, graded-commutative rings or graded, commutative rings over fields of characteristic 2. There does not seem to be an adequate (for our purposes) definition of graded-commutative Witt vectors for graded-commutative rings, nor will it be necessary, in light of Prop. 1.4

Throughout, let p be a fixed prime. For a graded abelian group M and an integer $i \geq 0$, we write $M(i)$ for the graded abelian group with $M(i)_n = M_{p^i n}$.

We assume the reader is familiar with the ungraded theory of p-typical Witt vectors, cf. [Wit37, Haz09, Hes08].

Definition. Let A be a p-polar ring. As a graded set, the p-typical Witt vectors of A of length $0 \leq n \leq \infty$ are defined as

$$W_n(A) = \prod_{i=0}^{n} A(i), \quad \text{i.e.} \quad W(A)_j = \prod_{i=0}^{n} A_{jp^i}.$$

There is a ghost map

$$w\colon W_n(A) \to \prod_{i=0}^{n} A(i), \quad w(a_0, a_1, \dots) = (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \dots).$$

Lemma 3.1. *Let A be a graded p-polar ring and $1 \leq n \leq \infty$.*

(1) There is a unique natural graded p-polar ring structure on $W_n(A)$ making the ghost map w into a p-polar ring map.

(2) *There are unique natural additive maps $F\colon W_{n+1}(A) \to W_n(A)(1)$ (Frobenius) and $V\colon W_n(A)(1) \to W_{n+1}(A)$ (Verschiebung) such that the following diagram commutes ($x_{-1} = 0$ by convention):*

$$\begin{array}{ccccc} W_n(A)(1) & \xrightarrow{V} & W_{n+1}(A) & \xrightarrow{F} & W_n(A)(1) \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ \prod_{i=0}^{n-1} A(i+1) & \xrightarrow{(x_i)_i \mapsto (px_{i-1})} & \prod_{i=0}^{n} A(i) & \xrightarrow{(x_i)_i \mapsto (x_{i+1})_i} & \prod_{i=0}^{n-1} A(i+1) \end{array}$$

Explicitly, $V(a_0, \ldots, a_{n-1}) = (0, a_0, \ldots, a_{n-1})$ and F is uniquely determined by $F(\underline{a}) = \underline{a}^p$ and $FV = p$, where $\underline{a} = (a, 0, \ldots, 0)$ denotes the Teichmüller representative of $a \in A$ in $W_n(A)$.

(3) *If A is a p-polar algebra over a perfect field k of characteristic p then $W_n(A)$ is a p-polar $W(k)$-algebra. For a graded $W(k)$-module M, give $M(i)$ the $W(k)$-module structure defined by*

$$\lambda.m = \mathrm{frob}^n(\lambda)m,$$

where frob *is the Frobenius F on $W(k)$. Then F and V are $W(k)$-linear.*

Proof. This was proved in the ungraded case in [Bau21, Lemma 3.3]. If A^{ungr} denotes the p-polar algebra A with the grading forgotten, then we have a commutative diagram

$$\begin{array}{ccc} W_n(A)_j & \longrightarrow & W_n(A^{\mathrm{ungr}}) \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ \prod_{i=0}^{n} A_{jp^i} & \longrightarrow & \prod_{i=0}^{n} A^{\mathrm{ungr}}, \end{array}$$

where the bottom and top maps are the standard inclusions. Along with the other three maps, the top map is a homomorphism. Since the right-hand map is injective for torsion free A, so is the left-hand map, together showing that there is a unique natural abelian group structure on $W_n(A)_j$ making w into a homomorphism. Like the polynomials occuring in w, the polynomials defining p-polar multiplication are p-polar, showing that $W_n(A)$ is a graded p-polar algebra and w is a homomorphism of such. As we do not need the p-polar algebra structure on $W_n(A)$ in this paper, we leave checking the details to the reader. This proves (1). Parts (2) and (3) follow easily. ☐

If A^{ungr} denotes A as an ungraded p-polar algebra then $W(A^u)$ and $W(A)^u$ are in general distinct:

Example 3.2. $W(k[u]) \cong W(k)[u]$ if $|u| = d > 0$. This is false if $d = 0$: the (ungraded) p-typical Witt vectors of $W(\mathbf{F}_p[x])$ are more complicated (cf. [Bor16, Exercise 10]).

The functor $M \mapsto M(1)$ is not an equivalence. But if k is a perfect field (and hence frob is invertible), it has a right inverse (-1) given by

$$M(-1)_n = \begin{cases} 0; & p \nmid n \\ M_{\frac{n}{p}}; & p \mid n \end{cases}$$

with the $W(k)$-linear structure given by $\alpha.m = \mathrm{frob}^{-1}(\alpha)m$. Confusingly, (-1) being a right inverse means that $M(-1)(1) \cong M$.

Definition. For a graded p-polar ring A, the group of unipotent co-Witt vectors is defined by

$$CW^u(A) = \operatorname{colim}(W_0(A) \xrightarrow{V} W_1(A(-1)) \xrightarrow{V} W_2(A(-2)) \xrightarrow{V} \cdots),$$

where $V\colon W_n(A) = W_n(A(-1))(1) \to W_{n+1}(A(-1))$ is induced by the Verschiebung $V\colon W_n(A)(1) \to W_{n+1}(A)$.

If A is a graded p-polar k-algebra for a perfect field k or characteristic p then $CW^u(A)$ is naturally a $W(k)$-modules.

3.1. **Representability of Witt and co-Witt vectors.** Since for graded k-algebras A and $n \leq \infty$, $W_n(A)_j \cong \prod_{i=0}^n A_{jp^i}$ as sets, this set-valued functor is represented in graded k-algebras by

$$(\Lambda_p^{(n)})_j = k[\theta_{j,0}, \dots, \theta_{j,n}]; \quad (\Lambda_p^{(\infty)})_j = (\Lambda_p)_j = k[\theta_{j,0}, \theta_{j,1}, \dots]$$

where $|\theta_{j,i}| = jp^i$, and $W_n(A)$, as a graded object, is represented by the bigraded k-algebra $\Lambda_p^{(n)} = (\Lambda_p^{(n)})_*$. Each $(\Lambda_p^{(n)})_j$ obtains a Hopf algebra structure by the natural Witt vector addition on $W_n(A)_j$, and $\Lambda_p^{(n)}$ becomes a Hopf ring (cf. [Wil00]) with a comultiplication

$$\Lambda_p(A)_j \to \bigoplus_{j_1+j_2=j} \Lambda_p(A)_{j_1} \otimes \Lambda_p(A)_{j_2}.$$

In other words, Λ_p represents a graded ring object, even a plethory, in affine schemes. This is a graded version of the p-typical symmetric functions of [BW05, II.13].

The co-Witt vectors are not representable, but their restriction to alg_k, i.e. finite-dimensional k-algebras, is ind-representable. That is, CW_k^u is a formal group. In the ungraded case, this is described in [Fon77, §II.3–4]. In our graded case, $(W_n)_j$ is represented by $k[\![\theta_{j,0}, \dots, \theta_{j,n}]\!]$ as before; we choose a power series notation as we would in the ungraded case, although for graded algebras of finite type, there is arguably no difference.

Thus $CW^u(A)_j$ is represented by the profinite ring

$$(\mathcal{O}_{CW_k^u})_j = \lim_{n \geq 0} k[\![x_{j,0}, x_{j,-1}, \dots, x_{j,-n}]\!],$$

with $|x_{j,i}| = jp^i$, which in the case $jp^i \notin \mathbf{Z}$ is to be understood as $x_{j,i} = 0$. Note that for $j = ap^l \neq 0$ with $p \nmid a$, the above formula simplifies to

$$(\mathcal{O}_{CW_k^u})_j = k[\![x_{j,0}, \dots, x_{j,-l}]\!].$$

By naturality of the co-Witt vector addition, CW_k thus becomes a (graded) formal group.

Example 3.3. Let $k = k_0$ be perfect of characteristic p and $A = k\langle x^{p^i} \mid i \geq 0\rangle$ the free p-polar algebra on a single generator x in degree 2. Then

$$W_n(A)_{2p^i} = \{(a_0 x^{p^i}, a_1 x^{p^{i+1}}, \dots, a_n x^{p^{i+n}}) \mid a_i \in W(k)\} \cong W_n(k)$$

and $W_n(A)_j = 0$ if j is not twice a power of p. The Verschiebung is given by

$$V\colon W_n(A)_{2p^i} \to W_{n+1}(A(-1))_{2p^i}, \quad (a_0, \dots, a_n) \mapsto (0, a_0, \dots, a_n).$$

We have that

$$CW^u(A)_{2p^i} = \{(\dots, a_{-1}, a_0) \mid a_j \in A_{2p^{i+j}}\},$$

which is understood to mean $a_j = 0$ if $2p^{i+j} \notin \mathbf{Z}$. Thus $CW^u(A)_{2p^i} = W_i(k)$, and the Frobenius and Verschiebung maps are given by

$$V\colon CW^u(A)_{2p^i} \to CW^u(A)_{2p^{i-1}} \text{ the restriction map } W_i(k) \to W_{i-1}(k)$$

and

$$F\colon CW^u(A)_{2p^i} \to CW^u(A)_{2p^{i+1}}, \text{ the multiplication-by-}p\text{ map } W_i(k) \to W_{i+1}(k).$$

4. Graded Dieudonné theory

Throughout, let k be a perfect field of characteristic p.

Definition. A *graded Dieudonné module* over k is a graded $W(k)$-module M together with maps of $W(k)$-modules

$$F\colon M \to M(1) \quad \text{and} \quad V\colon M(1) \to M$$

satisfying $FV = p$ and $VF = p$. We denote the category of Dieudonné modules (with the obvious definition of morphism) by Dmod_k.

We call a Dieudonné module M *unipotent* if for any $x \in M$, $V^n(x) = 0$ for $n \gg 0$. We denote the full subcategories of unipotent Dieudonné modules by $\mathrm{Dmod}_k^{V,\mathrm{nil}}$. Note that for degree reasons, the unipotence condition can only possibly fail for x of degree 0.

Moreover, a Dieudonné module M is called *connected* if M is profinite as a $W(k)$-module and has a fundamental system of neighborhoods consisting of (finite length) $W(k)$-modules N such that $F^n(N) = 0$ for $n \gg 0$.

Proof of Prop. 1.3. Note that the formal and the affine parts of the statement are Cartier dual to one another, thus it suffices to prove the claim in the affine case. Let $G \in \mathrm{AbSch}_k$ be an affine group scheme with representing graded commutative Hopf algebra $H = \mathcal{O}_G$.

By [Bou96, Prop. A.4], any bicommutative Hopf algebra (and dually, every bicommutative complete Hopf algebra) over a field of characteristic $p > 2$ splits naturally into an even part and an odd part:

$$H = H_{\mathrm{ev}} \otimes H_{\mathrm{odd}},$$

where H_{ev} is concentrated in even degrees and H_{odd} is an exterior algebra on primitive generators in odd degrees. Thus we obtain a splitting

$$\mathrm{AbSch}_k \simeq \mathrm{AbSch}_k^{\mathrm{odd}} \times \mathrm{AbSch}_k^{\mathrm{ev}}$$

Considering G^{ev}, the even part of G, as an ungraded group scheme, it is shown in [Fon77, §I.7] that there is a natural splitting

$$G^{\mathrm{ev}} \simeq G^u \times G^m$$

into a unipotent group and a group of multiplicative type. The grading is compatible with this splitting, and the proof is complete by observing that a graded group of multiplicative type, by definition, has to be concentrated in degree 0.

For the p-adic version of the statement, note that odd groups and unipotent groups are necessarily p-adic. □

Remark 4.1. There is actually a natural splitting $G \cong G_0^m \otimes G_0^u \otimes G_{\neq 0}^{\mathrm{ev}} \otimes G^{\mathrm{odd}}$, where the three last factors are unipotent and the first two are concentrated in degree 0.

Proof of Prop. 1.4. The affine case is basically a reformulation of Bousfield's result [Bou96, Prop. A.4], which says that the functor of primitives gives an equivalence between odd affine groups and oddly graded k-modules, inverse to the exterior algebra functor. To see that $\mathrm{Fr}^{\mathrm{odd}}(A)$, defined as represented by $\bigwedge(A^{\mathrm{odd}})$, indeed is adjoint to the forgetful functor, we calculate for any odd Hopf algebras $H = \bigwedge(M)$ with M some oddly graded k-vector space:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Alg}_k}(H, A) &= \mathrm{Hom}_{\mathrm{Alg}_k}(\textstyle\bigwedge(M), A)\\ &= \mathrm{Hom}_{\mathrm{Mod}_k^{\mathrm{odd}}}(M, A^{\mathrm{odd}}) = \mathrm{Hom}_{\mathrm{Hopf}_k^{\mathrm{odd}}}(H, \textstyle\bigwedge(A)).\end{aligned}$$

The formal case follows by dualization, noting that $\bigwedge(M^*) \cong (\bigwedge(M))^*$ as formal Hopf algebras. □

Lemma 4.2. *The forgetful functors* $\mathrm{AbSch}_k^p \to \mathrm{Alg}_k^{\mathrm{op}}$ *and* $\mathrm{Fgps}_k \to (\mathrm{Pro}-\mathrm{alg}_k)^{\mathrm{op}}$ *have left adjoints.*

Proof. It suffies to construct the various componentss of the adjoints Fr according to the splittings of Prop. 1.3. For $\mathrm{Fr}^{\mathrm{odd}}$, this is done in Prop. 1.4. For Fr^e and Fr^m, this was done in [Bau21, Proof of Lemma 1.2] as these are ungraded. The Hopf algebra representing $\mathrm{Fr}^u(A)$ is

$$\mathrm{Cof}^u(A) = \bigoplus_{n\geq 0} (A^{\otimes_k n})^{\Sigma_n},$$

the cofree cocommutative conilpotent Hopf algebra first constructed by Takeuchi [Tak74]. His (ungraded) construction is compatible with the grading.

Dually, the Hopf algebra dual to the formal Hopf algebra representing $\mathrm{Fr}^c(A)$ is the unipotent part of the free commutative Hopf algebra on a cocommutative coalgebra [Tak71]. □

Theorem 4.3. *There are exact natural equivalences*

$$\mathbb{D}^{\mathrm{f}}\colon\ \mathrm{Fgps}_k^c \to \mathbb{D}\,\mathrm{mod}_k^F \quad \textit{and} \quad D\colon\ \mathrm{AbSch}_k^u \to \mathrm{Dmod}_k^{V,\mathrm{nil}}.$$

The functor $\mathbb{D}^{\mathrm{f}}$ *is represented by the formal group* CW_k^c*, while the functor* $D(G)$ *is given by* $D(G) = \mathrm{colim}_n \mathrm{Hom}_{\mathrm{AbSch}_k}(G, W_n)$*.*

Proof. Since every affine group splits naturally into its degree-0 part and a part represented by a Hopf algebra H that is connected in the sense that $H_0 = k$, the result follows from the conjunction of [Fon77, §III Théorème 1] and [Sch70], cf. also [Bau21, Theorem 4.2] and [Goe99, Rav75].

The equivalence $\mathbb{D}^{\mathrm{f}}$ is given by the composition

$$\mathrm{Fgps}_k^c \xrightarrow{(-)^*} \mathrm{AbSch}_k^u \xrightarrow{D} \mathrm{Dmod}_k^{V,\mathrm{nil}} \xrightarrow{I} (\mathbb{D}\,\mathrm{mod}_k^F),$$

where $(-)^*$ denotes Cartier duality and I denotes Matlis (Poincaré) duality

$$M \mapsto \mathrm{Hom}^c_{W(k)}(M, CW(k)),$$

the group of continuous homomorphisms into $CW(k)$. Since both $(-)^*$ and I are anti-equivalences, the affine case follows from the formal case. The fact that $\mathbb{D}^{\mathrm{f}}$ is isomorphic to the claimed form is proved in [Dem86, Ch. III,6]. □

Proof of Thm. 1.6. Consider first the formal case. For a finite-dimensional k-algebra A, we have that

$$\mathbb{D}^{\mathrm{f}}(\mathrm{Fr}^c(A)) = \mathrm{Hom}_{\mathrm{Fgps}_k}(\mathrm{Fr}^c(A), CW_k^c) = \mathrm{Hom}_{\mathrm{FSch}_k}(\mathrm{Spec}\, A, CW_k^c) = CW^c(A),$$

as claimed.

In the affine case, for a k-algebra A, we have

$$D(\mathrm{Fr}^u(A)) = \operatorname*{colim}_n \mathrm{Hom}_{\mathrm{AbSch}_k}(\mathrm{Fr}(A), W_n) = \mathrm{colim}\, W_n(A) = CW^u(A). \qquad \square$$

Proof of Thm. 1.1. In the formal case, the existence of the adjoint $\mathrm{Fr} = \mathrm{Fr}^{\mathrm{odd}} \times \mathrm{Fr}^c \times \mathrm{Fr}^e$ follows from Lemma 4.2. The factorization $\tilde{\mathrm{Fr}}(A)$ for $A \in \mathrm{pol}_p(k)$ is given by:

- $\tilde{\mathrm{Fr}}^{\mathrm{odd}}(A)$ is defined as in Prop. 1.4;
- $\tilde{\mathrm{Fr}}^e(A) = \mathrm{Fr}^e(A_0)$ is defined as in [Bau21];
- $\tilde{\mathrm{Fr}}^c(A) = (\mathbb{D}^{\mathrm{f}})^{-1}(CW^c(A))$, using Thm. 1.6.

The affine case is completely analogous. $\square$

5. Properties and applications

The factorization of the free formal group functor (Thm. 1.1) induces a factorization

$$\begin{array}{ccc} (\mathrm{Pro}-\mathrm{alg}_k)^{\mathrm{op}} & \xrightarrow{\mathrm{Fr}} & \mathrm{Fgps}_k \\ & \searrow^{\mathrm{pol}} & \uparrow{\scriptstyle \tilde{\mathrm{Fr}}} \\ & & (\mathrm{Pro}-\mathrm{pol}_p(k))^{\mathrm{op}}, \end{array}$$

where the functors denoted by Fr and $\tilde{\mathrm{Fr}}$ are the unique extension of the functors from Thm. 1.1 that commute with directed colimits. In this section, we will concentrate on the free unipotent, resp. connected, construction only.

Lemma 5.1. *The functor $\hat{\mathrm{Fr}}^c \colon (\mathrm{Pro}-\mathrm{pol}_p(k))^{\mathrm{op}} \to \mathrm{Fgps}_k$ commutes with all colimits and has a right adjoint V.*

Proof. For the odd part, the functor $\hat{\mathrm{Fr}}^{\mathrm{odd}}$ factors as

$$\hat{\mathrm{Fr}}^{\mathrm{odd}} \colon (\mathrm{Pro}-\mathrm{pol}_p(k))^{\mathrm{op}} \xrightarrow{U} (\mathrm{Pro}-\mathrm{mod}_k^{\mathrm{odd}})^{\mathrm{op}} \xrightarrow{\bigwedge} \mathrm{AbSch}_k^p$$

(cf. Prop. 1.4), where U is the forgetful functor. Then an adjoint is given by the composition of the adjoint of the exterior Hopf algebra functor $\bigwedge$ (which is the functor of primitives) and an adjoint of U. The latter is the objectwise free p-polar algebra functor, which works because the free p-polar algebra on an odd finite-dimensional k-module is again finite dimensional (it is a sub-p-polar algebra of the exterior algebra).

So we can restrict our attention to even formal groups. For the free connected formal group Fr^c, it suffices to show that its composition with the Dieudonné equivalence $\mathbb{D}^{\mathrm{f}}$ commutes with the stated colimits, and by Theorem 1.6, it is therefore enough to show that $CW_k^c \colon \mathrm{Pro}-\mathrm{pol}_p(k) \to \mathbb{D}\,\mathrm{mod}_k^F$ commutes with all limits. But CW_k^c is a formal group, i.e. representable.

The existence of an adjoint follows from Freyd's special adjoint functor theorem once we show that $\mathrm{Pro}-\mathrm{pol}_p(k)$ is complete, well-powered, and possesses a cogenerating set. Any pro-category of a finitely complete category, such as $\mathrm{pol}_p(k)$, is complete, and any pro-category has the constant objects as a cogenerating class. Since $\mathrm{pol}_p(k)$ has a small skeleton, the condition on a cogenerating set is satisfied. To see that $\mathrm{Pro}-\mathrm{pol}_p(k)$ is well-powered, observe that a subobject $S < A$ for $A \in \mathrm{Pro}-\mathrm{pol}_p(k)$ is in particular a sub-pro-vector space. By [AM69, Prop. 4.6], a monomorphism in $\mathrm{Pro}-\mathrm{Mod}_k$ can be represented by a levelwise monomorphism. Thus if $A\colon I \to \mathrm{Mod}_k$ represents a pro-finite k-module with $\#\,\mathrm{Sub}(A(i)) = \alpha_i$ for some (finite) cardinals α_i then $\#\,\mathrm{Sub}(A) \leq \prod_{i\in I} \alpha_i$; in particular, it is a set. □

Lemma 5.2. *The functor $\hat{\mathrm{Fr}}^u\colon \mathrm{Pol}_p(k)^{\mathrm{op}} \to \mathrm{AbSch}_k^u$ commutes with all colimits and filtered limits, and has a right adjoint V.*

Proof. As in Lemma 5.1, the right adjoint on odd affine groups is given by the functor of primitives follow by the free p-polar algebra functor, so we will restrict our attention to even p-adic affine groups. By Theorem 1.6, it suffices to show that the functors $CW^u\colon \mathrm{Pol}_p(k) \to \mathrm{Dmod}_k^{V,\mathrm{nil}}$ commutes with all limits, which looks wrong until one realizes that limits in $\mathrm{Dmod}_k^{V,\mathrm{nil}}$ are not the same as limits in Dmod_k.

The functor $CW^u(A)$ commutes with finite limits (it is ind-representable), so it suffices to show it commutes with infinite products. Indeed, the natural map

$$CW^u(\prod_i A_i) \to \prod_i CW^u(A_i)$$

is an isomorphism; an element in the right hand side is a set of elements ($x_i \in CW^u(A_i)$) such that there is an $n \gg 0$ such that $V^n(x_i) = 0$ for all i.

The commutation with filtered colimits is straightforward: CW^u commutes with them because it is a colimits of functors represented by small objects (polarizations of finitely presented algebras).

For the existence of an adjoint, we apply again the special adjoint functor theorem in the form of [AR94, Thm. 1.66]. The category $\mathrm{Pol}_p(k)$ is locally presentable and the functor $D \circ \hat{F}r^u$, as just shown, is accessible (commutes with ω-filtered colimits) and commutes with all limits. □

Note that this implies, by taking adjoint functors in Thm. 1.1, that the algebra underlying a unipotent Hopf algebra H is always of the form $\mathrm{hull}(V(H))$, i.e. free over a p-polar k-algebra, and the pro-finite algebra underlying a complete connected Hopf algebra H is always of the form $\mathrm{hull}(V(H))$, i.e. free over a profinite p-polar k-algebra. Of course, this is also a direct corollary of Borel's work on the structure of algebras underlying Hopf algebras [Bor54, MM65] .

Lemma 5.3. *Let A be a p-polar k-algebra and $H = \mathrm{Cof}^u(A)$ the unipotent Hopf algebra representing the p-adic affine group $\hat{\mathrm{Fr}}^u(A)$. Then H is isomorphic, as a pointed coalgebra, to the symmetric tensor coalgebra on the k-vector space A.*

Proof. The symmetric tensor coalgebra on a vector space V is given by

$$S(V) = \bigoplus_{i\geq 0} S^i(V) \quad \text{with} \quad S^i(V) = \left(V^{\otimes i}\right)^{\Sigma_i},$$

and it is a pointed coalgebra by the inclusion $k \cong S^0(V) \subset S(V)$. A pointed coalgebra C is conilpotent if for each $x \in C$, $\psi^N(x) \in C^{\otimes(N+1)}$ maps to 0 in $\bar{C}^{\otimes(N+1)}$, where $\bar{C}$ is the cokernel of the pointing. The coalgebra $S(V)$ is conilpotent and, indeed, the right adjoint to the forgetful functor U from the category Coalg_k^u

of pointed, conilpotent, cocommutative coalgebras to k-vector spaces, mapping a coalgebra C to $\bar{C}$.

Note that a Hopf algebra is unipotent if and only if its underlying pointed coalgebra is conilpotent. The claim is that the diagram

$$\begin{array}{ccc} \mathrm{Pol}_p(k) & \xrightarrow{\mathrm{Cof}^u} & \mathrm{Hopf}_k^u \\ \big\downarrow{\scriptstyle U_1} & & \big\downarrow{\scriptstyle U_2} \\ \mathrm{Mod}_k & \xrightarrow{S} & \mathrm{Coalg}_k^u \end{array}$$

2-commutes. By taking left adjoint functors, this is equivalent to the 2-commutativity of the square in the diagram

$$\begin{array}{ccc} \mathrm{Pol}_p(k) & \xleftarrow{V} & \mathrm{Hopf}_k^u \\ {\scriptstyle \mathrm{Fr}}\big\uparrow & & {\scriptstyle \mathrm{Fr}}\big\uparrow \\ \mathrm{Mod}_k & \xleftarrow[U]{} & \mathrm{Coalg}_k^u\,. \end{array}$$

The free commutative Hopf algebra on $C \in \mathrm{Coalg}_k^u$ is given by the symmetric algebra $\mathrm{Sym}(\bar{C})$, and hence there is a natural map of k-modules $C \to \mathrm{Sym}(\bar{C})$ given in degree 0 by the augmentation and in degree one by the projection $C \to \bar{C}$. This map $\phi\colon U(C) \to U_1(V(C))$ is adjoint to a map $\phi\colon \mathrm{Fr}(U(C)) \to V(\mathrm{Fr}(C))$ in $\mathrm{Pol}_p(k)$. To see that this map is an isomorphism, we consider its image under the conservative functor

$$\mathrm{hull}\colon\ \mathrm{Pol}_p(k) \to \mathrm{Alg}_k\,.$$

Since $\mathrm{hull} \circ V\colon\ \mathrm{Hopf}_k^u \to \mathrm{Alg}_k$ is the forgetful functor and $\mathrm{hull} \circ \mathrm{Fr}\colon\ \mathrm{Mod}_k \to \mathrm{Alg}_k$ is the symmetric algebra functor, we see that $\mathrm{hull}(\phi)$ is the identity on $\mathrm{Sym}(C)$. □

Corollary 5.4. *Let H be a unipotent Hopf algebra which is unipotent cofree on a p-polar k-algebra A. Then A is isomorphic to the vector space of primitive elements PH.*

Proof. If H is as in the statement then $U_2(H) \cong S(PH)$ as coalgebras, but by the preceding Lemma, $U_2(H) \cong S(U_1(A))$. Applying the functor P and noting that $P(S(M)) \cong M$, we find that $PH \cong U_1(A)$. □

Remark 5.5. It is not true that $V(H) = P(H)$ in general, or that $P(H)$ is a p-polar algebra. Also, if H is a unipotent Hopf algebra whose underlying pointed unipotent coalgebra is cofree, H is not necessarily cofree over a p-polar k-algebra. For example, consider the graded Hopf algebra H dual to $H^* = k[x,y]$ with $|x| = j > 0$, $|y| = p^2 j$, x primitive and $\psi(y) = y\otimes 1 + 1\otimes y + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} x^{pi} \otimes x^{p(p-i)}$. Then the primitives PH are dual to the indecomposables $Q(H^*) = \langle x, y\rangle$, i.e. $PH = \langle a, b\rangle$ with $|a| = j$, $|b| = p^2 j$. Suppose H was cofree, so that PH is a p-polar k-algebra by the corollary above. For degree reasons, PH cannot carry any but the trivial p-polar algebra structure, $PH \cong \langle a\rangle \times \langle b\rangle$. As a right adjoint, Cof^u commutes with products and hence $\mathrm{Cof}^u(PH) = \mathrm{Cof}(\langle a\rangle) \otimes \mathrm{Cof}(\langle b\rangle) = (k[x] \otimes k[y])^* = H'$. But $H \ncong H'$ as Hopf algebras since $P(H^*) = \langle x, x^p, x^{p^2}, \dots\rangle$, while $P((H')^*) = \langle x, x^p, \dots, y, y^p, \dots\rangle$, a contradiction.

Proof of Thm. 1.7. Let $\Lambda_p = k[\theta_{j,0}, \theta_{j,1}, \dots]$ be the Hopf algebra representing the functor of p-typical Witt vectors. Denote by $\eta\colon \Lambda_p \to \mathrm{Cof}^u(V(\Lambda_p))$ the unit of the adjunction. Applying the functor of primitives, since $\mathrm{Cof}^u(V(\Lambda_p))$ is cofree as a coalgebra by Lemma 5.3, we obtain a map

$$P\eta\colon P(\Lambda_p) \to V(\Lambda_p).$$

Now $V(\Lambda_p) = \mathrm{pol}_{(j)}(\Lambda_p) = k\langle \theta_{j,i}^{p^k} \mid i, k \geq 0\rangle$ and $P(\Lambda_p) = k\langle \theta_{j,0}^{p^k} \mid k \geq 0\rangle$, and $P\eta$ is the inclusion map. We see that $P(\Lambda_p)$ is in fact a direct factor of $V(\Lambda_p)$ as a p-polar algebra, with a retraction $p\colon V(\Lambda_p) \to P(\Lambda_p)$ given by

$$p(\theta_{j,i}) \mapsto \begin{cases} \theta_{j,0}; & i = 0 \\ 0; & \text{otherwise} \end{cases}$$

Taking adjoints, we obtain a map of Hopf algebras $q\colon \Lambda_p \to \mathrm{Cof}^u(P\Lambda_p)$. A map of unipotent Hopf algebras is injective iff it is injective on primitives, and $Pq\colon P\Lambda_p \to P(\mathrm{Cof}^i(P\Lambda_p)) \cong P\Lambda_p$ is the identity, so q is injective. By dimension considerations, it must also be surjective. □

5.1. F-modules and p-polar rings.

Definition. An *F-module* is a positively graded profinite k-vector space N with a k-linear map $F\colon N \to N(1)$. Denote their category by Mod_F.

Note that because of the grading, we could equivalently have defined an F-module to be positively graded and of finite type.

The category Mod_F is anti-equivalent (by taking continuous duals) to the category Mod_V of k-vector spaces M with a k-linear map $V\colon M(1) \to M$.

Kuhn showed in [Kuh20, Thm. 2.11] that every V-module of finite type over a perfect field k is a sum of modules $M(j,m)$ as follows:

$$M(2j,m) = \langle x, V^{-1}x, \dots, V^{-m}x\rangle \quad \text{with } |V^{-i}x| = 2p^i j, V(V^i x) = V^{i+1}x, Vx = 0,$$

$$M(2j,\infty) = \langle x, V^{-1}x, \dots\rangle \quad \text{with } |x| = 2j,$$

and

$$M(2j+1,0) = \langle y\rangle \quad \text{with } |x| = 2j+1,\ V = 0.$$

Remark 5.6. Finite type can be relaxed to countable generation [Web85, Thm. 2 and the remarks following the proof], but there are uncountably generated modules that do not decompose in this way. An example is given by $M_{2p^n} = \prod_{i\geq n} k$ with $V\colon M_{2p^n} \to M_{2p^{n-1}}$ being the canonical inclusion.

Corollary 5.7. *If k is perfect, any F-module of finite type is isomorphic to a product of $N(j,m) = M(j,m)^*$.* □

There is a forgetful functor $U_F\colon \mathrm{Pro}-\mathrm{pol}_p(k) \to \mathrm{Mod}_F$ from profinite p-polar k-algebras to F-modules, given by $U_F(A) = (A, x \mapsto x^p)$.

Lemma 5.8. *For every $M \in \mathrm{Mod}_F$ of finite type, there exists an $A \in \mathrm{Pro}-\mathrm{pol}_p(k)$ such that $U_F(A) \cong M$.*

Proof. Modules of the form $N(j,m)$ carry a unique p-polar algebra structure $A(j,m)$. For an arbitrary F-module M, which by Cor. 5.7 is isomorphic to a product $\prod_{i\in I} N(j_i, m_i)$, define $A = \prod_{i\in I} A(j_i, m_i)$. Then $U_F(A) \cong M$. □

Note that this construction is not functorial because it depends on the chosen decomposition of M. It is not true that any two p-polar algebras with isomorphic underlying F-module are isomorphic.

We obtain the following slight variation of Kuhn's theorem:

Corollary 5.9. *Given any F-module M of finite type, there is a formal connected Hopf algebra H which is cofree over a profinite p-polar algebra and such that $PH \cong M$, unique up to isomorphism of formal Hopf algebras cofree over profinite p-polar algebras.*

Proof. Given M, choose a p-polar algebra structure on M as in Lemma 5.8 and define $H = \mathrm{Cof}^u(M)$. Then $PH \cong M$ as F-modules. Kuhn showed in [Kuh20, Thm. 1.20] that H is unique among all formal Hopf algebras which are cofree as formal connected coalgebras and split. Here, a formal Hopf algebra H is called split if the inclusion $PH \to \tilde{H}$ has a retraction as F-modules, where $\tilde{H}$ is the augmentation coideal.

Lemma 5.3 shows that $\mathrm{Cof}^u(A)$ is always cofree as a coalgebra, and by Cor. 5.4, the composition

$$A \xrightarrow{\cong} P\,\mathrm{Cof}^u(A)) \to \widetilde{\mathrm{Cof}^u(A)} \to A$$

is the identity. This shows that Hopf algebras which are cofree over profinite p-polar algebras satisfy Kuhn's condition. ☐

6. The cohomology of free iterated loop spaces

Throughout this section, let X be a pointed, connected CW-complex and, for simplicity, $k = \mathbf{F}_p$. All homology and cohomology is taken with $\mathbf{F}_p$-coefficients.

Classical iterated loop space theory tells us that

$$H^*(\Omega^n\Sigma^n X) \cong \bigoplus_{k\geq 0} H^*(C_k \times_{\Sigma_k} X^{\wedge k}),$$

where C_k is the kth space of the little n-disks operad. This splitting is induced from the stable Snaith splitting. It is emphatically *not* a splitting of Hopf algebras (not even of algebras). By [CLM76, Chapter III], $H^*(\Omega^n\Sigma^n X)$ is a functor D_n of the algebra $H^*(X)$ as Hopf algebras. For instance, $D_1(A) = \mathrm{Cof}^{nc}(A)$ is the cofree *non-cocommutative* Hopf algebra on A.

In [Kuh20], Kuhn showed that for connected spaces X of finite type, $H^*(\Omega\Sigma X)$ only depends on the stable homotopy type of X in the sense that if $H^*(X) \cong H^*(Y)$ as F-modules then $D_1(H^*(X)) \cong D_1(H^*(Y))$. To each F-module M of finite type, he assigns a Hopf algebra $H(M)$ that is cofree with primitives $P(H(M)) \cong M$. It is, however, not true that H is a functor (and therefore that D_1 factors through the category of F-modules), as the following example shows.

Example 6.1. (This example arose from a discussion with Nick Kuhn.)

Let $M = \mathbf{F}_p\langle x, Fx\rangle$ be the two-dimensional $\mathbf{F}_p$-module generated by a class x in degree 2 and a class Fx in degree $2p$ such that $F(x) = Fx$ and $F(Fx) = 0$. Then $H(M) = \mathbf{F}_p[x_{(0)}, x_{(1)}, \dots]/(x_{(i)}^{p^2})$ with comultiplication

$$\Delta(x_n) = \sum_{i+j=n} x_i \otimes x_j,$$

where $x_{a_0+a_1p+\cdots+a_np^n} = \frac{1}{a_0!\cdots a_n!} x_{(0)}^{a_0} x_{(1)}^{a_1} \cdots x_{(n)}^{a_n}$ [Kuh20, Theorem 3.1]. Furthermore, $H(M^l) \cong \prod_{i=1}^l H(M)$, where the product is taken in the category of non-cocommutative Hopf algebras. If H could be made functorial with $P \circ H \cong \mathrm{id}$, we would have the following sequence of functors:

$$\mathrm{Mod}_F \xrightarrow{H} \mathrm{Hopf}^{nc}_{\mathbf{F}_p} \xrightarrow{\mathrm{ab}} \mathrm{Hopf}_{\mathbf{F}_p} \xrightarrow{D} \mathrm{Dmod}_{\mathbf{F}_p},$$

where ab denotes the functor associating to a non-cocommutative Hopf algebra $H \in \mathrm{Hopf}_k^{nc}$ its maximal cocommutative Hopf algebra. This functor is right adjoint to the inclusion functor and hence sends products of non-cocommutative Hopf algebras to products of cocommutative Hopf algebras, i.e. tensor products. We have that

$$D(H(M)) \cong \left\{ \mathbf{Z}/p \underset{1}{\overset{p}{\rightleftarrows}} \mathbf{Z}/p^2 \underset{1}{\overset{p}{\rightleftarrows}} \mathbf{Z}/p^2 \underset{1}{\overset{p}{\rightleftarrows}} \cdots \right\}$$

where arrows to the left are V and arrows to the right are F.

Note that $\mathrm{End}_{\mathrm{Mod}_F}(M) \cong \mathbf{Z}/p$ and $\mathrm{End}_{\mathrm{Dmod}_{\mathbf{F}_p}}(D(H(M))) \cong \mathbf{Z}/p^2$, and hence $\mathrm{Aut}_{\mathrm{Mod}_F}(M^l) \cong \mathrm{GL}_l(\mathbf{Z}/p)$ and $\mathrm{Aut}_{\mathrm{Dmod}_{\mathbf{F}_p}}(D(H(M^l))) \cong \mathrm{GL}_l(\mathbf{Z}/p^2)$. Since $P \circ H \cong \mathrm{id}$, functoriality would imply a section of the mod-p reduction homomorphism $\mathrm{GL}_l(\mathbf{Z}/p^2) \to \mathrm{GL}_l(\mathbf{Z}/p)$. However, by [Sah77, p. 22], this map splits exactly when $l = 1$ or when $l = 2$ and $p \leq 3$ or when $l = 3$ and $p = 2$. So there is a counterexample for every prime.

Our result gives a functorial factorization through a category that retains a little more structure than just a Frobenius.

To prove Thm. 1.8, we need to study the algebraic structure of $H_*(\Omega^{n+1}\Sigma^{n+1}X)$ more closely. For the reader's convenience, we recall the necessary details from [CLM76, III.1–4], with some corrections and additions from [Wel82]. An $(n+1)$-fold loop structure on a space X gives $H = H_*(\Omega^{n+1}\Sigma^{n+1}X)$ the structure of an algebra over the Dyer-Lashof algebra satisfying certain unstability conditions, quite analogously to the structure of cohomology rings as algebras over the Steenrod algebra. However, for finite n, there is an additional piece of structure: the so-called Browder operations $[-,-]\colon H_q \otimes H_r \to H_{q+r+n}$. Additionally, the top Dyer-Lashof $Q^{\frac{q+n}{2}}$ operation is not linear; its nonlinearity is measured by the Browder operations. The Browder operations together with the top Dyer–Lashof operation gives H_* the structure of a (n-shifted) restricted Lie algebra. We now outline the algebraic structure on the homology of an $(n+1)$-fold loop space, omitting details about signs to avoid too much clutter. The exact formulas are of little relevance here − what is mostly important is what part of the algebraic structure depends on what other parts. For simplicity, we assume that $p > 2$; the case of $p = 2$ is similar but slight simpler.

Definition. Let $p = \mathrm{char}(k) > 2$ and let M be a graded k-vector space. An R_n-structure on M consists of:

(1) Operations $Q^r\colon M_q \to M_{q+2r(p-1)}$ for $0 \leq 2r \leq q+n$;
(2) Operations $\beta Q^r\colon M_q \to M_{q+2r(p-1)-1}$ for $1 \leq 2r \leq q+n$;
(3) A *Browder operation* $[-,-]\colon M_q \otimes M_r \to M_{q+r+n}$

satisfying the following axioms:

(4) $[-,-]$ is bilinear;
(5) $[x,y] = \pm[y,x]$;
(6) $[x,[y,z]] \pm [y,[z,x]] \pm [z,[x,y]] = 0$;
(7) $[x, Q^r y] = 0$ for $2r < q+n$ and $[x, \beta Q^r y] = 0$ for $2r \leq q+n$;
(8) $[x, Q^{\frac{q+n}{2}} y] = [y,[y,\cdots,[y,x]\cdots]$ (p-fold bracket) for $q+n$ even;
(9) $Q^r x = 0$ if $2r < |x|$; $\beta Q^r x = 0$ if $2r \leq |x|$;
(10) All Q^r and βQ^r are additive except for $Q^{\frac{q+n}{2}}$ when $q+n$ is even;
(11) $Q^r(\lambda x) = \lambda^p Q^r(x)$; $\beta Q^r(\lambda x) = \lambda^p \beta Q^r(x)$

(12) $Q^{\frac{q+n}{2}}(x+y) = Q^{\frac{q+n}{2}}(x) + Q^{\frac{q+n}{2}}(y) + (x^p + y^p - (x+y)^p)$ when $q+n$ is even; here the parenthesized summand is understood as evaluated in the universal enveloping (noncommutative) algebra of the Lie bracket $[-,-]$ and can be shown to lie in M;

(13) $\beta^\epsilon Q^r Q^s = \sum_{i>0} \pm\binom{(p-1)(r-i)-1}{i-ps-1}\beta^\epsilon Q^i Q^{r+s-i}$ for $r > ps$, $\epsilon \in \{0,1\}$;

(14) $Q^r\beta Q^s = \sum_{i>0} \pm\left(\binom{(p-1)(r-i)}{i-ps}\beta Q^i Q^{r+s-i} \pm \binom{(p-1)(r-i)-1}{i-ps} Q^i \beta Q^{r+s-i}\right)$ for $r \geq ps$;

(15) $\beta Q^r \beta Q^s = \sum_{i>0} \pm\binom{(p-1)(r-i)-1}{i-ps}\beta Q^i \beta Q^{r+s-i}$ for $r \geq ps$.

An R_n-*algebra* is an R_n-module M with a commutative, unital multiplication satisfying

(16) $[x, 1] = 0$;

(17) $[x, yz] = [x, y]z \pm y[x, z]$;

(18) $Q^r 1 = 0$ and $\beta Q^r 1 = 0$ for $r > 0$;

(19) $Q^r x = x^p$ if $2r = |x|$;

(20) $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$ for $2r < q+n$;

(21) $\beta Q^r(xy) = \sum_{i+j=r}(\beta Q^i(x)Q^j(y) + Q^i(x)\beta Q^j(y))$;

(22) $Q^{\frac{q+n}{2}}(xy) = \sum_{i+j=\frac{q+n}{2}} Q^i(x)Q^j(y) + \Gamma(x,y)$ when $q+n$ is even, where Γ is a certain function of x and y constructed from the multiplication and Browder bracket in M;

An R_n-*Hopf algebra* is an R_n-algebra M with a cocommutative comultiplication Δ making M into a Hopf algebra and satisfying

(23) $\Delta Q^r(x) = Q^r(\Delta(x))$, where $Q^r\colon M\otimes M \to M \otimes M$ is given by the external Cartan formula: $Q^r(x\otimes y) = \sum_{i+j=r} Q^i(x)\otimes Q^j(y)$;

(24) $\Delta\beta Q^r(x) = \sum_{i+j=r}\sum_{(x)}(\beta Q^i(x')\otimes Q^j(x'') + Q^i(x')\otimes\beta Q^j(x''))$

(25) $\Delta[x,y] = \sum_{(x)}\sum_{(y)}\Big(\pm[x',y']\otimes y''x'' \pm x'y'\otimes[x'',y'']\Big)$.

(Axiom (23) seems to be formulated in a convoluted way, but the formula analogous to (24) would be wrong when $r = \frac{q+n}{2}$ because of the nonlinearity of Q^r.)

Denote by Hopf_{R_n} the category of R_n-Hopf algebras thus equipped. There is a forgetful functor

$$\mathrm{Hopf}_{R_n} \xrightarrow{U} \mathrm{Hopf}_k \to \mathrm{Coalg}_k, \tag{6.2}$$

It is shown in [CLM76, Thm. 3.2] that this composite has a left adjoint W such that

$$W(H_*(X)) \cong H_*(\Omega^{n+1}\Sigma^{n+1}X) \quad \text{for } X \text{ connected, } n \geq 1.$$

However, the left adjoint is not constructed as a composition of adjoint functors according to (6.2). Thus we need to show:

Lemma 6.3. *The functor* $U\colon \mathrm{Hopf}_{R_n} \to \mathrm{Hopf}_k$ *has a left adjoint* F.

Proof. While it is possible to give an explicit construction, we give a proof using Freyd's adjoint functor theorem.

The category Hopf_k has all small limits and we will show that U creates (and thus preserves) limits in Hopf_{R_n}. Given two R_n-Hopf algebras H_1 and H_2, the usual tensor product $H_1\otimes_k H_2$, which is the biproduct in Hopf_k, becomes an R_n-Hopf algebra by the external Cartan formulas for Q^i, βQ^i, and the Browder bracket. Given two maps $f, g\colon H_1 \to H_2$ of R_n-Hopf algebras, their equalizer (as graded sets) is a k-Hopf algebra and also the equalizer in Hopf_k. In fact, it is an R_n-Hopf algebra because f and g commutes with the R_n-structure. Since U is conservative (reflects isomorphisms), it thus creates finite limits. Finally, given an inverse system $H\colon I \to \mathrm{Hopf}_{R_n}$, its limit in Hopf_k is given as

$$\lim_I (U \circ H) = \{x \in \lim H \mid \Delta(x) \in \lim H \otimes \lim H\},$$

where the limits on the right hand side are limits of graded sets (equivalently, graded abelian group or graded commutative algebras). The diagonal Cartan formulas show that this limit is closed under Q^r, βQ^r, and the Browder bracket, so that $\lim_I H$ is created by $\lim_I U \circ H$.

Furthermore, U commutes with filtered colimits, which are also created in graded sets.

To apply Freyd's special adjoint functor theorem, we show that Hopf_k and Hopf_{R_n} are locally presentable categories. This is well-known for Hopf_k (it is, after all, equivalent to Dieudonné modules). Since Coalg_k is locally presentable and Hopf_{R_n} is monadic over it (either by the explicit form of Cohen's construction of the free R_n-Hopf algebra on a coalgebra or an application of Beck's monadicty theorem), it is also locally presentable [AR94, 2.78]. □

Proof of Thm. 1.8. By [CLM76, Thm. 3.2],

$$H_*(\Omega^{n+1}\Sigma^{n+1}X) \cong W(H_*(X)) \underset{\textit{Lemma } 6.3}{\cong} F(\mathrm{Fr}(H_*(X))).$$

Dually,

$$H^*(\Omega^{n+1}\Sigma^{n+1}X) \cong F^*(\mathrm{Cof}(H^*(X))),$$

where F^* is the functor F on opposite categories. By the formal case of Thm. 1.1, the desired factorization is given by

$$A \mapsto F^*(\mathrm{Cof}(A)) \quad \text{for } A \in \mathrm{pol}_p(k).$$

□

We obtain the following generalization of Kuhn's result, a strengthening of Cor. 1.9 from the introduction:

Corollary 6.4. *For $n \geq 0$, and spaces X for finite type, the R_n-Hopf algebra $H^*(\Omega^{n+1}\Sigma^{n+1}X)$ only depends on the F-module $H^*(X)$ (up to noncanonical isomorphism), and in particular only on the stable homotopy type of X.*

Proof. The case $n = 0$ was proved in [Kuh20, Cor. 7.2]. For $n \geq 1$, [Kuh20, Thm. 1.12] shows that $\mathrm{Cof}^{nc}(A)$ only depends on the structure of A as an F-module, up to noncanonical isomorphism. Here $\mathrm{Cof}^{nc}(A)$ is the cofree non-cocommutative formal Hopf algebra on A, but the cocommutative case follows since the inclusion of cocommutative into not necessarily cocommutative Hopf algebras has a right adjoint (cf. [Kuh20, Remark 3.13]). Since $H^*(\Omega^{n+1}\Sigma^{n+1}) = F^*(\mathrm{Cof}(H^*X))$, the result follows. □

In [Kuh20, Remark 7.3], Kuhn asks whether stable equivalence of spaces X and Y implies that $H^*(\Omega\Sigma X) \cong H^*(\Omega\Sigma Y)$ as Hopf algebras over the Steenrod algebra. While we cannot settle this case with our methods, note that this is true for $H^*(\Omega^{n+1}\Sigma^{n+1}X)$ for $n \geq 1$ because F^* and Cof are compatible with Steenrod operations, as follows:

The Steenrod operations on $H^*(X)$ extend to $\mathrm{Cof}(H^*(X))$ by the external Cartan formula. Given a (homology) Hopf algebra H with a compatible Steenrod action, this module structure extends to $F(X)$ by:

(1) the Cartan formula (for operations on products);
(2) the duals of the Nishida relations (for operations on $Q^i(x)$);
(3) the relation $\beta(Q^i(x)) = \beta Q^i(x)$ for $i < \frac{q+n}{2}$, where β on the left is the Steenrod algebra Bockstein, whereas βQ^i is the a priori indecomposable operation of the Dyer-Lashof algebra;
(4) the relation $\beta(Q^{\frac{q+n}{2}}(x)) = \beta Q^{\frac{q+n}{2}}(x) + [\beta(x), [x, \cdots, [x, x] \cdots]$ (p factors);
(5) the relations $P^s_*[x, y] = \sum_{i+j=s}[P^i_* x, P^j_* y]$ and $\beta[x, y] = [\beta x, y] \pm [x, \beta y]$ (for operations on $[x, y]$).

We see thus that $H^*(\Omega^{n+1}\Sigma^{n+1}X$, as an R_n-Hopf algebra with Steenrod operations, is determined up to noncanonical isomorphism by the Steenrod module $H^*(X)$ as long as X is of finite type.

References

[AM69] M. Artin and B. Mazur. *Etale homotopy*. Lecture Notes in Mathematics, No. 100. Springer-Verlag, Berlin, 1969.

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Bau21] Tilman Bauer. Affine and formal abelian group schemes on p-polar rings. preprint, to appear in Math. Scand., 2021.

[Bor54] Armand Borel. Sur l'homologie et la cohomologie des groupes de Lie compacts connexes. *Amer. J. Math.*, 76:273–342, 1954.

[Bor16] James Borger. Witt vectors, lambda-rings, and arithmetic jet spaces. Lecture notes and exercises, available at https://maths-people.anu.edu.au/~borger/classes/copenhagen-2016/, 2016.

[Bou96] A. K. Bousfield. On p-adic λ-rings and the K-theory of H-spaces. *Math. Z.*, 223(3):483–519, 1996.

[BW05] James Borger and Ben Wieland. Plethystic algebra. *Adv. Math.*, 194(2):246–283, 2005.

[CLM76] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[Dem86] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original.

[Fon77] Jean-Marc Fontaine. *Groupes p-divisibles sur les corps locaux*. Société Mathématique de France, Paris, 1977. Astérisque, No. 47-48.

[Goe99] Paul G. Goerss. Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 115–174. Amer. Math. Soc., Providence, RI, 1999.

[Haz03] Michiel Hazewinkel. Cofree coalgebras and multivariable recursiveness. *J. Pure Appl. Algebra*, 183(1-3):61–103, 2003.

[Haz09] Michiel Hazewinkel. Witt vectors. I. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 319–472. Elsevier/North-Holland, Amsterdam, 2009.

[Hes08] Lars Hesselholt. Lecture notes on Witt vectors. preprint at http://web.math.ku.dk/~larsh/papers/s03/wittsurvey.pdf, 2008.

[Kuh20] Nicholas J. Kuhn. Split Hopf algebras, quasi-shuffle algebras, and the cohomology of $\Omega\Sigma X$. *Adv. Math.*, 369:107183, 30, 2020.

[McG81] C. A. McGibbon. Stable properties of rank 1 loop structures. *Topology*, 20(2):109–118, 1981.

[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[Rav75] Douglas C. Ravenel. Dieudonné modules for abelian Hopf algebras. In *Conference on homotopy theory (Evanston, Ill., 1974)*, volume 1 of *Notas Mat. Simpos.*, pages 177–183. Soc. Mat. Mexicana, México, 1975.

[Sah77] Chih Han Sah. Cohomology of split group extensions. II. *J. Algebra*, 45(1):17–68, 1977.

[Sch70] Colette Schoeller. Étude de la catégorie des algèbres de Hopf commutatives connexes sur un corps. *Manuscripta Math.*, 3:133–155, 1970.

[Sul74] Dennis Sullivan. Genetics of homotopy theory and the Adams conjecture. *Ann. of Math. (2)*, 100:1–79, 1974.

[Tak71] Mitsuhiro Takeuchi. Free Hopf algebras generated by coalgebras. *J. Math. Soc. Japan*, 23:561–582, 1971.

[Tak74] Mitsuhiro Takeuchi. Tangent coalgebras and hyperalgebras. I. *Japan. J. Math.*, 42:1–143, 1974.

[Web85] Cary Webb. Decomposition of graded modules. *Proc. Amer. Math. Soc.*, 94(4):565–571, 1985.

[Wel82] Robert J. Wellington. The unstable Adams spectral sequence for free iterated loop spaces. *Mem. Amer. Math. Soc.*, 36(258):viii+225, 1982.

[Wil00] W. Stephen Wilson. Hopf rings in algebraic topology. *Expo. Math.*, 18(5):369–388, 2000.

[Wit37] Ernst Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1937.