Mapping class groups of spin surfaces and their homology

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by

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Abstract

In this MSc thesis, mapping class groups of spin surfaces are studied. A spin surface is a Riemann surface equipped with a spin structure on the tangent bundle, and the corresponding spin mapping class group is the group of isotopy classes of automorphisms of these surfaces. The main new result is Theorem 3.3.5 which states that the spin mapping class groups, when stabilized with respect to the genus of the underlying surface, have the homology of an infinite loop space. The proof relies on the methods used by Tillmann to show the same statement for ordinary mapping class groups.

In addition to proving this theorem the notions and properties of higher categories, n-simplicial objects and loop space machines are reviewed. Moreover, since the set-up Harer used to show homology stabilisation for spin mapping class groups does not exactly fit into the framework of this thesis, a section is devoted to showing that the two approaches are equivalent.

Contents

Introduction

Especially in the last twenty years, mapping class groups and moduli spaces of surfaces have enjoyed a great interest among topologists as well as algebraic geometers. The diversity of settings in which mapping class groups (or moduli spaces) occur is remarkable. A large variety of methods are used to tackle problems in connexion with mapping class groups.

A mapping class is simply an isotopy class of automorphisms of a given Riemann surface. The mapping classes form a group which is dependent on the genus of the surface, and if it is not a compact closed surface, then also on the number of boundary components (which one usually requires to stay fixed pointwise under a diffeomorphism) and on the number of punctures. However, it depends neither on the differentiable structure nor on the complex structure, so there are equivalent definitions of mapping class groups on differentiable, orientable surfaces or just topological, orientable surfaces.

Moreover, at least for closed surfaces, it is easy to define these topological symmetry groups of surfaces in a purely group-theoretic way without even mentioning surfaces: they are just outer automorphism groups of free groups (i.e. the full automorphism group modulo the inner automorphisms). A convenient, finite set of generators for them is also at hand, with the relations being less than convenient. But this point of view does not prove to be the most fruitful one for computational purposes. Many important properties can be derived from the operation of the mapping class group on the Teichmüller space of Riemann surfaces, which has as its orbit set the moduli space of surfaces. Since this operation is properly-discontinuous (finite isotropy groups) and Teichmüller space contractible, the homology of the groups on the one hand and the moduli space on the other is very similar (rationally equal). By compactifying Teichmüller space in such a way that the action of the mapping class group extends to the boundary (for example, by considering stable curves), one can see that the mapping class group contains a subgroup of finite index which behaves homologically like a compact manifold, according to the fact that moduli space is an orbifold. So, mapping class groups are virtually of finite cohomological dimension, virtual duality groups, and so on.

It is striking that mapping class groups share all these and many other properties with arithmetic groups without being arithmetic (except in the case of genus 1, where the mapping class group of a torus is just $SL_2(\mathbb{Z})$). Led by this analogy and using similar techniques, Harer proved what is probably the most important result with respect to the homology of mapping class groups: namely that it stabilises with increasing genus. This had been shown before for groups like $Sp_{2n}(R)$, $O_n(R)$ for certain kinds of ring R by Vogtman, Wagoner and others by considering simplicial complexes of subspaces of R^n, the module on which these groups act. Such systems of subspaces form a partially ordered set by inclusion, and to a partially ordered set one associates a simplicial complex X, where the n-simplices are chains $V_0 < V_1 < \cdots < V_n$, and the boundary maps are the obvious ones by omitting elements of the chain. The arithmetic group then acts on X and therefore gives rise to a spectral sequence, built from the double complex consisting of a resolution of $\mathbb{Z}$ over the group ring, tensored with the chain complex of X. The reason why this helps is that X turns out to be homology equivalent to a bouquet of high-dimensional spheres, and therefore the spectral sequence converges to zero in small dimensions (compared to n).

For mapping class groups, a rich combinatorial theory of simplicial complexes of embedded curves in the surface replaces the systems of subspaces. These are highly connected as well, as Harer shows using the theory of train tracks. In this way, one can infer that the i-th homology of mapping class groups is independent of the compact surface as long as its genus is higher than $3i+\text{const}$ (improved by Ivanov to $2i+\text{const}$). In particular, it is independent also of the number of boundary components, but not of the number of punctures, if there are any.

Recently, Ulrike Tillmann succeeded in applying the theory of infinite loop spaces, commonly referred to as “machinery”, and higher categories, to mapping class groups.

In [**Til**], she shows that their homology, if stabilised with respect to the genus, has the structure of an infinite loop space homology. To prove this, she constructs a 2-category, consisting of circles, bordisms between them (i.e. surfaces), and their automorphisms, and exhibits a strictly symmetric monoidal structure on it. Therefore, by infinite loop space theory, its realisation is an infinite loop space, and Tillmann shows by an application of the group completion theorem that its first deloop is homology equivalent to the stable mapping class group.

The aim of this thesis is twofold: on the one hand to explain in greater detail how the methods used by Tillmann work, in particular, what higher categories are, how you make the step from these categories to topology, and into the simplicial world. This is mostly well-known. On the other hand, to apply these methods to mapping class groups of surfaces with an additional spin structure. These groups are finite-index subgroups of the ordinary ones, and play an important rôle in modern physics, namely string theory. A spin structure on a complex curve can be defined to be a chosen square root of the canonical bundle, and the spin mapping class group is then the subgroup that preserves this bundle; but a more combinatorial description is more useful for these purposes: we regard a spin structure as a quadratic form on the first homology group with coefficients in the integers modulo 2. It was also Harer who showed that the spin mapping class group homology stabilises in the same way as the unspinned, except for a slightly worse stability range. Unfortunately, he explicitly only shows this for surfaces with one boundary which is not sufficient for my aim to transfer Tillmann's arguments to these groups to get the result that also spin mapping class groups are homologically infinite loop spaces. But a rather simple study of inclusions of surfaces and their effect on the mapping classes shows that everything carries over into the context Harer used.

I would like to thank my supervisor Ulrike Tillmann for many very helpful discussions and her steady interest in the progress of my work and this thesis.
Finally, I would like to thank Gavin Harper for his careful proof-reading.

CHAPTER 1

Homology of spin mapping class groups

Everywhere in this work, let $F_{g,n}$ be a connected, compact, oriented surface of genus g with n boundary components. Up to homeomorphism, for every $g, n \in \mathbb{N}_0$, there exists a unique such surface. The mapping class group of $F = F_{g,n}$ is defined to be the isotopy classes of orientation-preserving homeomorphisms of F onto itself, fixing the boundary pointwise: $\Gamma(F) := \Gamma_{g,n} := \pi_0 \operatorname{Homeo}^+(F, \partial F)$. Of course, orientation-preserving means in this case that every mapping class acts on $H_2(F; \partial F) \cong \mathbb{Z}$ as the identity. Alternatively, one can consider differentiable surfaces with self-diffeomorphisms, or in the case of a closed surface also homotopy equivalences. All these descriptions yield the same group $\Gamma(F)$ (cf. [**Mis94**] and the references given there.) The mapping class groups are generated by a finite set of Dehn-Twists ([**Deh38, Lic64**], later simplified in [**HT80**]), and indeed they are finitely presented. This was shown by McCool [**McC75**]; the first explicit and rather small complete set of relations was given by Wajnryb in [**Waj83**]. But these presentations known today are quite complicated and do not give much insight into the structure of the groups. It turns out that the mapping class groups share many properties with arithmetic groups, e.g., they are virtually duality groups, and their homology stabilises as the genus tends to infinity [**Har85**]. Nevertheless, it is known that the mapping class groups are *not* arithmetic.

It is also possible to consider mapping class groups of surfaces F that carry additional structure, e.g., a *spin structure*, which I will study here. By a spin structure we will here just mean a quadratic form Q on the first homology group of the surface with $\mathbb{Z}_2$ coefficients. Why this deserves the name "spin structure" will be explained in section 1.2. We call (F, Q) a spin surface (every orientable surfaces admits spin structures), and the spin mapping class group is the subgroup of all mapping classes of F which, in addition, preserve the form Q.

My aim in the next sections is to give a clear statement and proof of the stabilisation of the homology of mapping class groups of spin surfaces with several boundary components, using Harer's result [**Har90**] to show that the attachment of one spin surface to another of genus at least k yields an isomorphism between homology groups of spin mapping class groups up to dimension $k/4$.

1.1. Surfaces and Mapping Class Groups

In this first chapter, we will set up the right category to work in, and show some elementary but important properties of surfaces and mapping class groups.

We consider maps of the following kind between surfaces:

DEFINITION. Let F,G be surfaces, $f\colon F \longrightarrow G$. Call f a **weak embedding** if the restriction to the interior of F

$$f|_{\mathring{F}}\colon \mathring{F} \longrightarrow \mathring{G}$$

is an embedding.

• Strange things can happen on the boundary!

DEFINITION. The **mapping class group** of a surface F is

$$\begin{aligned}\Gamma(F) &:= \pi_0 \operatorname{Homeo}^+(F;\partial F)\\ &= \pi_0\{f\colon F \xrightarrow{\cong} F \mid f|_{\partial F} = \mathrm{id} \text{ and } f \text{ is orientation-preserving}\}\\ &= \frac{\operatorname{Homeo}^+(F;\partial F)}{\operatorname{Homeo}_0^+(F;\partial F)}.\end{aligned}$$

where Homeo_0^+ is the 1-component of Homeo^+. If $X \subseteq F$, then put

$$\Gamma(F;X) := \pi_0 \operatorname{Homeo}^+(F;X \cup \partial F).$$

Remark. Let $(F - X)^\vee$ denote the "closure" of $F - X$ whenever this makes sense: that is, if X is a neighbourhood retract of some open neighbourhood $V \supseteq X$, then $(F-X)^\vee := F - V$. If this is again a surface, then we have: $\Gamma(F-X)^\vee \cong \Gamma(F;X)$. We will need this notion for X an embedded graph.

• Note that for disconnected surfaces, this is only one of two natural definitions. Suppose that the surface contains two components without boundary. Then, in this definition, the homeomorphism that exchanges these two components is a mapping class; one could also postulate that every mapping class induces the identity on π_0. However, since we are mainly concerned with surfaces with boundaries in every component, there is no difference — if the boundary is fixed, then the surface is automatically fixed componentwise.

• Two homeomorphisms $f, g \colon F \longrightarrow F$ fall into the same Γ-class if and only if there is an isotopy $H \colon \mathbb{I} \times F \longrightarrow F$ satisfying $H_0 = f$, $H_1 = g$, and all H_t are homeomorphisms.

• "Homeomorphism" can be replaced by "diffeomorphism" everywhere above.

In this paragraph, the relationship between mapping class groups of subsurfaces of a given surface and stabilisers of subsets in the mapping class group will be investigated.

LEMMA 1.1.1. *If* $f : F \longrightarrow G$ *is a weak embedding then* f *induces a map*

$$f_* \colon \Gamma(F) \longrightarrow \Gamma(G).$$

So Γ *becomes a functor from the category of surfaces and weak embeddings into the category of groups.*

Proof: For $\phi \in \mathrm{Homeo}^+(F;\partial F)$ define:

$$(f_*\phi)(x) := \begin{cases} x; & x \notin \mathrm{im}(f)^\circ \\ f(\phi(f^{-1}(x))); & x \in \mathrm{im}(f)^\circ \end{cases}$$

$f_*\phi|_{\mathrm{im}(f)^\circ}$ and $f_*\phi|_{(G-\mathrm{im}(f))^\circ}$ are injective, extend to a map on the common boundary $\partial\,\mathrm{im}(f)$ and coincide there, so $f_*\phi$ is again a homeomorphism. f_* is well-defined on $\Gamma(F)$: If $\phi\colon \mathbb{I}\times F \longrightarrow F$ is an isotopy between ϕ_0 and ϕ_1 then $f_*\phi_t$ is an isotopy between $f_*\phi_0$ and $f_*\phi_1$. $\square$

Not only can we deform $\phi \in \mathrm{Homeo}^+(F;\partial F)$ without changing the image in $\Gamma(G)$ but also f: If we have an isotopy of weak embeddings (i.e. a map $f\colon \mathbb{I}\times F \longrightarrow G$ where each f_t is a weak embedding) the induced maps f_{t*} do not change with t:

LEMMA 1.1.2. *Isotopic weak embeddings induce equal maps on the mapping class groups.*

Proof: Let $f\colon \mathbb{I}\times F \longrightarrow G$ be an isotopy of weak equivalences f_0 and $f_1\colon F \longrightarrow G$, and let $\phi \in \mathrm{Homeo}^+(F;\partial F)$. Then it is easy to check the continuity of $t \mapsto f_{t*}\phi$, so there is a path in $\mathrm{Homeo}^+(G;\partial G)$ from $f_{0*}\phi$ to $f_{1*}\phi$. $\square$

One method to generate subgroups of $\Gamma(F)$ is to consider stabilisers:

DEFINITION. Let $X \subseteq F$ be a subset of a surface F. Define $\mathrm{Stab}_{\Gamma(F)} X$ to be the components of $\mathrm{Homeo}^+(F;\partial F)$ that intersect $\mathrm{Stab}_{\mathrm{Homeo}^+(F;\partial F)} X$ nontrivially. In formulae,

$$\mathrm{Stab}_{\Gamma(F)} X = \frac{\mathrm{Homeo}_0^+(F;\partial F)\cdot \mathrm{Stab}_{\mathrm{Homeo}^+(F;\partial F)} X}{\mathrm{Homeo}_0^+(F;\partial F)}.$$

• It is easy to see that $\mathrm{Stab}_{\Gamma(F)} X$ indeed is a subgroup of $\Gamma(F)$: If ϕ_0, ψ_0 are two representatives of elements in $\mathrm{Stab}_{\Gamma(F)} X$ then one can find, for both maps, a path in $\mathrm{Homeo}^+(F)$ to a homeomorphism that fixes X, say ϕ_t to ϕ_1, ψ_t to ψ_1. Then $t \mapsto \phi_t\circ\psi_t$ is obviously a path from $\phi_0\circ\psi_0$ to a homeomorphism fixing X, namely $\phi_1\circ\psi_1$. Therefore, $\phi_0\circ\psi_0$ is a representative of a map in $\mathrm{Stab}_{\Gamma(F)} X$.

• Warning: $\mathrm{Stab}_{\Gamma(F)} X \neq \Gamma(F;X)$! Indeed, $\Gamma(F;X)$ is not even a subset of $\Gamma(F)$. The reason is that in $\Gamma(F;X)$ we only divide out by homotopies that fix X *at any time*. But there is a surjection

$$\Gamma(F;X) = \frac{\mathrm{Homeo}^+(F;X\cup\partial F)}{\mathrm{Homeo}_0^+(F;X\cup\partial F)} \twoheadrightarrow \frac{\mathrm{Homeo}_0^+(F;\partial F)\cdot \mathrm{Stab}_{\mathrm{Homeo}^+(F;\partial F)} X}{\mathrm{Homeo}_0^+(F;\partial F)} = \mathrm{Stab}_{\Gamma(F)} X. \tag{1.1.3}$$

Let us introduce the following notation: if we have two surfaces F_1, F_2 and a homeomorphism α between a subset of the boundaries of F_1 and F_2 then we write $F_1 \sqcup_\alpha F_2$ for the result of gluing them together via f. The domain and image of f are called the *inner boundaries* of F_1 and F_2, respectively, and written $\partial_{\text{in}} F_i$, the complemental boundaries are $\partial_{\text{out}} F_i$.

We will now try to find a link between stabilisers of certain sets and mapping class groups of weakly embedded surfaces. Our first Lemma covers the case of *surjective* weak embeddings:

LEMMA 1.1.4. *If* $f\colon F \twoheadrightarrow G$ *is a surjective weak embedding, then*

$$f_*\colon \Gamma(F) \longrightarrow \mathrm{Stab}_{\Gamma(G)} f(\partial F)$$

is also surjective.

Proof: We have an isomorphism

$$\mathrm{Homeo}^+(G, \partial G \cup f(\partial F)) \cong \mathrm{Homeo}^+((G - f(\partial F))^\vee, \partial) \underset{\text{f surj.}}{\cong} \mathrm{Homeo}^+((F-\partial F)^\vee, \partial) = \mathrm{Homeo}^+(F).$$

In the same way, we get an isomorphism between the 1-components. So we get

$$\Gamma(F) \cong \Gamma(G, f(\partial F)) \twoheadrightarrow \mathrm{Stab}_{\Gamma(G)} f(\partial F) \quad \text{(by (1.1.3))}$$

$\square$

The following result links the notion of stabilisers with mapping classes of subsurfaces and shows that we did not treat too special a case in the above lemma:

PROPOSITION 1.1.5. *Let* F_1,F_2 *be two surfaces,* $F = F_1 \sqcup_\alpha F_2$ *for some* α*, and let*

$$f_i := f|_{F_i} \colon F_i \hookrightarrow F \quad (i = 1, 2)$$

be the inclusions. Suppose that F *is connected. Then* F_i *is isotopic to a* surjective *weak embedding* $g_i \colon F_i \twoheadrightarrow F$*, and the following diagram commutes, where* $A_i := g_i(\partial F_i) - \partial F$*:*

$$\begin{array}{ccc} \mathrm{Stab}_{\Gamma(F)} A_i & \xrightarrow{\textit{incl.}} & \Gamma(F) \\ {\scriptstyle g_{i*}} \uparrow & & \| \\ \Gamma(F_i) & \xrightarrow{f_{i*}} & \Gamma(F) \end{array}$$

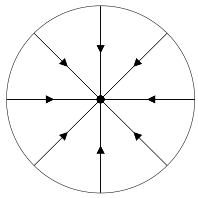
Proof: Since F is connected, at least one of F_1 and F_2 has to be connected. Without loss of generality, let $i = 1$ and F_2 be connected. Inductively, we can assume that F_2 is a pair of pants or a disc because we can decompose F_2 in such.

We construct a flow ϕ on F_2 such that $\phi_0(x) = x$ and $\partial_{out} F_2 \subseteq \phi_1(\partial_{in} F_2)$. This flow induces an isotopy between the inclusion $f_1 \colon F_1 \hookrightarrow F$ and a surjective map in the following way:
Let $\coprod_{j=1}^{k} \mathbb{S}^1 \times \mathbb{I} \hookrightarrow F_1$ be a collar around the inner boundary of F_1. Extend the flow ϕ on F_2 to this collar by defining on each component of the collar:

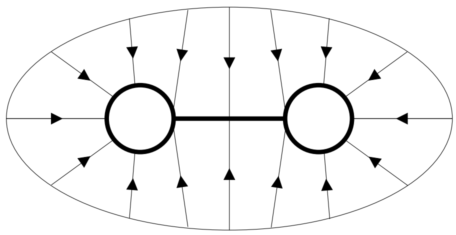
$$\begin{array}{ccc} \mathbb{I} \times \mathbb{S}^1 \times \mathbb{I} & \longrightarrow & F \\ (t, (\theta, s)) & \mapsto & \begin{cases} \phi_{s+t-1}(\theta); & \text{if } s + t > 1, \\ (\theta, s + t); & \text{if } s + t \leq 1. \end{cases} \end{array}$$

Extend it by the identity further on the whole of F.

Case 1. $F_2 = \text{disc} = \mathbb{D}^2$. The flow ϕ may be taken to be:

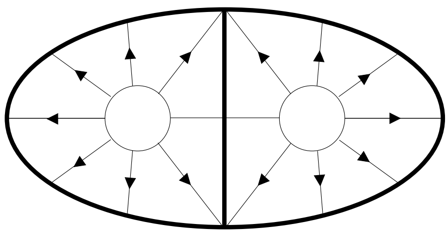


Case 2. $F_2 = \text{pants}$, glued along one component:



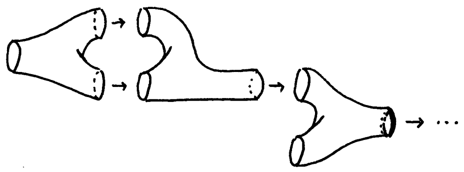
Thick lines in these pictures denote the image of the inner boundary of F_1 at the end.

Case 3. $F_2 = \text{pants}$, glued along two components:

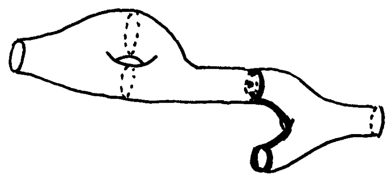


The resulting isotopy agrees with $f_1 \colon F_1 \hookrightarrow F$ at $t = 0$ and with $g_1 \colon F_1 \twoheadrightarrow F$ at $t = 1$. Therefore $f_{1*} = g_{1*}$ and the commutativity of the diagram is shown. $\square$

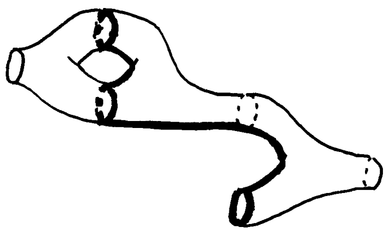
The following set of pictures illustrates the above procedure. They show the surface F_2, composed of three tori, and the image of the inner boundary of F_1 at each step (thick lines).



Slide 1



Slide 2



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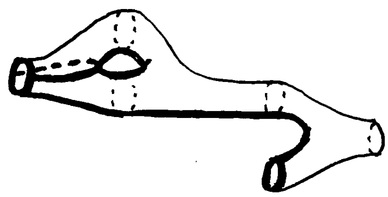


Slide 4

DEFINITION. An **arc system** on a surface F is a collection of simple arcs $\{\gamma_i\}$ in F such that:

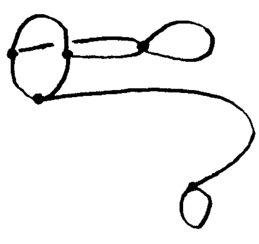
1. Two arcs intersect at most in their endpoints
2. All endpoints lie in ∂F
3. In every component of ∂F, there is at most one intersection point with the whole arc system
4. No arc is trivial, i.e. homotopic to a point *rel* endpoints, and no two arcs are isotopic to each other.

• The reason why we are interested in such special graphs in F is that there is a simplicial complex made up of such arcs. The vertices are simply non-trivial isotopy classes of arcs with fixed ends, and the n-simplices are collections of n such mutually non-isotopic arcs such that the complement is still connected. Arc systems play a central rôle in the study of mapping class groups. There exist quite a lot of different variants of arc complexes, all of which share the property that they are highly connected (see, eg, [**Har85**]), but ours agrees with the one Harer [**Har90**] uses in the case of only one boundary component to prove homology stability for spin mapping class groups.

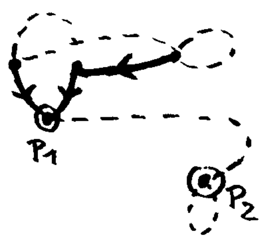
SUPPLEMENT. *The construction in Proposition 1.1.5 reveals that the set* A_1 *is a connected graph on* F *which contains every outer boundary component of* F_2*. If* ϕ *is a homeomorphism that fixes* A_1 *pointwise then we can find a small neighbourhood* U_1 *of* A_1 *in* F *such that* ϕ *can be deformed isotopically into a homeomorphism* ϕ' *which fixes all of* U_1 *pointwise. Having done this, we can deform* A_1 *into an arc system* $\{\gamma_j\}$ *with endpoints in* ∂F_2*, and this deformation can be chosen to be the identity outside* U_1*. Then we have:*

(1) $\mathrm{Stab}_{\Gamma(F)} A_1 = \mathrm{Stab}_{\Gamma(F)} \gamma_j$,
(2) *Every* γ_j *is a simple arc with endpoints in* ∂F_2*,*
(3) $F - \{\gamma_j\}$ *is still connected.*

Proof of the supplement: Let A be our graph and T an arbitrary spanning forest with as many components as $\partial_{out} F_2$, and roots p_i in these boundary components.
Example: In the case of our above example, the graph and its spanning forest look like:

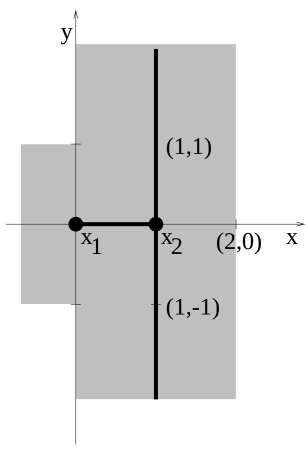


$A = \text{Image of } \partial F_1$



$T = \text{spanning forest of } A$

We now construct a flow ψ in F such that $\psi_0 = \mathrm{id}_F$ and $\psi_1(x) = p_i$ for every vertex x in the same component as p_i.
Let x_1, x_2 be two adjacent vertices in our graph such that x_1 is nearer to the root than x_2 (in the sense of the ordering induces by the forest). Choose a neighbourhood of the graph as described. Then we can find coordinates $a\colon \mathbb{R}^2 \longrightarrow F$ such that $a([0,1]\times\{0\}\cup\{1\}\times[-1,1]) \subseteq A$ and $a(0,0) = x_1$, $a(1,0) = x_2$, and no more components of A meet the set $\mathbb{R}\times[-1,1]$:

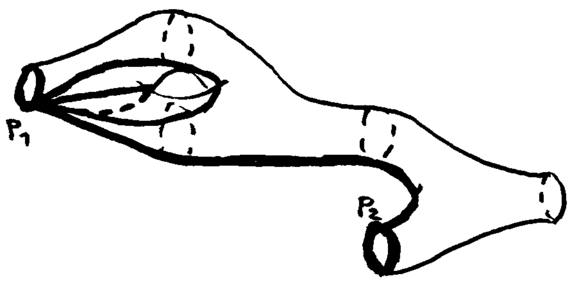


Now let $r\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a map which is 1 on $]-\infty, 1]$ and 0 on $[2, \infty[$. Furthermore, let

$$\psi(x,y) = \begin{cases} (x,y); & \text{if } x < 0 \text{ or } |y| \geq 1 \\ (1 - r(x) + |y| r(x))x; & \text{otherwise} \end{cases}$$

This map extends to all of F and deforms the graph such that the edge between x_1 and x_2 is mapped onto a point but the map is injective on all other points. It is homotopic to the identity via a linear homotopy on $\mathbb{R}^2$. If we nest the maps for all adjacent pairs x_1, x_2 in such an order that there is only one vertex afterwards, we get the desired transformation into an arc system. Since the homotopies can be nested in the same way, we see that the stabilisers of A and of $\{\gamma_i\}$ have not changed. $\square$

In our favourite example, the resulting arc system is shown in the next picture:



It is now natural to ask under which conditions an embedding $f\colon F \hookrightarrow G$ induces an inclusion of the mapping class groups.

Example. Let F be the standard annulus in $\mathbb{R}^2$, i.e. $F = \{x \in \mathbb{R}^2 \mid \frac{1}{2} \le \|x\| \le 1\}$, and G the unit disc. Then the inclusion $i\colon F \hookrightarrow G$ certainly cannot induce an inclusion of mapping class groups since $\Gamma(G) = 0$ because G is contractible, but $\Gamma(F) \cong \mathbb{Z}$ (the generator is given by the Dehn-twist along the circle $\|x\| = \frac{3}{4}$).

We will see that this is in a certain sense the only counter-example:

PROPOSITION 1.1.6. *Let* $f\colon F \hookrightarrow G$ *be an inclusion such that each component of* $G - \mathrm{im}(f)$ *contains at least one component of* ∂G *(hence we exclude the above example), and let* G *be connected. Then*

$$f_*\colon \Gamma(F) \longrightarrow \Gamma(G)$$

is injective.

Proof: Let $P := G - \mathrm{im}(f)^\circ$. Without losing generality, we can suppose that P is connected because otherwise we can repeat the argument with every component of P. Furthermore, it is even sufficient to show the proposition in the case where P is a pair of pants that shares one or two boundary components with $\mathrm{im}(f)$. This is because P

can otherwise be assembled from such atoms and we can use induction.
In the first case, when sewing along one component, the statement is trivial: if $g\colon G \longrightarrow F$ is the map which identifies one of the two remaining boundary components of P to a point, then $g \circ f \simeq \mathrm{id}_F$, thus $g_* f_* = \mathrm{id}_{\Gamma(F)}$, showing that f_* is injective.
The second case is slightly more difficult. Let us first assume that we have isotoped f into a surjective weak embedding, as described in Proposition 1.1.5. Then, by case 3 of our construction, $A := f(\partial F) - \partial G$ is a single arc, indeed we can assume a closed curve, and we know that $\Gamma(F) = \Gamma(G; A)$.
Now consider the fibration

$$\mathrm{Homeo}^+(G;A) \hookrightarrow \mathrm{Homeo}^+(G) \twoheadrightarrow \frac{\mathrm{Homeo}^+(G)}{\mathrm{Homeo}^+(G;A)}.$$

Let $\mathfrak{I}(G)$ be the space of all embeddings of circles with a fixed endpoint p (we take $p = \partial A = A \cap \partial G$). Let $a\colon (\mathbb{S}^1, *) \hookrightarrow (G, p)$ be a basepoint for this space with $\mathrm{im}(a) = A$.

Consider the map

$$\mathrm{Homeo}^+(G) \to \mathfrak{I}(G)$$
$$\phi \mapsto \phi \circ a$$

The kernel of this map is the set of all $\phi \in \mathrm{Homeo}^+(G)$ that fix A pointwise, i.e. $\mathrm{Homeo}^+(G;A)$. Therefore, we have an embedding

$$\frac{\mathrm{Homeo}^+(G)}{\mathrm{Homeo}^+(G;A)} \hookrightarrow \mathfrak{I}(G)$$

This embedding is extremely well-behaved in the sense that if a point of $\mathfrak{I}(G)$ is in the image then the whole component is. To see this, take two embeddings in the same component — without loss of generality a and another one, a'. Then they are linked by a path in $\mathfrak{I}(G)$, i.e. an isotopy of embeddings. Any such isotopy can be extended to the ambient surface (cf. **[Eps66]**, Theorem 4.1), so the preimage of a' we are looking for comes with a path in $\mathrm{Homeo}^+(G)$ connecting it with id_G.
This tells us that $\pi_1\left(\frac{\mathrm{Homeo}^+(G)}{\mathrm{Homeo}^+(G;A)}\right) = \pi_1(\mathfrak{I}(G))$, and it is known (see, for example,

[Gra73]) that the latter group is trivial.
Hence the last bit of the exact fibre sequence looks like:

$$\begin{array}{ccccccc} \ldots \longrightarrow & \pi_1(\mathfrak{I}(A)) & \longrightarrow & \pi_0(\text{Homeo}^+(G;A)) & \longrightarrow & \pi_0(\text{Homeo}^+(G)) & \longrightarrow \ldots \\ & \| & & \| & & \| & \\ & 1 & & \Gamma(G;A) & & \Gamma(G) & \end{array}$$

Therefore the map

$$\Gamma(G;A) \longrightarrow \text{Stab}_{\Gamma(G)}(A) \subseteq \Gamma(G)$$

is injective. □

1.2. Spin Structures on Surfaces

In this section I will study the mapping class groups of surfaces which carry a *spin structure*. Using all the results of the previous section, we obtain a functorial statement of Harer's theorem on the stability of the homology of these groups.

Generalities on spin structures.

Recall that for $n \geq 2$, the group $SO(n)$ is not simply connected, but has a two-sheeted (universal if $n \geq 3$) covering called Spin(n). Note that in the case $n = 2$ we are concerned with, Spin(2) $= SO(2) = \mathbb{S}^1$. If $E \longrightarrow M$ is an orientable n-dimensional vector bundle with a Riemannian metric on a manifold M then it has $SO(n)$ as its structure group. If we consider the associated principal $SO(n)$-bundle of orthonormal n-frames in E, let us call it $P(E)$, then we can try to find a two-sheeted covering of this bundle. A spin structure on the bundle $E \longrightarrow M$ is defined to be a principal Spin(n)-bundle $Q \longrightarrow E$ together with a 2-sheeted covering map $Q \longrightarrow P(E)$ that is the standard covering map Spin$(n) \longrightarrow SO(n)$ on each fibre. We are only concerned with spin structures on the tangent bundle.
Remark. The choice of a metric on E is not essential for the definition of spin structures. If we replace $SO(n)$ by $GL^+(n)$ (the 1-component of $GL(n)$) and Spin(n) by the

universal cover of this, we get isomorphic bundles. But it is often convenient to work with compact fibres.

Not every orientable vector bundle admits a spin structure. Neither is it normally unique if it does. But there are cohomological criteria for both: The fibration

$$SO(n) \longrightarrow P(E) \longrightarrow M$$

gives us a long exact sequence in $\mathbb{Z}_2$-cohomology:

$$0 \to H^1(M;\mathbb{Z}_2) \to H^1(P(E);\mathbb{Z}_2) \to H^1(SO(n);\mathbb{Z}_2) \xrightarrow{d^*} H^2(M;\mathbb{Z}_2).$$

To see this, we look at the lower left hand corner of the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(M; \mathfrak{H}^q(SO(n);\mathbb{Z}_2)) \Longrightarrow H^{p+q}(P(E)).$$

The local coefficient system $\mathfrak{H}^q(SO(n);\mathbb{Z}_2)$ is simple because

$$H^q(SO(n);\mathbb{Z}_2) \leq \mathbb{Z}_2,$$

so it does not have any nontrivial automorphisms. Therefore the E_2-term looks like (omitting $\mathbb{Z}_2$-coefficients):

$$\begin{array}{c|cccc}
0 & \vdots & \vdots & & \\
0 & H^0(M;H^1(SO(n))) & H^1(M;H^1(SO(n))) & \cdots & \\
0 & H^0(M;H^0(SO(n))) & H^1(M;H^0(SO(n))) & H^2(M;H^0(SO(n))) & \cdots \\
\hline
0 & 0 & 0 & 0 & 0
\end{array}$$

Since the first differential d_2 already has degree $(-1,2)$, $E_2^{0,1} = E_\infty^{0,1}$, and

$$E_\infty^{1,0} = E_3^{1,0} = \ker\left(d_2 \colon E_2^{1,0} \longrightarrow E_2^{0,2}\right)$$

and so we get the exact sequence

$$0 \to E_2^{0,1} = H^1 M \xrightarrow{\pi^*} H^1(P(E)) \xrightarrow{\iota^*} E_2^{1,0} = H^1 SO(n) \xrightarrow{d^*} E_2^{0,2}$$

which is exactly the above one.

Now, two-sheeted coverings of a manifold X (in our case, $P(E)$) are in one-to-one correspondence with elements of $H^1(X;\mathbb{Z}_2)$ (they are classified by maps $X \longrightarrow B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$, so their homotopy classes are just $[X; K(\mathbb{Z}_2, 1)] = H^1(X;\mathbb{Z}_2)$). But not every element of $H^1(P(E);\mathbb{Z}_2)$ defines a spin structure: the condition that it is the standard covering of $SO(n)$ on each fibre means exactly that this class maps to 0 in $H^1(SO(n);\mathbb{Z}_2)$. Such an element exists if and only if the map ι^* is surjective, i.e. if d^* is 0, i.e. the image of the generator of $H^1(SO(n);\mathbb{Z}_2)$ (which is the second Stiefel-Whitney class of E) is 0.

In our case of a surface F, the principal $SO(2)$-bundle can just be taken to be the unit tangent vector bundle $\mathbb{S}\mathbb{T}F$, and $\text{im}(d^*) = 0$ in $H^2(M;\mathbb{Z}_2)$ because the Stiefel-Whitney class is just the Euler class reduced modulo 2, and since F is orientable, its Euler characteristic is even. So we get a short exact sequence (coefficients in $\mathbb{Z}_2$ omitted):

$$0 \longrightarrow H^1(F) \xrightarrow{\pi^*} H^1(\mathbb{S}\mathbb{T}F) \xrightarrow{i^*} \mathbb{Z}_2 \longrightarrow 0.$$

Therefore we can define a spin structure on F to be an element ξ of $H^1(\mathbb{S}\mathbb{T}F;\mathbb{Z}_2)$, and we require that ξ is nontrivial on the fibres, which means that it does not map to 0 under i^*.

Atiyah **[Ati71]**, and later, in a quite different way, Johnson **[Joh80]** have shown that the set of all spin structures on a fixed surface correspond bijectively to *quadratic forms* on $H_1(F;\mathbb{Z}_2)$, therefore we will use this algebraic description:

DEFINITION. A **spin structure** on a (connected) surface F is a quadratic form

$$Q : H_1(F;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

i.e. it satisfies $Q(x+y) = Q(x) + Q(y) + \langle x, y\rangle$ where

$$\langle\cdot,\cdot\rangle\colon H_1(F;\mathbb{Z}_2) \otimes H_1(F;\mathbb{Z}_2) \longrightarrow H_0(F;\mathbb{Z}_2)$$

is the intersection form and $H_0(F;\mathbb{Z}_2)$ is identified with $\mathbb{Z}_2$.
If F and G carry a spin structure then we define a weak embedding of (F, Q_F) in (G, Q_G) to be a weak embedding $f\colon F \longrightarrow G$, and we require that $Q_F = f^*Q_G$, i.e. $Q_F(x) = Q_G(f_*(x))$ for all $x \in H_1(F;\mathbb{Z}_2)$. The inclusion map of a surface embedded in a bigger one allows us to speak about sub-spin surfaces.
The mapping class group of a spin surface $(F; Q)$ is, of course,

$$G(F, Q) := \pi_0 \operatorname{Aut}(F; \partial F) = \pi_0\{f \in \operatorname{Homeo}^+(F; \partial F) \mid f^*Q = Q\}.$$

• We can also define $G(F, Q)$ as the stabilizer of Q in $\Gamma(F)$ because isotopic maps f_1, f_2 induce the same map f_1^*, f_2^*.
• In our definition, we do not require that F is closed. Indeed, Q can take arbitrary values on the homology classes of the boundary components, subject only to the necessary condition that $Q(\partial_1 + \cdots + \partial_r) = 0$ if $\{\partial_i\}$ are all r boundary components of F (since $\partial_1 + \cdots + \partial_r \sim 0$.)

1.2.1. In the case of a closed surface, there are only two spin structures up to isomorphism (isomorphisms defined in the obvious way), classified by their *Arf invariant*

$$\alpha(F, Q) := \sum_{i=1}^{g} Q(a_i)Q(b_i)$$

where a_i and b_i are a symplectic basis of $H_1(F;\mathbb{Z}_2)$ with respect to the intersection form. However, this is no longer true if F has more than one boundary component, since if say $r' < r$ boundary components have $Q-$ value 1 then obviously this is also the case for every isomorphic image. But r' is always an even integer because $Q(\partial F) = 0$

and $Q(\partial_i + \partial_j) = Q(\partial_i) + Q(\partial_j)$. It is easy to see that indeed, we have a bijection between the set of equivalence classes of spin structures and the set $\mathbb{Z}_2 \times \{0, \dots, \lfloor \frac{r}{2} \rfloor\}$.

For our demands, it is often convenient to require that the spin form evaluates to 0 on every boundary component. This makes it, for example, possible to glue spin surfaces together along a single boundary without need to care about the consistency of the spin structures. In this case, we again get a complete classification of spin structures on the surface by their Arf invariant. Write $G^i_{g,n}$ for $G(F_{g,n}, Q)$ with $\alpha(Q) = i \in \mathbb{Z}_2$.

LEMMA 1.2.2. *There are* $2^{g-1}(2^g+1)$ *even and* $2^{g-1}(2^g-1)$ *odd spin structures on a surface of genus g, if we require that* $Q(\partial_i) = 0$ *for every boundary component* ∂_i.

Proof: Let n^i_g $(i = 0, 1)$ be the number of spin structures on a genus g surface with Arf invariant i. On a torus $F_{1,n}$ with generators $a, b \in H_1(F_{1,n}; \partial)$, there is only one odd spin structure, namely $Q(a) = Q(b) = 1$, the remaining three combinations are even. Since gluing of two surfaces means addition of the Arf invariants, we have the following recursive formulae:

$$
\begin{aligned}
n^0_g &= 3n^0_{g-1} + n^1_{g-1} \\
n^1_g &= n^0_{g-1} + 3n^1_{g-1}
\end{aligned}
$$

which easily yields $n^0_g = 2^{g-1}(2^g + 1)$ and $n^1_g = 2^{g-1}(2^g - 1)$. □

In particular, since $\Gamma_{g,n}$ acts transitively on the set of all spin structures of Arf invariant i and $G^i_{g,n}$ is the stabilizer of any one of them, we see that the index of $G_{g,n}$ in $\Gamma_{g,n}$ is finite, namely:

$$
\begin{aligned}
[\Gamma_{g,n} : G^0_{g,n}] &= 2^{g-1}(2^g + 1) \\
[\Gamma_{g,n} : G^1_{g,n}] &= 2^{g-1}(2^g - 1).
\end{aligned}
$$

In view of this result, one would expect that the spin mapping class groups have similar homological properties to the mapping class groups without spin. Indeed, the transfer map

$$\mathrm{tr}\colon H_*(G_g^i;\mathbb{Q}) \longrightarrow H_*(\Gamma_g;\mathbb{Q})$$

is injective with right inverse $\frac{1}{[\Gamma_g : G_g^i]}\iota_*$, where $\iota\colon G_g^i \longrightarrow \Gamma_g$ is the inclusion map. Hence, we get a splitting of homology groups

$$H_k(G_g^i;\mathbb{Q}) \cong H_k(\Gamma_g;\mathbb{Q}) \oplus \text{something}$$

where it is yet unknown if "something" is actually anything, at least for g big compared to k.

We can now restate Harer's stabilization theorem in the following way:

THEOREM 1.2.3. *If* (F, Q) *is a connected embedded sub-spin-surface of the connected spin surface* (F', Q') *such that every component of* $F' - F$ *contains at least one boundary component of* F'*, then the inclusion* $f\colon (F, Q) \hookrightarrow (F', Q')$ *induces an isomorphism*

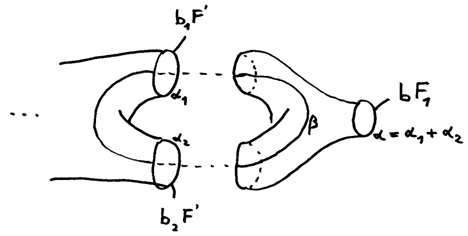
$$f_*\colon H_k(G(F, Q)) \xrightarrow{\cong} H_k(G(F', Q')) \textit{ for genus}(F) \geq 4k + 2$$

Proof: In [**Har90**, Theorem 3.1], Harer showed that if we have a loop system γ in a surface F with exactly one boundary component and we add another loop to obtain a system γ' then the inclusion $\mathrm{Stab}_{G(F,Q)}\,\gamma' \hookrightarrow \mathrm{Stab}_{G(F,Q)}\,\gamma$ gives us an isomorphism

$$i\colon H_k(\mathrm{Stab}_{G(F,Q)}\,\gamma') \xrightarrow{\cong} H_k(\mathrm{Stab}_{G(F,Q)}\,\gamma)$$

for $\mathrm{genus}(F - \gamma) \geq 4k + 2$. So, if $\gamma' - \gamma$ contains more than one arc, we can apply this theorem repeatedly and we get as a sufficient condition for the induced map being an isomorphism in the k-th homology: $\mathrm{genus}(F - \gamma') \geq 4k + 2$.
Let F' have genus g' and r' boundary components. Then F' can be included into the surface $F_1 = F_{g'+r'-1,1}$ of genus $g' + r' - 1$ and with only one boundary component by repeated attachment of a pair of pants to a pair of boundary components of F'. F_1 can

be given a spin structure such that this inclusion is an embedding of spin surfaces. To see this, take two boundary components b_1F' and b_2F' of F' with $Q(b_iF) = \alpha_i$. When attaching a pair of pants, we have to define two new Q-values in a compatible way: the Q-value α of the new boundary bF_1 has to be $Q(b_1F) + Q(b_1F) \pmod 2$ because b_1 is homologous to $b + b_2$, and the Q-value of the created new longitude β that transverses b_1F and b_2F can be chosen arbitrarily.



This inclusion of spin surfaces can be deformed into a surjective weak embedding i', according to Proposition 1.1.5.
In the same way, let g be a surjective version of f and i be the composition $i' \circ g$. This is again a surjective weak embedding of spin surfaces. Then we get two loop systems γ and γ' which are the images of ∂F and $\partial F'$ in F_1, respectively. In the last section, we have seen that $i_*\colon \Gamma(F) \longrightarrow \mathrm{Stab}_{\Gamma(F_1)}\gamma$ is surjective (by Lemma 1.1.4) and also injective (by Proposition 1.1.6), and the same is true for i' and γ'. Of course, this is still true if we intersect everything with the stabilizer of the quadratic form. By definition, $\gamma' \subseteq \gamma$, and therefore we have the commutative diagram

$$\begin{tikzcd}
G(F,Q) \arrow[r, "i", "\cong"'] \arrow[d, "f"] & \mathrm{Stab}_{G(F_1,Q_1)}\gamma \arrow[d, "\text{incl.}"] \\
G(F',Q') \arrow[r, "i'", "\cong"'] & \mathrm{Stab}_{G(F_1,Q_1)}\gamma'
\end{tikzcd}$$

which yields the stability of the k-th homology for $\text{genus}(F_1 - \gamma) = \text{genus}(F) \geq 4k + 2$. □

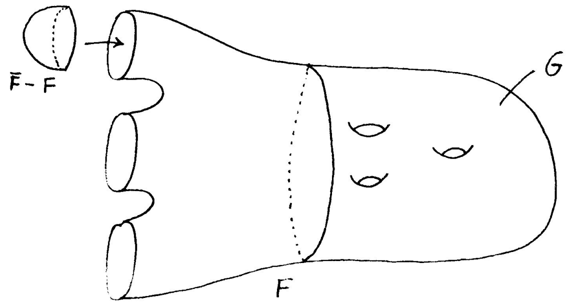
COROLLARY 1.2.4. *Let* (F, Q) *be a connected spin surface,* P *a pair of pants. Then the attachment of* P *to* F*, along one or along two boundary components, together with an arbitrary extension of the spin structure to* P *induces an isomorphism in the* k*-th homology groups if genus*$(F) \geq 4k + 2$.

This is only a special case of the above theorem.

We still do not know what happens to the homology when we attach a disc to a boundary component (of course, necessarily, the quadratic form must evaluate to 0 on this boundary) and thereby decrease the number of boundary components by 1. Harer also showed in his paper **[Har90]** that the attachment of a disc to a surface with exactly one boundary induces an isomorphism in H_k for $g \geq 4k + 7$. This is also true for surfaces with initially more than one boundary component:

COROLLARY 1.2.5. *Let* (F, Q) *be a connected spin surface,* ∂ *any boundary component of* F *with* $Q(\partial) = 0$*, and* D *a 2-disc. Then the attachment of* D *to* F *along* ∂ *induces a homomorphism of spin mapping classes and an isomorphism in their* k*-th homology if genus*$(F) \geq 4k + 7$.

Proof: The proof for one boundary component was done by Harer **[Har90]**. So let F have at least two boundary components, $\overline{F}$ be the surface with a cap attached to one of the boundary components, and G be a subsurface of F with only one boundary component but of the same genus:



Then the inclusions are compatible:

$$\begin{array}{ccc} G & \longrightarrow & F \\ & \searrow & \downarrow i \\ & & \overline{F} \end{array}$$

where i is the map induced by attachment of the disc. Theorem 1.2.3 applies to the other two inclusions, and so, $i_*\colon H_k(G(F)) \longrightarrow H_k(G(\overline{F}))$ is an isomorphism for genus $\geq 4k+2$. This result, although better than that stated above, is of course only true in the case where $\overline{F}$ is not closed. $\square$

CHAPTER 2

Methods

2.1. Higher Categories

In this section I will give an overview of the definitions of (strict) higher categories. The reader might want to skip this section if he or she feels familiar with the notion of strict n-categories, monoidal categories, and category operations. For the chapters to follow, nothing more than a monoidal 2-category is actually needed — nevertheless it seems to me to be instructive to treat the general case sometimes.

We begin by defining n-categories and n-functors for every $n \in \mathbb{N}_0$, which agree with conventional categories for $n = 1$. These n-categories are what is sometimes called "strict". Basically, there are two ways of defining them: one is recursive, by saying that an n-category is a category "enriched" over the $(n-1)$-categories; the other way is iterative, by giving a set of n suitably compatible composition operations. The first definition is more conceptual, whereas the second one is much easier to work with. We start with the recursive definition.

DEFINITION. A **(recursive) 0-category** is a class, and a **0-functor** is a map between two 0-categories.
Any n-category is called **small** if the underlying set of objects is small. A small 0-category is just a set.

Suppose we know what a recursive $(n-1)$-category and an $(n-1)$-functor are. Then an **n-category** $\mathcal{C}$ consist of a nonempty class $\mathcal{C}_0$ of objects and, for each pair of objects x, y, of a small $(n-1)$-category $\mathcal{C}(x, y)$ or $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of morphisms, together with a

composition $(n-1)$**-functor**

$$\circ\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z)$$

which is associative and has, in each $\mathcal{C}(x,x)$, a right and left identity element id_x.
Note that here, in requiring an equality of different applications of the composition functor, we use the assumption that the morphism $(n-1)$-category is small in order not to get entangled in set-theoretic problems. If one wants to get rid of this restriction, one has to replace the equalities by suitable and "coherent" natural transformations and is quickly led to the notions of "weak categories" that have become popular recently.
In this definition, in writing $\mathcal{C}(y,z) \times \mathcal{C}(x,y)$, we have used the fact that for two $(n-1)$-categories (in fact for *any* set of $(n-1)$-categories $\mathcal{C}_i$) one can construct the **product category** $\prod_i \mathcal{C}_i$ whose objects are the products of the classes of objects of the $\mathcal{C}_i$, and whose morphisms are the products of the morphism categories one dimension below. Likewise, we can define the **coproduct category** $\coprod_i \mathcal{C}_i$, using disjoint union.

For any n-category $\mathcal{C}$, $n \geq 1$, we define the **total morphism category** $\mathcal{C}'$ to be

$$\mathcal{C}' := \coprod_{x,y \in \mathcal{C}_0} \mathcal{C}(x,y).$$

This category is then an $(n-1)$-category. As a shortcut, $\mathcal{C}^{(n)} := (\mathcal{C}^{(n-1)})'$.
We call the elements of $(\mathcal{C}^{(k)})_0$ the k-morphisms of the n-category $\mathcal{C}$.

An **n-functor** F between two n-categories $\mathcal{C}, \mathcal{D}$ consists of a map

$$F_0\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$$

and an $(n-1)$-functor

$$F'\colon \mathcal{C}' \longrightarrow \mathcal{D}' \quad \text{where} \quad F'(\mathcal{C}(x,y)) \subseteq \mathcal{D}(F_0(x), F_0(y))$$

such that for $x \in \mathcal{C}_0$, $F'(\mathrm{id}_x) = \mathrm{id}_{F_0(x)}$ and such that it takes compositions to compositions:

$$\begin{tikzcd}
\mathcal{C}(y,z) \times \mathcal{C}(x,y) \arrow[r, "F\times F"] \arrow[d, "\circ"'] & \mathcal{D}(y',z') \times \mathcal{D}(x',y') \arrow[d, "\circ"'] \\
\mathcal{C}(x,z) \arrow[r, "F"] & \mathcal{D}(x',z')
\end{tikzcd}$$

where the dashed letters are the images under F_0. It is usual to write just F instead of F_0 and F'.

2.1.1. A k-morphism f will be identified with the $(k+1)$-morphism id_f, the $(k+2)$-morphism $\mathrm{id}_{\mathrm{id}_f}$, and so on; whereas it gives also rise to $(k-1)$-morphisms source(f) and target(f). So, the most general morphism is an n-morphism. A "morphism" is just any one of the above.

As we are used to in 1-categories, we can work with diagrams of objects and morphisms. But since there may also be higher morphisms in addition to objects and 1-morphisms, one is led to draw diagrams like

$$\begin{tikzcd}
x_0 \arrow[r, bend left, "x_1"] \arrow[r, bend right, "y_1"'] \arrow[r, phantom, "\Downarrow x_2"] & y_0
\end{tikzcd}$$

where the x_i and y_i are i-morphisms, and, for example, source$(x_2) = x_1$. It should be intuitively clear what it means that diagrams composed of such pieces commute. By suppressing some information, the diagram can also be drawn as $x_0 \xrightarrow{x_2} y_0$.

For the iterative definition of higher categories, we need a few preliminary definitions:

DEFINITION. A **partial monoid** is a set X together with a partial map

$$\Box\colon X \times X \longrightarrow X$$

and a subset $\mathcal{I}$ of X, the identities, satisfying:

$$(2.1.2)\qquad \begin{cases} \text{(i)} & (x\Box y)\Box z = x\Box(y\Box z) \text{ whenever this is defined;} \\ \text{(ii)} & i\Box x = x \text{ and } y\Box i = y \text{ whenever } i \in \mathcal{I} \text{ and this is defined;} \\ \text{(iii)} & \text{for every } x \in X\text{, there are (automatically unique) } i, j \in \mathcal{I} \text{ such} \\ & \text{that } i\Box x \text{ and } x\Box j \text{ are defined.} \end{cases}$$

A **homomorphism** between two partial monoids $(X, \Box, \mathcal{I})$ and $(Y, \Box, \mathcal{J})$ is a function $f\colon X \longrightarrow Y$ such that:

1. $f(\mathcal{I}) \subseteq \mathcal{J}$
2. If $x\Box y$ is defined, then so is $f(x)\Box f(y)$, and $f(x\Box y) = f(x)\Box f(y)$.

• If $\mathcal{I}$ consists of a single element, we get the usual definition of a monoid back.
• A partial monoid is nothing but the total morphism set of a small 1-category. For the converse direction (to get from a partial monoid to a category $\mathcal{C}$), one defines

$$\begin{aligned} \mathcal{C}_0 &:= \mathcal{I} \text{ and} \\ \mathcal{C}(i,j) &:= \{x \in X \mid j \circ x \circ i \text{ is defined}\} \end{aligned}$$

and considers $\mathcal{C}_0$ and $\mathcal{C}'$ as disjoint. The composition map is the monoid multiplication, and the category axioms are easily checked. Therefore, we have seen that partial monoids and 1-categories are the same concepts. In order to generalize this and to define iterative higher categories, we need:

DEFINITION. Let X be a set with k different partial monoidal structures $\Box_1, \ldots, \Box_k$ on it. Let us call this sequence of structures **compatible** if it satisfies:

$$(2.1.3)\qquad \begin{cases} 1. & \text{Whenever } (x_1\Box_i y_1)\Box_j(x_2\Box_i y_2) \text{ is defined, then so is} \\ & (x_1\Box_j x_2)\Box_i(y_1\Box_j y_2)\text{, and the latter is equal to the former;} \\ 2. & \text{if } x \text{ is a left or right identity for } \Box_i \text{ then it is also a left or right} \\ & \text{identity for } \Box_j \text{ if } j > i. \end{cases}$$

DEFINITION. An **(iterative) n-category** is a set X together with n compatible partial monoidal structures. An n-functor is a map between two such sets X, Y that is a homomorphism with respect to every monoidal structure.

PROPOSITION 2.1.4. *We can identify iterative and small recursive n-categories. More precisely, the function $^{(n)}$ that assigns to a small recursive n-category the set of n-morphisms takes values in n-fold compatible monoids; and it has an inverse $\mathcal{B}^n$.*

The proof is deferred to the next section.

Examples of higher categories. The prototypical example of an n-category is the category $(n-1)-\mathcal{CAT}$ of all small $(n-1)$-categories, where the morphisms are $(n-1)$-functors. To see that this is indeed an n-category, we define:

DEFINITION. When we have two n-functors $F_1, F_2 \colon \mathcal{C} \longrightarrow \mathcal{D}$ of recursive n-categories $\mathcal{C}, \mathcal{D}$, we can define the $(n-1)$-category of **morphisms** (or **natural transformations**) between these functors to be the set of all $(n-1)$-functors $H \colon \mathcal{C} \longrightarrow \mathcal{D}'$ that satisfy $\operatorname{source} H(x) = F_1(x)$ and $\operatorname{target} H(x) = F_2(x)$ and make the following diagram commute for all $(n-1)$-morphisms x,y and every n-morphism $f \colon x \longrightarrow y$:

$$\begin{tikzcd} F_1(x) \arrow[r, "H(x)"] \arrow[d, "F_1(f)"'] & F_2(x) \arrow[d, "F_2(f)"] \\ F_1(y) \arrow[r, "H(y)"] & F_2(y). \end{tikzcd}$$

If $(n-1)-\mathcal{CAT}(\mathcal{C}, \mathcal{D}')$ is an $(n-1)$-category, then so is the subcategory of morphisms from F_1 to F_2; therefore, inductively, we get that this is true (since this is obviously true for $n = 1$).

NOTATION. I shall use the following symbols for categories:

Ens	— the 1-category of sets
$\mathcal{CAT}, n - \mathcal{CAT}$	— the 2-category of small categories, the $(n+1)$-category of small n-categories
$\mathcal{C}at, n - \mathcal{C}at$	— the 1-category of small n-categories, with morphisms the n-functors; $\mathcal{C}at = 1 - \mathcal{C}at$

Another simple example comes from groups. If G is a group (or just a monoid) then G is a 0-category, and we have basically two possibilities of understanding G as a 1-category: the first one is $\mathcal{B}G$, the 1-category that has one object and one (iso-)morphism g for every $g \in G$.
The other method is to take G as the object set of the category and have one morphism $g \in G$ from h to hg. In a way, the information here is more redundant. This category has #G elements and exactly one morphism between each pair of objects. We call it the translation category of G and write $\mathcal{E}G$.
We can now ask whether G is also a higher category in a natural way. In lack of any other structure in sight, we could take one object, one 1-morphism, and G as the set of 2-morphisms, where $\circ_1$ as well as $\circ_2$ is the group multiplication. Since any two morphisms are composible in both ways, we must ensure that

$$(g \circ_1 g') \circ_2 (h \circ_1 h') = (g \circ_2 h) \circ_1 (g' \circ_2 h')$$
$$\iff gg'hh' = ghg'h' \qquad \text{for all } g, g', h, h' \in G.$$

So G has to be abelian. In fact, if G is abelian, we could in the same way also construct a 3-, 4-, ... category from G.

2.2. Mixing the two concepts: monoidal categories

In practice, one often has to switch between the monoidal, iterative point of view, and the recursive definition of categories. Moreover, these two approaches are sometimes combined; in particular, the notion of a *monoidal category* is very important especially in the theory of loop spaces. Therefore we extend the range of definition for monoids from sets to recursive n-categories:

DEFINITION. A **monoidal structure** on a small recursive n-category $\mathcal{C}$ is an associative n-functor $\Box\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ together with a right and left identity element 0 in $\mathcal{C}$. $\mathcal{C}$ is then called monoidal. A **partial monoidal structure** on $\mathcal{C}$ is the obvious generalization of the definition of partial monoids to n-categories: a "partial" n-functor $\Box\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ together with a subset $\mathcal{I} \subseteq \mathcal{C}_0$ satisfying (2.1.2). Here, "partial" means that the functor is possibly not defined on all objects, but if it is defined on objects (X, Y) and (X', Y') then also on every k-morphism $(f, g)\colon (X, Y) \longrightarrow (X', Y')$, for every $k \geq 1$. If we have k partial monoidal structures of $\mathcal{C}$, they are called **compatible** if the axioms (2.1.3) hold.

The following proposition is a stronger result than Proposition 2.1.4, and implies it:

PROPOSITION 2.2.1. *Let* $k\,\mathrm{Mon}\,n\mathcal{C}at$ *be the* 1-category *of all small* k*-fold compatibly partial monoidal* n*-categories, where the morphisms are functors which are homomorphisms with respect to the monoidal structures. For* $k \geq 1$*, there are functors*

$$k\,\mathrm{Mon}\,n\mathcal{C}at \underset{\mathrm{mor}}{\overset{\mathcal{B}}{\rightleftarrows}} (k-1)\,\mathrm{Mon}(n+1)\mathcal{C}at \ ,$$

inverse to each other up to natural isomorphism, where mor *is taking total morphism sets:* $\mathrm{mor}(\mathcal{C}) = \mathcal{C}'$*. In particular (iterating this), we can identify* n*-fold compatible partial monoids with* n*-categories.*

• $\mathcal{B}$ is called the **classifying-functor** for a monoidal category. Some authors use a 'B' letter differently to denote the nerve of a category, which I will call $\mathcal{N}$.
Proof: First of all, we must observe that mor does indeed take values in k-fold compatibly partial monoidal n-categories. Define

$$\begin{array}{rccc} \mathrm{mor}: & (k-1)\,\mathrm{Mon}(n+1)\mathcal{C}at & \longrightarrow & k\,\mathrm{Mon}\,n\mathcal{C}at \\ & ((\mathcal{C}, \circ), \Box_1, \ldots, \Box_{k-1}) & \mapsto & (\mathcal{C}', \Diamond_1, \ldots, \Diamond_k) \\ & (F\colon X \longrightarrow Y) & \mapsto & F' \end{array}$$

where

$$\diamond_1 = \circ, \diamond_2 = \square'_1, \ldots, \diamond_k = \square'_{k-1}.$$

and the identities for $\diamond_1$ are $\{\mathrm{id}_X \mid X \in \mathcal{C}_0\}$.
We have to check the compatibility conditions (2.1.3). Since $\diamond_2, \ldots, \diamond_k$ are compatible by definition, it remains to show:

1. $(x \diamond_j y) \diamond_1 (x' \diamond_j y') = (x \diamond_1 x') \diamond_j (y \diamond_1 y') \quad$ for all $j > 1$;
2. if x is a right or left identity for $\diamond_1$, then so it is for every $\diamond_j$, $j \leq k$.

The first assertion is clear because $\square_j$ is an $(n+1)$-functor: it takes compositions to compositions. The second assertion is not less obvious since the property of being a left or right identities for composition is preserved under functors.

For the inverse map, define:

$$\begin{array}{rccc} \mathcal{B}: & k\,\mathrm{Mon}\, n\mathcal{C}at & \longrightarrow & (k-1)\,\mathrm{Mon}(n+1)\mathcal{C}at \\ & (\mathcal{C}, \diamond_1, \ldots, \diamond_k) & \mapsto & ((\mathcal{D}, \circ), \square_1, \ldots, \square_{k-1}) \\ & (F\colon X \longrightarrow Y) & \mapsto & F \end{array}$$

where

$$\mathcal{D}_0 := \mathcal{I}, \text{ the identities for } \diamond_1;$$

$$\mathcal{D}(I, J) := \{X \in \mathcal{C} \mid J \diamond_1 X \diamond_1 I \text{ is defined }\};$$

$$\circ := \diamond_1;$$

$$\square_j := \diamond_j + 1 \quad (j = 1, \ldots, k-1).$$

The composition function is well-defined because $X \in \mathcal{D}(J, K), Y \in \mathcal{D}(I, J)$ implies $K \diamond X \diamond J \diamond J \diamond Y \diamond I = X \diamond Y$ is defined. It is by definition an associative n-functor, and so, we get an $(n+1)$-category. But we have to show that the remaining $\square_1, \ldots, \square_{k-1}$ still form a $(k-1)$-fold compatibly partial monoidal structure on $\mathcal{D}$.
It is enough to show: *For all* j, $\square_j$ *is an associative* $(n+1)$*-functor.* For if we remove the

first member of a compatible sequence, the remaining sequence is still compatible. The associativity is clear. For the functoriality, observe that $\Diamond_j$ maps identities to identities:

$$\mathrm{id}_{I\times J} = I \times J \in \mathcal{D}' \times \mathcal{D}'$$

and if defined, then

$$I \Diamond_j J = I = J \quad \text{by Axiom (ii) of compatibility (2.1.3).}$$

If we have composible morphisms $X \times X', Y \times Y' \colon \mathcal{D}' \times \mathcal{D}' \longrightarrow \mathcal{D}' \times \mathcal{D}'$, then

$$(X \circ Y) \Diamond_j (X' \circ Y') \underset{\text{Def}}{=} (X \Diamond Y) \Diamond_j (X' \Diamond Y') \underset{2.1.3(\text{i})}{=} (X \Diamond_j Y) \Diamond (X' \Diamond_j Y').$$

□

In the following sections, we will be particularly interested in monoidal structures that are "commutative", at least up to a natural isomorphism. More precisely:

DEFINITION. A slightly weaker condition for the monoidal structure than being commutative is being **symmetric**. This means that there exists a natural transformation $s \colon \Box \longrightarrow \Box \circ \mathrm{Tw}$, where $\mathrm{Tw} \in \mathrm{End}(\mathcal{C} \times \mathcal{C})$ is the twist functor $X \times Y \mapsto Y \times X$, satisfying the following compatibility conditions with the monoidal structure:

1. $s(X, 0) = \mathrm{id}_X = s(0, X)$;
2. $s(X, Y \Box Z) = \big(\mathrm{id}_Y \Box s(X, Z)\big) \circ \big(s(X, Y) \Box \mathrm{id}_Z\big)$;
3. $s(Y, X) \circ s(X, Y) = \mathrm{id}_{X \Box Y}$.

If the last condition does not necessarily hold, we speak of a **braiding**.

Whereas there are hardly any *commutative* monoidal categories in nature, the *symmetric* ones occur quite often. For example, any category having strictly monoidal finite products or coproducts is symmetric monoidal. However, with the completely strict definitions given up to now, the category of modules over a commutative ring, with the tensor product, is not monoidal because we have no right to say that $(A \otimes B) \otimes C$ *equals* $A \otimes (B \otimes C)$.

2.2.1. Category Operations. Now, we want to generalize the notion of an action of a group (or a monoid) on a set X, or a space, or a similar object, to actions of a whole small category. If we regard a monoid action of G on X as a functor from $\mathcal{B}G$ to the category Ens where X is just the image of the unique object of $\mathcal{B}G$, it seems natural to define:

DEFINITION. An **operation** of a small 1-category $\mathcal{C}$ on sets (or spaces etc.) is a functor $\rho\colon \mathcal{C} \longrightarrow \mathrm{Ens}$ (or Top etc.). We say that $\mathcal{C}$ operates on the collection $\{\rho(X) \mid X \in \mathcal{C}_0\}$, and we call the subcategory $\mathrm{im}(\rho)$ a **$\mathcal{C}$-diagram**.

If we have an operation ρ of $\mathcal{C}$, we can construct the following important object, the **translation category** $\mathcal{E}\rho$ of ρ:

$$(\mathcal{E}\rho)_0 := \{(C, x) \mid C \in \mathcal{C}_0 \text{ and } x \in \rho(C)\}$$
$$(\mathcal{E}\rho)((C, x), (D, y)) := \{f \in \mathcal{C}(C, D) \mid \rho(f)(x) = y\}$$

Examples: If ρ is the trivial functor which assigns the same set to every object and the identity morphism to every morphism, we get $\mathcal{E}\rho \cong \mathcal{C}$. This does not seem to be an interesting example.
If G is a monoid, then G operates on itself by left multiplication. This operation gives us a functor $\rho\colon \mathcal{B}G \longrightarrow \mathrm{Ens}$ whose translation category $\mathcal{E}\rho$ is the translation category $\mathcal{E}G$ defined before. So, we use $\mathcal{E}G$ as a shortcut.

There is a functor $\pi\colon \mathcal{E}\rho \longrightarrow \mathcal{C}$ given "by projection onto the first factor":

$$\pi(C, x) := C \quad \text{and} \quad \pi(f) := f.$$

If we take nerves of all categories and functors (as to be defined in the next section), we will see that the functor $\mathcal{E}\rho \xrightarrow{\pi} \mathcal{B}G$ for ρ as defined in the example above becomes the principal G-bundle on the classifying space BG with contractible total space EG.

2.3. Multi-Simplicial Categories, Nerves, and Geometric Realisation

I assume that the reader is familiar with the notions of simplicial objects in 1-categories, the basic theory of simplicial sets, and the nerve construction that assigns a simplicial set to a small category. For background information on simplicial objects, consult May's book [**May72**] or [**Jar97**]. The nerve functor and its properties were introduced by Segal [**Seg68**]

There is a generalisation of simplicial objects, multi-simplicial objects, which are important whenever one deals with fibrations in the simplicial category, or with category operations as described above, or higher categories. The natural generalisation of the nerve functor to n-categories takes values in n-simplicial sets. Let me first define a few notations I use:

NOTATION. $[n]$ is the ordered set $\{0, \dots, n\}$; Δ is the category with objects $[n], n \in \mathbb{N}_0$, and maps the monotonic maps.

I denote the category of simplicial objects in a category $\mathcal{C}$ with $\Delta\mathcal{C}$; remember this is just the category of functors $\Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ with morphisms the natural transformations.

$\mathcal{N}\colon \mathcal{C}at \longrightarrow \Delta\,\mathrm{Ens}$ is the nerve functor that assigns to a small category $\mathcal{C}$ the nerve, regarded as a simplicial set.

For a monoid G, BG is defined to be $\mathcal{N}\mathcal{B}G$ and is the **(simplicial) classifying space** of G. Its realisation is homotopy equivalent to the Milnor construction with join-products.

DEFINITION. An n-**simplicial object** in a category $\mathcal{C}$ is a simplicial object in the category of $(n-1)$-simplicial object in $\mathcal{C}$. Equivalently, it is a functor $(\Delta^{\mathrm{op}})^n \longrightarrow \mathcal{C}$ from the nth Cartesian power of the category Δ^{op} to $\mathcal{C}$. The n-simplicial objects in $\mathcal{C}$ form a category which I call $\Delta^n\mathcal{C}$.

Examples. n-simplicial sets are the objects of the category Δ^n Ens; based n-simplicial sets are the objects of the category Δ^n Ens_* of n-simplicial objects in Ens_*, pointed sets. Note that if the category $\mathcal{C}$ has products (coproducts), then so has the category $\Delta^n\mathcal{C}$: for two functors $F, G\colon (\Delta^{\mathrm{op}})^n \longrightarrow \mathcal{C}$, define

$$(F \times G)(-) := F(-) \times G(-), \quad (F \amalg G)(-) := F(-) \amalg G(-).$$

In particular, for based simplicial sets X, Y, we can form the simplicial sets $X \times Y$, $X \amalg Y$, $X \vee Y$, and $X \wedge Y$.

Any object X in a category $\mathcal{C}$ can be interpreted as a "discrete" or "constant" simplicial object const(X) where const$(X)([n]) = X$ for all n and const$(X)(f) = \mathrm{id}_X$ for all $f\colon [m] \longrightarrow [n]$. Moreover, for every n-simplicial object X, the $(n+1)$-simplicial object const(X) is defined in the same way.

For the other direction, the **diagonal simplicial object** of an n-simplicial object X is the 1-simplicial object diag(X) defined by

$$\mathrm{diag}(X)[k] := X([k], \ldots, [k]).$$

If we have a morphism $f\colon [k] \longrightarrow [l]$ in Δ, we define

$$\mathrm{diag}(X)(f) := X(f \times \cdots \times f).$$

It will turn out later that the diagonal simplicial set is something very natural to consider.

It is straightforward that $\mathcal{N}$ commutes with products and coproducts. That is, whenever $\mathcal{C}$ and $\mathcal{D}$ are two small categories, there is a natural isomorphism of bifunctors $\mathcal{N}(\mathcal{C} \times \mathcal{D}) \longrightarrow \mathcal{N}\mathcal{C} \times \mathcal{N}\mathcal{D}$, and the same for coproducts.

Now let us assume that our small category is the topmost morphism category $\mathcal{C}^{(n-1)}$ of an n-category $\mathcal{C}$. We would like to define the nerve of $\mathcal{C}$ to be the $(n-1)$-category obtained by taking the nerve of $\mathcal{C}^{(n-1)}$. However, it is not entirely clear how to construct this new category because since we have changed the n-morphisms, we must redefine the lower composition operations. We adopt the "partially monoidal" point of view for this: Interpret $\mathcal{C}$ as an $(n-1)$-fold partially monoidal 1-category. Taking nerves, our multiplications $\circ_i$, $i = 1, \dots, n-1$ translate to $\mathcal{N}\circ_i$, and the necessary identities translate as well: e.g.,

$$(-\mathcal{N} \circ_i -)\mathcal{N} \circ_j (-\mathcal{N} \circ_i -) = (-\mathcal{N} \circ_i -)\mathcal{N} \circ_j (-\mathcal{N} \circ_i -) \in \Delta\,\mathrm{Ens}((\mathcal{N}\mathcal{C})^4, \mathcal{N}\mathcal{C})$$

because this is true for 1-simplices, and therefore for all simplices.
So the following definition is justified:

DEFINITION. If $\mathcal{C}$ is an n-category, $n \geq 1$, we define its 1-nerve $\mathcal{N}^{(1)}\mathcal{C}$ to be the $(n-1)$-category obtained by taking nerves of the categories of n-morphisms. The k-nerve, $k \leq n$, is then defined to be $\mathcal{N}^{(k)}\mathcal{C} := \mathcal{N}^{(1)}\mathcal{N}^{(k-1)}\mathcal{C}$, and $\mathcal{N}\mathcal{C} := \mathcal{N}^{(n)}\mathcal{C}$ is called the **total nerve** of $\mathcal{C}$, which is an n-simplicial set.

Looking at the categories in the iterative, monoidal way, one sees immediately that taking nerves of categories is a functor

$$\mathcal{N}\colon n - \mathcal{C}\mathrm{at} \longrightarrow \Delta^n\,\mathrm{Ens}.$$

LEMMA 2.3.1. *If* $(\mathcal{C}, \square)$ *is a monoidal* n*-category, then* $\mathcal{N}^{(1)}$ *is a monoidal* $(n-1)$*-category with monoidal structure* $\mathcal{N}\square$*.*

Proof: This is clear since a monoidal n-category is the same as a $(n+1)$-category with one object; here we can take $\mathcal{N}^{(1)}$ to get an n-category with one object, which is in turn the same as a monoidal $(n-1)$-category. □

2.3.1. Geometric Realisation. Now we proceed from the category of n-simplicial sets to the topological category Top of compactly generated Hausdorff spaces. There is an elegant way of defining the geometric realisation of an n-simplicial set X by using the *simplex category* $\Delta^n \downarrow X$ which is simply the category Δ^n over X.

For $\mathbf{d} \in \mathbb{N}_0^n$, let $\Delta_{\mathbf{d}}$ be the simplicial standard prism, represented by the functor $\mathrm{Hom}_{\Delta^n}(-, [d_1] \times \cdots \times [d_n])$. For any n-simplicial set X, define the category $\Delta^n \downarrow X$ to have as objects the n-simplicial maps from $\Delta_{\mathbf{d}}$ to X, for any $\mathbf{d} \in \mathbb{N}_0^n$, and as morphisms the commuting triangles

$$\begin{tikzcd} \Delta_{\mathbf{d}} \arrow[d, "f"'] \arrow[dr, "\sigma"] & \\ \Delta_{\mathbf{d}'} \arrow[r, "\sigma'"'] & X \end{tikzcd}$$

of n-simplicial maps. Any n-simplicial set X can be reconstructed from the simplex category as a limit over the simplex category:

$$X = \varinjlim_{\Delta_d \longrightarrow X} \Delta_{\mathbf{d}}$$

DEFINITION. The **geometric realisation** of the standard prism $\Delta_d = \Delta_{(d_1,\ldots,d_n)}$ is

$$|\Delta_{\mathbf{d}}| := |\Delta_{d_1}| \times \cdots \times |\Delta_{d_n}|$$

where for simplices Δ_n,

$$|\Delta_n| := \left\{ x \in \mathbb{R}_+^{n+1} \mid \sum x_i = 1 \right\}$$

is the standard topological n-simplex.

Now consider, for any n-simplicial set X, the simplex category $\Delta^n \downarrow X$ and the functor F from there to topological spaces defined by $F(\Delta_d \xrightarrow{s} X) := |s|$. Define the realisation of any X to be the colimit of this functor:

$$|X| := \varinjlim_{\Delta_d \to X} |\Delta_d|.$$

• In other words, regarding the simplex category $\Delta^n \downarrow X$ as a big diagram, the realisation functor $|\Delta^n \downarrow X|$ gives us a big diagram of prisms which are to be glued according to the diagram.

The realisation functor preserves finite products if we work in a suitably nice category of topological spaces (compactly generated Hausdorff spaces will do, with the compactly generated product (Kelley-product)). Milnor **[Mil57]** proved that the projections $X_1 \times X_2 \longrightarrow X_i$ induce a homeomorphism $|X_1 \times X_2| \cong |X_1| \times |X_2|$.

The realisation functor has a well-known adjoint, the functor that assigns to each space X the n-simplicial set of all singular d-prisms:

$$S_d(X) := \{f \colon |\Delta_d| \longrightarrow X \text{ continuous } \}$$

where the structural maps $S_d(X) \to S_{d'}(X)$ are given by the inclusion of faces and degeneracy maps $S_{d'}(X) \to S_d(X)$.

LEMMA 2.3.2. *The following realisations are homeomorphic:*

(i) *the ordinary realisation* $|X|$ *as defined above;*
(ii) $|\operatorname{diag}(X)|$;
(iii) *the realisation of the* k*-simplicial space obtained by the realisation of* X*, regarded as a* k*-simplicial* $(n-k)$*-simplicial object.*

The idea of the proof is to observe that all above realisations preserve all colimits, and since $X = \varinjlim_{\Delta_d \to X} \Delta_d$, it is enough to show the Lemma for the models Δ_d, all d. But here it follows easily from Milnor's observation that realisation preserves products. For details, see for example Quillen's famous article **[Qui73]**.

2.3.2. Closed Model Categories. The fact that n-simplicial sets form a good category to do homotopy theory in can be expressed by saying that it is a *closed model category*.

DEFINITION. Let $\mathcal{C}$ be a category together with subclasses $\mathsf{eq}\mathcal{C}$, $\mathsf{f}\mathcal{C}$, and $\mathsf{co}\mathcal{C}$ of so-called **equivalences, fibrations and cofibrations** in $\mathcal{C}'$. As a shortcut, arrows of the form $\rightarrowtail$, $\twoheadrightarrow$, and $\xrightarrow{\sim}$ mean cofibration, fibration, and equivalence,respectively. $\mathcal{C}$ is called a **(closed) model category** if the following five axioms hold:

(CM1) $\mathcal{C}$ is closed under all finite direct and inverse limits;

(CM2) if in a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

two morphisms are weak equivalences, then so is the third;

(CM3) a retract of an equivalence, fibration, or cofibration is again one;

(CM4) if in a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

either of the vertical arrows is an equivalence, then the dashed arrow exists;

(CM5) Any map f can be factored in two ways:

$$\begin{tikzcd} & A \arrow[dr, two heads] & \\ X \arrow[ur, tail, "\sim"] \arrow[rr, "f"] \arrow[dr, tail] & & Y \\ & B \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

Elements of $\mathsf{eq}\mathcal{C} \cap \mathsf{co}\mathcal{C}$ and $\mathsf{eq}\mathcal{C} \cap \mathsf{f}\mathcal{C}$ are called **trivial (co–)fibrations**.

Properties of closed model categories can be found in [**Qui67**], [**Jar97**] and many other sources. I will only state the fact that $\Delta\,\mathrm{Ens}$ is a closed model category if one defines a map $f\colon X \longrightarrow Y$ to be:

- an equivalence if $|f|\colon |X| \longrightarrow |Y|$ is a weak equivalence of spaces,
- a cofibration if it is a monomorphism of simplicial sets,
- a fibration if liftings always exist in diagrams of the form

$$\begin{tikzcd} \Lambda_n^k \arrow[r] \arrow[d, tail] & X \arrow[d, "f"] \\ \Delta_n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

  where $\Lambda_n^k = \bigcup_{j\neq k} \partial_j^{-1}\Delta_{n-1}$ is the k-th "horn".

Also, the category of n-simplicial sets is a closed model category. However, there exist different closed model structures. On the one hand, one can define properties "diagonally", meaning that the bisimplicial map f is a fibration, cofibration, or equivalence if $\mathrm{diag}(f)$ is, as a mono-simplicial map. On the other hand, there are also "pointwise" notions: f is a fibration, cofibration, or equivalence if for every $n \in \mathbb{N}_0$, $f(n,-)$ is, again as a mono-simplicial map. However, not all combinations really yield model structures. The following are known: ("llp" means left lifting property)

	equivalences	cofibrations	fibrations
Bousfield-Kan structure	pointwise equivalence	llp for pointwise trivial fibrations	pointwise fibration
Moerdijk structure	diagonal equivalence	llp for diagonal trivial fibrations	diagonal fibration
Reedy structure	pointwise equivalence	monomorphism	more complicated

In section 2.4.1, we will need the Moerdijk structure.

2.4. Infinite Loop Space Machines and their application to categories

Several types of "loop space machines" have been developed in the seventies. The general point is that, given a topological space X, to decide whether or not this space is homotopy equivalent to the loop space on another space Y, or the n-fold loop space of another space Z, or even the latter for every $n \in \mathbb{N}_0$, in which case we speak of an infinite loop space (this will be the only case of interest here). On the other hand, a method to approximate certain spaces X by infinite loop spaces is desired. Using this machinery, it is well known and not hard to see that the nerve of a symmetric monoidal category is an infinite loop space. I will briefly say how May's operad approach [**May72**] works, which has proved to be very fruitful, and, assuming his main results, I will outline how one derives that symmetric monoidal categories give rise to infinite loop spaces. This is all done in the context of simplicial sets; it would also work in suitable topological categories. "Space" in this section always means "simplicial set".

DEFINITION. An E_∞-**operad** o is a sequence of contractible free right $\mathfrak{S}_k$-spaces o_k ($k \in \mathbb{N}_0$), i.e. contractible simplicial sets equipped with a free simplicial right action ρ of the symmetric group, together with the additional structure specified below. o_0 is supposed to the one-point simplicial set. We use the following notation: write

$$o_k^n := \coprod_{j_1+\cdots+j_n=k} (o_{j_1} \times \cdots \times o_{j_n})\,.$$

The additional structure consists of:

1. *composition maps*

$$\gamma\colon o_k \times o_l^k \longrightarrow o_l \quad \text{where} \quad k, l \in \mathbb{N}_0\,.$$

Define the **shuffle map** to be

$$
\begin{array}{rcccl}
\mathrm{sh}: & \mathbb{N}_0^* & \times & \mathbb{N}_0^* \longrightarrow & \mathbb{N}_0^* \\
& & & & \begin{pmatrix} j_1, & k_1 & ,\dots, & k_{j_1} \\ j_2, & k_{j_1+1} & ,\dots, & k_{j_1+j_2} \\ \vdots, & \vdots & ,\dots, & \vdots \\ j_t, & k_{j_1+\cdots+j_{t-1}+1} & ,\dots, & k_{\sum j} \end{pmatrix} \\
& & j_1,\dots,j_t\,,\,k_1,\dots,k_{\sum j} \mapsto & &
\end{array}
$$

where $\mathbb{N}_0^*$ means the set of sequences in $\mathbb{N}_0$ with almost all elements 0. Using this map to permute the indices, we can immediately extended γ to a map

$$o_k^j \times o_l^k \xrightarrow{(\gamma\times\cdots\times\gamma)\circ\mathrm{shuffle}^*} o_l^j,$$

and we require the following associativity condition to be satisfied:

$$
\begin{tikzcd}
o_j \times o_k^j \times o_l^k \arrow[r, "\gamma\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\gamma"'] & o_k \times o_l^k \arrow[d, "\gamma"] \\
o_j \times o_l^j \arrow[r, "\gamma"] & o_l
\end{tikzcd}
$$

Also, we require that the action is $\mathfrak{S}$-*equivariant*: If $k_i \in \mathbb{N}_0$ for $1 \le i \le j$, $k = \sum k_i$, bl: $\mathfrak{S}_j \longrightarrow \mathfrak{S}_k$ is the "block permutation" homomorphism with respect to the partition k_i of k, and τ: $\mathfrak{S}_j \times o_k^j \longrightarrow o_k^j$ is the permutation operation on the j factors, then:

$$
\begin{tikzcd}
& o_j \times \mathfrak{S}_j \times \prod_{i=1}^j o_{k_i} \arrow[r, "\rho\times\mathrm{id}"] \arrow[dl, "\mathrm{id}\times\mathrm{id}\times\tau^{-1}"] & o_j \times \prod_{i=1}^j o_{k_i} \arrow[dr, "\gamma"] & \\
o_j \times \mathfrak{S}_j \times o_k^j \arrow[dr, "\gamma"] & & & o_k \\
& o_k \times \mathfrak{S}_j \arrow[r, hook, "\mathrm{bl}"] & o_k \times \mathfrak{S}_k \arrow[ur, "\rho"] &
\end{tikzcd}
$$

and

$$\begin{tikzcd}
o_j \times \prod_{i=1}^{j} o_{k_i} \times \prod_{i=1}^{j} \mathfrak{S}_{k_i} \arrow[r, "\gamma\times \mathrm{id}"] \arrow[d, "\mathrm{shuffle}"] & o_k \times \prod_{i=1}^{j} \mathfrak{S}_{k_i} \arrow[dr] & \\
o_j \times \prod_{i=1}^{j} (o_{k_i} \times \mathfrak{S}_{k_i}) \arrow[d, "\mathrm{id}\times\rho^j"] & & o_k \times \mathfrak{S}_k \arrow[dl, "\rho"'] \\
o_j \times \prod_{i=1}^{j} o_{k_i} \arrow[r, "\gamma"] & o_k &
\end{tikzcd}$$

commute.

2. an *identity vertex* $\mathbf{1} \in o_1(0)$ such that

$$\gamma(\mathbf{1}; d) = d \quad \text{for all } d \in o_j$$

and

$$\gamma(c; \mathbf{1}, \dots, \mathbf{1}) = c \quad \text{for all } c \in o_k.$$

(here I have identified $\mathbf{1}$ with its degeneracies in $o_1(j)$, $j > 0$)

The **endomorphism operad** of a simplicial set X consists of the simplicial function spaces $\mathrm{End}_k(X) := \mathrm{map}_*(X^k, X)$ (powers meaning iterated smash products), with the obvious $\mathfrak{S}_k$-action and composition morphisms. It is not an E_∞-operad (i.e. $\mathrm{End}_j(X)$ is not supposed to be connected, nor $\mathfrak{S}_k$-free). An E_∞-operad o *action* on a simplicial set X is a morphism of operads (in the obvious sense) $o \longrightarrow \mathrm{End}(X)$, and makes X into an E_∞*-space*.

The spaces o_n of an operad should be regarded as a parameter space for operations $X^n \longrightarrow X$. If an E_∞-operad o operates on X via ρ, X automatically has much structure:

1. For every vertex $x \in o_2(0)$, $\rho(x)$ is an H-space structure for X, and since o_2 is connected, all these structures are homotopic to each other;
2. X is homotopy associative because o_3 is connected;

3. X is homotopy commutative because the action of $\mathfrak{S}_2$ on o_2 is homotopic to the identity (again because o_2 is connected);
4. there also exist higher coherence maps.

E_∞-operads act on infinite loop spaces. To see this, one constructs the *little cubes-operad* [**May72, Ada78**]. On the other hand, a space on which an E_∞-operad acts is "nearly" a loop space: a loop space is always a group-like H-space, but an E_∞-space need not have homotopy inverses. However, the connection is as simple as it could possibly be:

THEOREM 2.4.1. *Let X be an E_∞-space. If X is group-like (which is equivalent to π_0 being a group), then it is homotopy equivalent to an infinite loop space.*
In general, there exists an infinite loop space ΩBX, called the "group completion" of X, and a simplicial map of E_∞-spaces $X \longrightarrow \Omega BX$. This map is a homotopy equivalence if X is connected.

Remark: By abuse of language, I will often say that X *is* an infinite loop space and mean that it *has the homotopy (or even weak homotopy) type* of one.
How can we interpret the group completion of an E_∞-space? For, say, abelian simplicial monoids, this question has a very simple answer: ΩBX does indeed agree with the loop space on the classifying space of X defined before with the same notation, $\pi_0(X)$ is an abelian monoid, and we have $\mathfrak{G}(\pi_0 X) \cong \pi_0(\Omega BX)$, where $\mathfrak{G}$ denotes the Grothendieck group or group completion. Furthermore, $\pi_0 X$ is a multiplicative subset of the Pontryagin ring $H_*(X)$, and we have:

$$H_*(\Omega BX) \simeq (\pi_0 X)^{-1} H_*(X), \quad \text{the localization at } \pi_0 X.$$

This result is known as the group-completion theorem. In the topological category, it was proven by McDuff and Segal using homology fibrations [**MS76**] (although it seems to have been known for a length of time before, e.g. [**Seg74**]), and in the simplicial category a proof can be found in [**Jar89**].

Remark. The assumption that X be abelian is far too strong — there also exist notions of localization in noncommutative rings. It is enough to have $\pi_0 X$ in the center of $H_*(X)$, or even only constructible by "right fractions".
Now we apply this machinery to a symmetric monoidal category. Recall that a weak initial (or final) object in a category is an object that has (not necessarily unique) arrows into (resp. out of) every object.

PROPOSITION 2.4.2. *Let $\mathcal{C}$ be a symmetric monoidal category. Then its nerve $\mathcal{NC}$ supports an operation of an E_∞-operad, and therefore its group completion is an infinite loop space. If $\mathcal{C}$ has a weak initial or final object, then $\mathcal{NC}$ itself is an infinite loop space.*

Proof: Define spaces $o_n := \mathcal{N}(\mathcal{E}\mathfrak{S}_n)$ (cf. section 2.2.1). These are contractible spaces with free $\mathfrak{S}_n$-actions: $\mathfrak{S}_n$ operates simplicially on a nerve by object-wise multiplication from the right, and o_n is contractible and connected because every element is initial and final in $\mathcal{E}\mathfrak{S}_n$. o is then an E_∞-operad operating on $\mathcal{N}C$:

$$\mathcal{N}\phi_j \colon o_j \times (\mathcal{NC})^j \longrightarrow \mathcal{NC}$$

This action comes from the functor

$$\phi_j \colon \mathcal{E}\mathfrak{S}_n \times \mathcal{C}^j \longrightarrow \mathcal{C}$$

defined on objects, for $\sigma \in \mathfrak{S}_j, X_1, \dots, X_j \in \mathcal{C}_0$, by

$$\phi_j(\mathcal{E}\sigma, X_1, \dots, X_j) := X_{\sigma^{-1}(1)} \square \dots \square X_{\sigma^{-1}(j)}$$

and on morphisms, for $\lambda \in \mathcal{E}\mathfrak{S}_j(\sigma, \tau)$ (unique), $f_1, \dots, f_j \in \mathcal{C}'$, by

$$\phi_j(\lambda, f_1, \dots, f_j) := c_\lambda \circ (f_{\sigma^{-1}(1)} \square \dots \square f_{\sigma^{-1}(j)}),$$

where c_λ ($\lambda = \tau\sigma^{-1}$) is the symmetry map

$$c_\lambda \colon X_1 \square \dots \square X_j \xrightarrow{\cong} X_{\lambda(1)} \square \dots \square X_{\lambda(j)}.$$

It is straightforward to check that this indeed defines an operad action. Theorem 2.4.1 therefore implies that $\Omega B(\mathcal{N}\mathcal{C})$ is an infinite loop space.
The second part of the theorem follows from the observation that $\mathcal{N}C$ is connected and therefore group-like if $\mathcal{C}$ has weak initial or final objects. $\square$

2.4.1. Group completion in the simplicial context. In the simplicial setting, the group completion theorem was proven by Jardine [**Jar89**] and has a very nice generalization for category actions on simplicial sets (see section 2.2.1) due to Moerdijk [**Moe89**] and Tillmann [**Til**]. To state it, let $\mathcal{C}$ be a *simplicial category* acting simplicially on simplicial sets. A simplicial category is simply a category enriched over $\Delta\,\mathrm{Ens}$, i.e., the morphisms $\mathcal{C}(X, Y)$ form a simplicial set such that composition is a simplicial map. (This is the special case of a category object in $\Delta\,\mathrm{Ens}$ for which the morphism set is discrete. The latter would just mean that morphisms and objects are simplicial sets with all structural maps (composition, source and target) being simplicial). The projection functor π of the operation ρ gives us a map on the nerves:

$$\mathcal{N}\pi\colon \mathcal{N}\mathcal{E}\rho \longrightarrow \mathcal{N}\mathcal{C}$$

Now we want to compare the homotopy fibre of $\mathcal{N}\pi$ with the actual fibre $(\mathcal{N}\pi)^{-1}(*)$ of a point. In general, there is certainly no direct relation between them because the actual fibres can be quite different from each other. However, if we take any generalized homology theory h_* or even homotopy, and we assume that the action of our category induces h_*-isomorphisms between the fibres, then the generalized group completion theorem tells us that these fibres are also h_*-equivalent to the homotopy fibre of $\mathcal{N}\pi$. The properties needed for the functor h_* are that h_*-isomorphisms are preserved under pushouts, and that for bisimplicial sets, the property of the diagonal simplicial set being an h_*-isomorphism can be checked pointwise. To begin with, I will show that this is indeed true for generalised homology theories and for π_*:

LEMMA 2.4.3. *We work in the category* $\Delta^n\,\mathrm{Ens}$ *of* n*-simplicial sets with the Moerdijk structure (if* $n \geq 2$*). Let* h_* *be a generalised homology theory or* π_**,* $A \overset{i}{\rightarrowtail} X$ *a cofibration, and* $f\colon B \longrightarrow A$ *be any map that induces an isomorphism* $f_*\colon h_*(B) \xrightarrow{\cong} h_*(A)$*. Let* Y *be the pushout:*

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

Then g *induces an isomorphism* $g_*\colon h_*(Y) \xrightarrow{\cong} h_*(X)$*.*

Proof: Extend the diagram:

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & \mathrm{cofib}(j) \arrow[d, "\sim"] \\ X \arrow[r, "i"] & Y \arrow[r] & \mathrm{cofib}(i) \end{tikzcd}$$

Apply the functor diag to this cofibre sequence; then the long exact sequence together with the Five Lemma yields the isomorphism $g_*\colon h_*(Y) \xrightarrow{\cong} h_*(X)$. □

LEMMA 2.4.4. *Let* $f\colon X \longrightarrow Y$ *be a map of bisimplicial sets with the property that for each* n*,* $f(n)\colon X(n) \longrightarrow Y(n)$ *induces an* h_**-isomorphism of mono-simplicial sets. Then so does* $\mathrm{diag}(f)\colon \mathrm{diag}(X) \longrightarrow \mathrm{diag}(Y)$.

Sketch of proof: For $h_* = \pi_*$, this can be proved inductively on suitable filtration, using gluing lemmata. See **[Jar89]** or **[Jar97]**. For homology, consider the bisimplicial abelian group $\mathbb{Z}X(p,q)$ of linear combinations of (p,q)-bisimplices. This gives rise to a double complex $C(\mathbb{Z}X(p,q))$ where the differentials are alternating sums of the images of all the structural maps $(p,q) \longrightarrow (p+1,q)$ and $(p,q) \longrightarrow (p,q+1)$, respectively. We then have a spectral sequence converging to the homology of the total complex $h_*(\mathrm{Tot}(X))$ with

$$E^0_{p,q} = C\mathbb{Z}X(p,q)$$

Moreover, there is a natural homotopy equivalence between Tot(X) and diag(X) (cf. [**Wei94**, Theorems 8.5.1 and 8.3.8]). Since a map $f\colon X \longrightarrow Y$ between bisimplicial sets which induces an h_*-isomorphism vertically is an isomorphism of E^1-terms, a spectral sequence comparison argument shows that $\mathrm{diag}(f)\colon \mathrm{diag}(X) \longrightarrow \mathrm{diag}(Y)$ has to be a h_*-equivalence. $\square$

THEOREM 2.4.5. *Let* ρ *be an action of a simplicial category* $\mathcal{C}$ *on* $\Delta\,\mathrm{Ens}$ *with associated projection* π. *Suppose that for every* 0*-simplex* α *in every morphism set* $\mathcal{C}(x, y)$, *the map*

$$\rho(\alpha)\colon \rho(\mathrm{source}(\alpha)) \longrightarrow \rho(\mathrm{target}(\alpha))$$

induces an h_**-isomorphism on nerves. Then the diagram of bisimplicial sets*

$$\begin{tikzcd} \mathrm{const}(\rho(x)) \arrow[r, hook] \arrow[d] & \mathcal{N}\mathcal{E}\rho \arrow[d, "\mathcal{N}\pi"] \\ * \arrow[r, "x", "*\mapsto x\in(\mathcal{N}\mathcal{C})_0"'] & \mathcal{N}\mathcal{C} \end{tikzcd}$$

is a h_**-pullback diagram, i.e.* $\rho(x)$ *is* h_**-equivalent to the homotopy fibre of* $\mathcal{N}\pi$.

Proof (due to Moerdijk [**Moe89**]*):* The proof uses techniques from homotopical algebra and the fact that $\Delta\,\mathrm{Ens}$ and $\Delta^2\,\mathrm{Ens}$ are model categories. The *homotopy fibre* of a map $p\colon E \longrightarrow B$ of bisimplicial sets is constructed as follows: take any $(0, 0)$-simplex x of B. The inclusion $\{*\} \overset{x}{\hookrightarrow} B$ factors, as any map, into a trivial cofibration and a fibration: $\{*\} \underset{\cong}{\overset{j}{\rightarrowtail}} P(B) \overset{q}{\twoheadrightarrow} B$ where the notation $P(B)$ suggests that this space is something like the path space of B. The homotopy fibre of p is then the middle pullback of the diagram:

$$\begin{tikzcd} p^{-1}(*) \arrow[r, "\tilde{j}"] \arrow[d] & \mathrm{hofib}(p) = P(B) \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{*\} \arrow[r, tail, "j\ \cong"] \arrow[rr, bend right, "x"'] & P(B) \arrow[r, two heads, "q"] & B \end{tikzcd}$$

Note that $P(B)$ is well-defined, this follows from the model category axioms: Suppose $P'(B)$ is another space with the above properties. Then in

$$
\begin{tikzcd}
\{*\} \arrow[r, tail, "\cong", "j"'] \arrow[d, tail, "\cong"', "j'"] & P(B) \arrow[d, two heads] \arrow[dl, dashed, shift left] \\
P'(B) \arrow[r, two heads] \arrow[ur, dashed, shift left] & B,
\end{tikzcd}
$$

the diagonal arrows exist by (CM4), and are weak equivalences by (CM2).
In our case ($E = \mathcal{N}\mathcal{E}\rho, B = \mathcal{N}\mathcal{C}$, $p = \mathcal{N}\pi$, I will retain these abbreviations), we want to show that $h_*\tilde{j}$ is an isomorphism.

LEMMA 2.4.6 (Moerdijk [**Moe89**]). *Every trivial cofibration* $c\colon X \rightarrowtail Y$ *in* $\Delta^2\,\mathrm{Ens}$ *is a direct limit of a sequence of cofibrations*

$$X = X_0 \overset{u_0}{\rightarrowtail} X_1 \overset{u_1}{\rightarrowtail} \cdots \longrightarrow Y$$

where each map u_i *is defined by a pushout diagram*

$$
\begin{tikzcd}
\coprod_i \Lambda^k_n \arrow[r, tail] \arrow[d] & \coprod_i \Delta_{(n,n)} \arrow[d] \\
X_i \arrow[r, tail, "u_i"] & X_{i+1}
\end{tikzcd}
$$

Here, the bisimplicial set $\Lambda^k_n := \bigcup_{j\neq k}(\partial_j \times \partial_j)^{-1}\Delta_{(n-1,n-1)}$ *is the "*k*-th horn".*

We apply this lemma to the trivial cofibration $j\colon * \rightarrowtail P(B)$:

$$j = \varinjlim_i \{u_i\colon X_i \rightarrowtail X_{i+1}\}$$

Now, pullbacks commute with direct limits, therefore:

$$x^*(p) = \varinjlim\,(u_i^*(X_{i+1} \to B)^*p)$$

So we need to show that every $\tilde{u}_i$ induces an h_*-isomorphism. However, our h_* is chosen such that it takes pushouts of isomorphisms to isomorphisms; therefore it is enough to show that in the diagram

$$\begin{tikzcd} u^*\sigma^*E \arrow[r, "\tilde{u}"] \arrow[d, "u^*\sigma^*p"'] & \sigma^*E \arrow[r, "\tilde{\sigma}"] \arrow[d, "\sigma^*p"'] & E \arrow[d, "p"] \\ \Lambda^k_n \arrow[r, tail, "u"] & \Delta_{(n,n)} \arrow[r, "\sigma"] & B, \end{tikzcd}$$

$\tilde{u}$ induces an h_*-isomorphism. Explicitly, we can write the map $\sigma\colon \Delta_{(n,n)} \longrightarrow B$ as a chain

$$B_0 \xrightarrow{\sigma_0} B_1 \xrightarrow{\sigma_1} \dots \xrightarrow{\sigma_{n-1}} B_n$$

where each $B_i \in B(n)$.
Then, σ^*E is the bisimplicial set

$$(\sigma^*E)(p,q) = (\Delta_{(n,n)} \times_B E)(p,q) = \\ \Big\{(\alpha,\beta,e) \mid [p] \xrightarrow{\alpha} [n],\ [q] \xrightarrow{\beta} [n],\ e \in \rho(B_{\alpha(0)})(q)\Big\}.$$

Literally, e should be a p-chain of composible arrows

$$\rho(\alpha(0))(q) \to \dots \rho(\alpha(p))(q),$$

but since in our case morphisms in E are exactly the morphisms in B, this chain is well-defined if we only specify a point $e \in \rho(\alpha(0))(q)$. Similarly,

$$(u^*\sigma^*E)(p,q) = \left\{(\alpha,\beta,e) \mid \begin{array}{l} [p] \xrightarrow{\alpha} \partial_i[n],\ [q] \xrightarrow{\beta} \partial_j[n] \\ \text{where } i,j \neq k,\ e \in \rho(B_{\alpha(0)})(q) \end{array}\right\}.$$

We now compare these two bisimplicial sets to the simpler ones:

$$\overline{(\sigma^*E)}(p,q) := \big(\Delta_{(n,n)} \times \mathrm{const}(\rho(B_0))\big)(p,q) \\ = \Big\{(\alpha,\beta,e) \mid [p] \xrightarrow{\alpha} [n],\ [q] \xrightarrow{\beta} [n],\ e \in \rho(B_0)(q)\Big\}$$

and

$$\overline{(u^*\sigma^*E)}(p,q) := \left(A_n^k \times \mathrm{const}(\rho(B_0))\right)(p,q)$$
$$= \left\{(\alpha,\beta,e) \mid [p] \xrightarrow{\alpha} \partial_i[n],\ [q] \xrightarrow{\beta} \partial_j[n],\ e \in \rho(B_0)(q)\right\}.$$

There exist maps $\mu\colon \overline{(\sigma^*E)} \longrightarrow \sigma^*E$ and $\mu|_{\overline{(u^*\sigma^*E)}}\colon \overline{(u^*\sigma^*E)} \longrightarrow u^*\sigma^*E$, given by:

$$(\alpha,\beta,e) \mapsto \beta^*\sigma^*_{\alpha(0)-1}\dots\sigma_1^*\sigma_0^*e \quad \in \rho(B_{\alpha(0)}).$$

We want to show that μ is an h_*-isomorphism, or more precisely, that $\mathrm{diag}(\mu)$ is an h_*-isomorphism. By Lemma 2.4.4, it is enough to show that $\mu_p(q) = \mu(p,q)$ is an isomorphism for every p. In dimension p,

$$\overline{(\sigma^*E)}(p) = \coprod_{[p]\xrightarrow{\alpha}[n]} \Delta_n \times \rho(B_0) \quad \text{and} \quad (\sigma^*E)(p) = \coprod_{[p]\xrightarrow{\alpha}[n]} \Delta_n \times \rho(B_{\alpha(0)})$$

and μ_p is a coproduct

$$\mu_p = \coprod_{[p]\xrightarrow{\alpha}[n]} \mathrm{id} \times \sigma^*_{\alpha(0)-1}\dots\sigma_1^*\sigma_0^*$$

which is, by hypothesis, an h_*-isomorphism, and the same is true for the restriction of μ_p to $\overline{u^*\sigma^*E}$. Therefore, the diagram

$$\begin{tikzcd}
\overline{u^*\sigma^*E} \arrow[r, tail, "u\times\mathrm{id}", "\cong"'] \arrow[d, "\mu"'] & \overline{\sigma^*E} \arrow[d, "\mu"] \\
u^*\sigma^*E \arrow[r, "\tilde{u}"] & \sigma^*E
\end{tikzcd}$$

becomes a diagram of h_*-isomorphisms if we apply the functor d, and the proof is complete. □

EXAMPLE 2.4.7. If $\mathcal{C}$ is any simplicial category and $Z \in \mathcal{C}_0$, then $\mathcal{C}$ acts from the right on its morphism sets $\mathcal{C}(-,Z)$: let

$$\rho_Z(X) := \mathcal{C}(X,Z) \quad \text{and} \quad \rho_Z(X \xrightarrow{f} Y) := \mathcal{C}(Y,Z) \xrightarrow{-\circ f} \mathcal{C}(X,Z).$$

In this case,

$$\mathcal{N}\mathcal{E}\rho_Z = \mathcal{N}\{(X, f) \mid X \in \mathcal{C}_0,\ f\colon X \longrightarrow Z\} = \mathcal{N}(\mathcal{C} \downarrow Z),$$

But the category $\mathcal{C} \downarrow Z$ of objects over Z has the initial element (Z, id_Z) and its nerve is therefore contractible (recall that a category with an initial or final object is equivalent to the final category with one object, and that nerves of equivalent categories are homotopy equivalent).

CHAPTER 3

Spin mapping class groups, revisited

I will transfer U. Tillmann's arguments [**Til**] that show that the group completion of the classifying space of mapping class groups is an infinite loop space to the case where the surfaces under consideration carry spin structures. This is made possible by Harer's stabilization results on spin mapping class group homology [**Har90**] in the formulation of Chapter 1. In constructing a quite simple monoidal spin surface 2-category, we obtain an input for the machinery developed by May [**May72**], Segal [**Seg74**], Boardman, Vogt, and Adams [**Ada78**] that turns certain categories into infinite loop spaces, and by using a generalized group completion theorem due to Moerdijk [**Moe89**] and Tillmann [**Til**], we see that this output is homology equivalent to a suitable notion of G_∞, a stable spin mapping class group.

In this section, a suitable symmetric monoidal category is constructed in order to apply the infinite loop space machinery to it and to prove that the group completion of the classifying space of the stable spin mapping class group is an infinite loop space. However, some care is necessary while constructing this category. I will first explain what the most naive approach would be and why it does not work.

3.1. Introduction to surface categories

Let us leave the realm of mapping class groups for a moment. One of the classical examples for the application of infinite loop space machines is the symmetric group. Let $\mathcal{C}$ be the category of finite sets and *isomorphisms* between them. Disjoint union makes this category into a symmetric monoidal category. Since $\mathcal{C}$ is not small (the finite sets do not form a set), there is no notion of "nerve" of $\mathcal{C}$. However, it is perfectly sensible and well-defined up to homotopy equivalence just to say that $\mathcal{NC}$ is the nerve of any small skeleton of $\mathcal{C}$ (any small full subcategory containing one representative of each isomorphism class). For any two skeleta are equivalent as categories and therefore, their nerves are homotopy equivalent. The nerve of $\mathcal{C}$ is then just $\coprod_{j\in\mathbb{N}_0} B\mathfrak{S}_j$, a simplicial monoid with the operation induced by the obvious homomorphism $\mathfrak{S}_j \times \mathfrak{S}_k \longrightarrow \mathfrak{S}_{j+k}$. The group completion of this monoid is:

$$\Omega B\left(\coprod_{j\in\mathbb{N}_0} B\mathfrak{S}_j\right) \simeq \mathbb{Z} \times B\mathfrak{S}_\infty^+.$$

where $B\mathfrak{S}_\infty^+$ denotes the Quillen plus construction with respect to the perfect (commutator) subgroup A_∞.

Therefore, since $\mathcal{C}$ is symmetric monoidal, $B\mathfrak{S}_\infty^+$ is an infinite loop space.

Let us attempt to transfer this construction to our mapping class groups. As shown by Harer [**Har90**], and with the generalization of his result to surfaces with many boundaries given in chapter 1, the homology of spin mapping class groups in a fixed dimension does not depend on the number of boundary components as long as the genus is high enough, the stabilization range depending linearly on the dimension. So we can hope to get information about the *stable* spin mapping class group by looking at suitable categories. Let $\mathcal{C}$ be the following category (this is the category Miller [**Mil86**] used to find a double loop space structure on the classifying space of the stable mapping class group):

- for objects, take a representing surface $F_{g,1}$ for every $g \in \mathbb{N}_0$ of genus g with one boundary component;
- morphisms do not exist between different surfaces; but the endomorphisms of $F_{g,1}$ are just all automorphisms of this surface, say homeomorphisms.

This category is monoidal if we define the multiplication to be the "pair of pants"-multiplication $\square$: attach a surface of type $F_{0,3}$ to the two surfaces F_1, F_2 which are to be

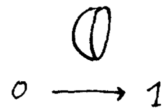
multiplied to get a new surface G with one boundary. Then define $F_1 \square F_2$ to be unique surface in $\mathcal{C}$ homeomorphic to F, via a homeomorphism $\phi = \phi_{F_1,F_2}$. On objects, this is just addition of natural numbers, i.e. the genus. Let f_1, f_2 be mapping classes on F_1, F_2. These give a mapping class on G, and via ϕ on F. We can choose the ϕ_{F_1,F_2} in such a way that $\square$ indeed becomes associative.
Since on objects, $F_1 \square F_2 = F_2 \square F_1$, it is tempting to believe that this category is commutatively monoidal, but this is not true — if we have two homeomorphisms f_i of F_i, there is no way to identify $f_1 \square f_2$ and $f_2 \square f_1$. $\mathcal{C}$ is *not even symmetric* monoidal, as shown by Fiedorowicz and Song [**FS97**].
Therefore, Tillmann's construction of a symmetric monoidal 2-category that can be used to produce an infinite loop space (in [**Til**]) came as a surprise. Applied to the spin world, it looks as follows. The trick is to consider disconnected surfaces as well, and surfaces with many boundaries, and to take disjoint union as the monoidal structure.

3.2. The spin surface category

We start with the $(1+1)$-dimensional bordism category that has closed compact 1-manifolds as objects and compact spin surfaces with two sets of boundary (incoming and outgoing) components as morphisms, and then to enrich this category with 2-morphisms between the morphisms, which shall be the spin diffeomorphisms between surfaces in our case. Moreover, we define a monoidal structure by simply taking disjoint unions of bordisms. However, some care is necessary because it is crucial that compositions are strictly associative, and we need a symmetry for our monoidal structure and not only a braiding, which would only yield a double loop structure on the nerve of the category and not give us an advantage over the construction in section 3.1.
It is not necessary to make the category so big that it comprises all possible spin surfaces in order to show the main result, Theorem 3.3.5. For example, all constructible

spin surfaces will have Arf invariant 0.
Let $\mathcal{C}$ be following category: as objects, pick representatives of every closed compact 1-manifold, so they are in 1-1 correspondence with the nonnegative integers. Throw in the following morphisms, most of which are spin surfaces with two sets of labels $\{1, \ldots, n\}, \{1, \ldots, m\}$ for the incoming and outgoing boundary components:

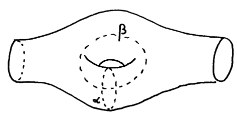
1. a morphism $0 \longrightarrow 1$, realized by a disk, numbered as it must:



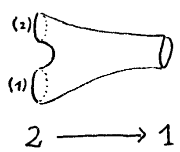
   Of course, since $H_1 = 0$, we have no choice concerning the spin structure.

2. morphisms $n \longrightarrow n$ with numbering $(1, \ldots, n)$ on the left and any numbering on the right. These morphisms act as permutations of the numbering of the boundaries and do not have representing surfaces; the morphism $n \longrightarrow n$ with the identity numbering acts as a strict identity for composition.

3. a morphism $1 \longrightarrow 1$, realized by a torus with two boundary components, and a spin structure which evaluates to zero on every element of H_1 *except* $\alpha + \beta$:



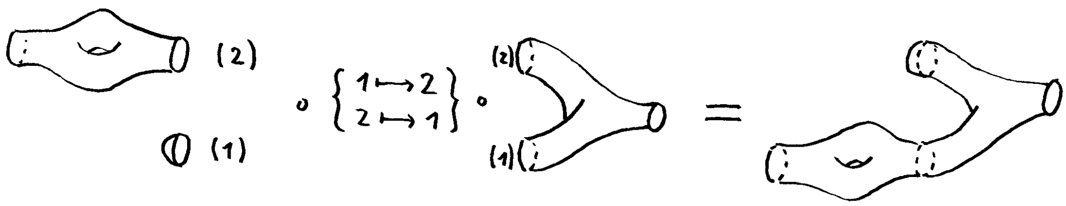
4. a morphism $2 \longrightarrow 1$, realized by a pair of pants, i.e. a surface of type $F_{0,3}$; the quadratic form defining the spin structure is 0 everywhere:



Then, add all morphisms that can be constructed from others by either

1. taking disjoint unions of two surfaces, thus producing a morphism $m_1 + m_2 \longrightarrow n_1 + n_2$ out of two morphisms $m_i \longrightarrow n_i$, and shifting the numbering of the second surfaces up; or
2. gluing two surfaces $k \longrightarrow m$, $m \longrightarrow n$ according to the labelling to give a surface $k \longrightarrow n$; if either of the surfaces is one of the abstract permutations, the surfaces itself stays the same, but the labelling changes; if both are permutations, they are composed. Note that this operation defines a unique spin structure on the compositum of two surfaces.

An example of a composition of three morphism is:



Now it is immediate that:

• this gives a strict monoidal category, the composition being gluing and the tensor product being disjoint union;
• Ignoring the spin structures, every surface with at least one boundary component in every component can be constructed (up to homeomorphism), but not uniquely. The spin structure constructed at the same time is unique: whenever there are two homeomorphic but distinct surfaces $F_1, F_2 \in \mathcal{C}_0$, these surfaces are isomorphic as spin surfaces (cf. 1.2.1).
Now we construct from this category the category $\mathcal{S}$ of spin mapping class groups by adding 2-morphisms. We choose all possible spin-*isomorphisms* as 2-morphisms; there are no 2-morphisms between non-isomorphic spin surfaces. Note that composition is

well-defined on mapping classes.

• There is a symmetry 1-morphism s, given by $s(m,n) \in \mathfrak{S}_{m+n} \subseteq \mathcal{S}(m \sqcup n, m \sqcup n)$ the element of the symmetric group which exchanges the first m with the last n circles. The axioms are easily checked:

0. it is a natural transformation:

$$\begin{tikzcd} m \sqcup n \arrow[r, "s(m,n)"] \arrow[d, "\phi\sqcup\psi"'] & n \sqcup m \arrow[d, "\psi\sqcup\phi"] \\ m' \sqcup n' \arrow[r, "s(m',n')"] & n' \sqcup m' \end{tikzcd}$$

   commutes for any morphisms ϕ in $\mathcal{S}(m,m')$, ψ in $\mathcal{S}(n,n')$;
1. $s(n,0) = s(0,n) = \mathrm{id} \in \mathfrak{S}_n$;
2. $s(n, m \sqcup k) = (\mathrm{id}_{\mathfrak{S}_m} \sqcup s(n,k)) \circ (s(n,m) \sqcup \mathrm{id}_{\mathfrak{S}_k})$; and
3. $s(n,m) \circ s(m,n) = \mathrm{id}_{\mathfrak{S}_{m+n}}$.

So, the important property of this category is:

LEMMA 3.2.1. *The category $\mathcal{S}$ is a symmetric monoidal 2-category.*

Now we want to apply the machinery of infinite loop spaces to the total nerve $\mathcal{NS}$ of this category.

3.3. Application of the group completion theorem to $\mathcal{S}$

$\mathcal{N}^{(1)}\mathcal{S}$ is a category which has as objects elements of $\mathbb{N}_0$, representing closed compact 1-manifolds. The morphisms from m to n form a simplicial set, namely the classifying space of the category of spin isomorphisms between surfaces with m incoming and n outgoing boundaries. In $\mathcal{S}$, there are many different 1-morphisms from m to n, corresponding to varying genus, spin structures, number of components of the surface, and, importantly, different constructions of homeomorphic surfaces. If we want to understand the morphism sets $\mathcal{N}^{(1)}\mathcal{S}(m,n)$, we need not consider all these surfaces.

As in the example of symmetric groups in section 3.1, it is enough to take a skeleton Σ of the category $\mathcal{S}(m,n)$ and apply the nerve functor to it since Σ and $\mathcal{S}(m,n)$, as equivalent categories, have homotopy equivalent nerves.
Now there is a morphism (and therefore isomorphism) between two morphisms $F, G \in \mathcal{S}(m,n)$ if and only if these surfaces are isomorphic as spin surfaces, i.e. if there is a bijection $\phi\colon \pi_0 F \xrightarrow{\cong} \pi_0 G$ such that for every connected component F_0 of F, F_0 and $\phi(F_0)$ have the same genus. Therefore, writing from now on $G_{g,n} = G^{(0)}_{g,n}$ for the spin mapping class group of a connected surface of Arf invariant 0, genus g, and n boundary components:

$$\mathcal{N}^{(1)}\mathcal{S}(m,n) \simeq \coprod_{\substack{k\in\mathbb{N}_0\\ (m_j,n_j)\in I_k}} \coprod_{g_1,\dots,g_k\in\mathbb{N}_0} (BG_{g,m_1+n_1} \sqcup \cdots \sqcup BG_{g,m_k+n_k}) \tag{3.3.1}$$

where m_j and n_j run over the k-partitions of m and n:

$$I_k = \left\{ (m_j,n_j) \in \mathbb{N}_0^k \times \mathbb{N}_0^k \;\middle|\; \begin{array}{cc} \sum_j m_j = m, & \sum_j n_j = n, \\ \text{all } m_j \geq 0, & \text{and all } n_j \geq 1 \end{array} \right\}.$$

In particular,

$$\mathcal{N}^{(1)}\mathcal{S}(m,1) \simeq \coprod_{g\in\mathbb{N}_0} BG_{g,m+1}. \tag{3.3.2}$$

Since, by Lemma 3.2.1, $\mathcal{S}$ is a symmetric monoidal 2-category, $\mathcal{N}^{(1)}\mathcal{S}$ inherits the symmetric monoidal structure and becomes a simplicial symmetric monoidal category (Lemma 2.3.1). Furthermore, $\mathcal{N}\mathcal{S} = \mathcal{N}\mathcal{N}^{(1)}\mathcal{S}$ is connected because the object $0 \in (\mathcal{N}^{(1)}\mathcal{S})_0$ is weakly initial: for every n, we can find a morphism $0 \to n$, for example, n disjoint caps. (Note that this is not true for any other object: we never find a morphism *to* the empty 1-manifold.) Therefore, Proposition 2.4.2 applies to $\mathcal{N}^{(1)}\mathcal{S}$ and we get:

COROLLARY 3.3.3. *$\mathcal{N}S$ is an infinite loop space.*

$\mathcal{N}\mathcal{S}$ is a bisimplicial set into which all the information of the classifying spaces of spin mapping class groups of all possible surfaces went, and it is not at all obvious in which way it did. Therefore, the rest of this section is devoted to the question of what it is that we have just computed: has $\mathcal{NS}$ an easier interpretation? For example, is it related to a suitable notion of "stable spin mapping class group G_∞" ?

DEFINITION. The **stable spin mapping class group** $G_{\infty,n+1}$ is defined to be the direct limit

$$G_{\infty,n+1} := \varinjlim \left\{ G_{0,n+1} \xrightarrow{t_0} G_{1,n+1} \xrightarrow{t_1} G_{2,n+1} \xrightarrow{t_2} \dots \right\} \tag{3.3.4}$$

where the maps t_i are induced by the attachment of a fixed chosen $F_{1,2}$ torus with the same spin structure of Arf invariant 0 as the tori in $\mathcal{S}$.

THEOREM 3.3.5. *Let $G_{\infty,1}$ be the stable spin mapping class group with one boundary component, as defined above. Then there is a homology equivalence*

$$H_*(\Omega(\mathcal{NS})) \cong H_*(BG_{\infty,1} \times \mathbb{Z}).$$

Therefore, $G_{\infty,1}$ has the homology of an infinite loop space.

Proof: The proof is analogous to Tillmann's proof in the non-spin case. It is a consequence of the generalized group completion theorem 2.4.5. The application goes as follows: $\mathcal{N}^{(1)}\mathcal{S}$ is certainly a simplicial category. We define two category actions of $\mathcal{NS}^{(1)}$ on simplicial sets. First, we define the action on objects:

$$\rho(n) := \mathcal{N}^{(1)}\mathcal{S}(n,1) \underset{(3.3.2)}{\simeq} \coprod_{g \in \mathbb{N}_0} BG_g \tag{3.3.6}$$

$$\rho_\infty(n) := \underrightarrow{\operatorname{holim}} \left\{ \mathcal{N}^{(1)}\mathcal{S}(n,1) \longrightarrow \mathcal{N}^{(1)}\mathcal{S}(n,1) \longrightarrow \dots \right\} \tag{3.3.7}$$

where the arrows in the homotopy limit refer to right translation by the morphism $1 \to 1$ of genus 1 on every connected component of $\mathcal{N}^{(1)}\mathcal{S}(n,1)$. In this simple case, the $\underrightarrow{\operatorname{holim}}$ can just be understood to be the telescope construction of the sequence. This defines the action on objects; on morphisms, it is simply the map induced by attachment of the surfaces on the left.

Before proceeding with the category action, let us investigate the homotopy of 3.3.7. Let Σ be a skeleton of the category $\mathcal{S}(n, 1)$ such that right translation with the morphism $1 \to 1$ does not lead out of Σ. That is, take the full subcategory of $\mathcal{S}(n, 1)$ with one chosen object $n \to 1$ and all the objects constructed from this by attaching a finite number of tori to the right of it. Σ is isomorphic to $\coprod_{g\in\mathbb{N}_0} \mathcal{B}G_{g,n+1}$, and the following diagram commutes, where the vertical arrows are the simplicial maps induced by the inclusion $\Sigma \hookrightarrow \mathcal{S}(n, 1)$; they are homotopy equivalences:

$$
\begin{tikzcd}
\mathcal{N}^{(1)}\mathcal{S}(n,1) \arrow[r] & \mathcal{N}^{(1)}\mathcal{S}(n,1) \arrow[r] & \cdots \arrow[r] & \rho_\infty(n) \\
\mathcal{N}^{(1)}\Sigma \arrow[r] \arrow[u, "f_0"', "\simeq"] \arrow[d, equal] & \mathcal{N}^{(1)}\Sigma \arrow[r] \arrow[u, "f_1"', "\simeq"] \arrow[d, equal] & \cdots & \\
\coprod_{g\in\mathbb{N}_0} BG_{g,n+1} \arrow[r, "\coprod t_i"'] & \coprod_{g\in\mathbb{N}_0} BG_{g,n+1} \arrow[r, "\coprod t_i"'] & \cdots \arrow[r] & \varinjlim\left\{\coprod_{g\in\mathbb{N}_0} BG_{g,n+1}\right\} \arrow[uu, "\phi"]
\end{tikzcd}
\tag{3.3.8}
$$

LEMMA 3.3.9. *There is a homeomorphism (= isomorphism of simplicial sets)*

$$\varinjlim\left\{\coprod_{g\in\mathbb{N}_0} BG_{g,n+1}\right\} \cong \mathbb{Z} \times BG_{\infty,n+1}.$$

Proof: Let $L := \varinjlim\left\{\coprod_{g\in\mathbb{N}_0} BG_{g,n+1}\right\}$ be the direct limit simplicial set. Since the number of boundary components neither changes nor matters for this lemma, I will suppress the index $n + 1$ everywhere. We can write explicitly:

$$L(k) = \{(i, g, x) \in \mathbb{N}_0 \times \mathbb{N}_0 \times BG_g(k)\}/\sim$$

where the equivalence relation $\sim$ is generated by

$$(i, g, x) \sim (i + 1, g + 1, t_g(x)).$$

Here, a point (i, g, x) refers to an element $x \in BG_g$ in the ith entry of the chain the direct limit is applied to.

The structural maps are defined by $L(f) = [\mathrm{id}_{\mathbb{N}_0}, \mathrm{id}_{\mathbb{N}_0}, BG_g(f)]$ for any morphism (= monotonic map of set $\{0, \dots, n\}$) $f \in \Delta'$. Let $j_g\colon G_g \longrightarrow G_\infty$ be the inclusion map into the direct limit in (3.3.4)). Then I claim that the map

$$\begin{aligned} \Psi(k): \qquad L(k) &\longrightarrow \mathbb{Z} \times BG_\infty(k) \\ [i, g, x] &\longmapsto (g - i, Bj_g(x)) \end{aligned}$$

is a homeomorphism.

Well-defined: It is enough to show that the two representatives $(i + 1, g + 1, t_g(x))$ and (i, g, x) have the same image. But

$$\Psi(i + 1, g + 1, t_g(x)) = [g - i, Bj_{g+1}(t_g(x))] = [g - i, Bj(x)] = \Psi(i, g, x).$$

Inverse map:

Any $x \in BG_\infty(k)$ is represented by a k-tuple of group elements $g_{\infty,1}, \dots, g_{\infty,k}$. Every $g_{\infty,i}$ has a preimage $g_{\gamma_i,i}$ in some G_{γ_i}, and, taking $\overline{\gamma} \geq \max\{\gamma_1, \dots, \gamma_k\}$, we can assume that the whole k-tuple lies in $G_{\overline{\gamma}}$. Consider $\overline{\gamma}$ as a function $\overline{\gamma}_k\colon BG_\infty(k) \longrightarrow \mathbb{N}_0$. Since $\overline{\gamma}$ can be chosen arbitrarily large on every point, we can define a map

$$\gamma_k\colon \mathbb{Z} \times BG_\infty(k) \longrightarrow \mathbb{N}_0 \quad \text{satisfying } \gamma_k(z, x) \geq \max\{\overline{\gamma}_k(x), z\}.$$

Define, using γ, a map $\delta(k)\colon \mathbb{Z} \times BG_\infty(k) \longrightarrow \coprod_{g\in\mathbb{N}_0} BG_g(k)$ such that

$$\delta(z, x) \in BG_{\gamma(z,x)}(k) \quad \text{and} \quad Bj_{\gamma(z,x)} \circ \delta = \mathrm{id}\,.$$

Note that at this stage, δ cannot be a simplicial map. Then define

$$\begin{aligned} \Psi^{-1}(k): \qquad \mathbb{Z} \times BG_\infty(k) &\longrightarrow L(k) \\ (z, x) &\longmapsto [\gamma_k(z, x) - z, \gamma_k(z, x)\delta_k(z, x)] \end{aligned}$$

It is a simple calculation to prove that this is well-defined, indeed the inverse map of Ψ, and simplicial. $\square$

Continuation of the proof of 3.3.5: It is not hard to see that the map

$$\phi\colon \varinjlim \left\{ \coprod_{g \in \mathbb{N}_0} BG_{g,n+1} \right\} \longrightarrow \rho_\infty(n) \quad \text{in diagram (3.3.8)}$$

must be a weak homotopy equivalence: it induces isomorphisms on homotopy groups. For injectivity, take a map $\gamma\colon \mathbb{S}^n \longrightarrow \rho_\infty(n)$ representing an element of π_n. Since $\mathbb{S}^n$ is compact, this map must factor through some space $\mathcal{N}^{(1)}\mathcal{S}(n,1)$ in the direct system; here, the vertical map f_i is a weak equivalence, so we get a map $\gamma'\colon \mathbb{S}^n \longrightarrow BG_{g,n+1} \longrightarrow BG_{\infty,n+1}$. Because of the commutativity of the diagram, $\phi_*[\gamma'] = [\gamma]$. For injectivity, repeat this argument with a null-homotopy.

By Example 2.4.7, $\mathcal{N}\mathcal{E}\rho$ is contractible. But since $\rho_\infty(n) = \underrightarrow{\operatorname{holim}}\,\rho(n)$, the same is true for the translation space $\mathcal{N}\mathcal{E}\rho_\infty \simeq \underrightarrow{\operatorname{holim}}\,\mathcal{N}\mathcal{E}\rho \simeq *$. Therefore the homology fibration $\mathcal{N}\mathcal{E}\rho_\infty \longrightarrow \mathcal{N}\mathcal{S}$ has contractible total space, and its homotopy fibre is homotopy equivalent to $\Omega\mathcal{N}\mathcal{S}$, the based loops on $\mathcal{N}\mathcal{S}$. In order to show that the group completion theorem 2.4.5 is applicable, we have to show that every 0-simplex in $\mathcal{N}^{(1)}\mathcal{S}$ induces an H_*-isomorphism. But a 0-simplex in $(\mathcal{N}^{(1)}\mathcal{S})(m,n)$ is just a surface constructed by iterated attachments of tori, pairs of pants, or discs, and some relabelling operations. By Corollaries 1.2.4 and 1.2.5, the action induces isomorphisms

$$H_*(\rho_\infty(n);\mathbb{Z}) \to H_*(\rho_\infty(m);Z),$$

and relabelling certainly too. Therefore, Theorem 2.4.5 applies and gives us that the inclusion map of the fibre into the homotopy fibre

$$\mathbb{Z} \times BG_{\infty,1} \simeq \rho_\infty(0) \longrightarrow \operatorname{hofib}(\mathcal{N}\mathcal{E}\rho_\infty \to \mathcal{N}\mathcal{S}) \simeq \Omega\mathcal{N}\mathcal{S}$$

is an $H_*(-,\mathbb{Z})$-equivalence. □

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