

THE REALIZABILITY OF SOME FINITE-LENGTH MODULES OVER THE STEENROD ALGEBRA BY SPACES

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ABSTRACT. The Joker is a finite cyclic module over the mod-2 Steenrod algebra \mathcal{A} which appears frequently in Adams resolutions as well as in Picard groups. In a previous paper, the first author commenced a study of when the Joker and its iterated Steenrod doubles are realizable as cohomologies of spectra or spaces. In this paper, we complete this study by showing that every version of the Joker is realizable by a space of as low a dimension as the unstability condition of modules over the Steenrod algebra permits. We construct these spaces using, on the one hand, spaces that have Dickson algebras as their cohomology rings (classifying spaces of the Lie groups $SO(3)$, G_2 , as well as the exotic 2-compact group DW_3), and on the other hand, ring spectra having with the cohomology of Hopf quotients if the Steenrod algebra (mod-2 cohomology, real K-theory, and topological modular forms).

1. INTRODUCTION

Let \mathcal{A} be the mod-2 Steenrod algebra, generated by the Steenrod squares Sq^1, Sq^2, Sq^4, \dots . A (left) \mathcal{A} -module M is *stably realizable* if there exists a spectrum X such that $H^*(X; \mathbf{F}_2) \cong M$ as \mathcal{A} -modules. For finite \mathcal{A} -modules, this is equivalent to the existence of a space X such that $H^*(X) \cong \Sigma^s M$ for some $s \geq 0$. This number s is bounded from below by the *unstability degree* $\sigma(M)$ of M , i.e. the minimal number t such that $\Sigma^t M$ satisfies the unstability condition for modules over \mathcal{A} . We say that M is *optimally realizable* if there exists a space X such that $H^*(X) \cong \Sigma^{\sigma(M)} M$.

We consider two constructions of new Steenrod modules from old. Firstly, for a left \mathcal{A} -module M , the linear dual $M^\vee = \text{Hom}(M, \mathbf{F}_2)$ becomes a left \mathcal{A} -module using the antipode of \mathcal{A} . Secondly, the *iterated double* $M(i)$ is the module which satisfies

$$M(i)^n = \begin{cases} M^{n/2^i} & \text{if } 2^i \mid n, \\ 0 & \text{otherwise,} \end{cases} \quad Sq^{2^k} x = \begin{cases} 0 & \text{if } k < i, \\ Sq^{2^{k-i}} & \text{if } k \geq i. \end{cases}$$

for $x \in M(i)^n$. We also set $M(0) = M$.

Let J be the quotient of \mathcal{A} by the left ideal generated by Sq^3 and $Sq^i \mathcal{A}$ for $i \geq 4$. The main result of this paper is the following.

Theorem 1.1. *The spectra $J(i)$ and $J(i)^\vee$ are optimally realizable for $i \leq 2$ and not stably realizable for $i > 2$.*

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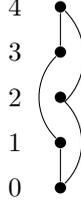
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The module J in this theorem is known as the *Joker*, although more colloquially than in actual written articles. Its dimension over \mathbf{F}_2 is 5, having dimension 1 in each degree $0 \leq d \leq 4$; a basis is given by

$$\{1, \text{Sq}^1, \text{Sq}^2, \text{Sq}^2 \text{Sq}^1, \text{Sq}^2 \text{Sq}^2 = \text{Sq}^1 \text{Sq}^2 \text{Sq}^1\},$$

or pictorially,



This picture, and others to follow, are to be read as follows. The numbers on the left denote the dimension. A dot denotes a copy of \mathbf{F}_2 in the corresponding dimension. A straight line up from a dot x to a dot y indicates that $\text{Sq}^1(x) = y$, and a curved line similarly indicates a nontrivial operation Sq^2 .

The Joker appears in several contexts in homotopy theory. In [AP76], Adams and Priddy showed that J generates the torsion on the Picard group of $\mathcal{A}(1)$ -modules. The Joker also appears regularly in projective resolutions of cohomologies of common spaces (such as real projective spaces) over \mathcal{A} or $\mathcal{A}(1)$. Its stable realizability is equivalent to the Toda bracket relation $\eta^2 \in \langle 2, \eta, 2 \rangle$, where $\eta \in \pi_1^s(\mathbf{S}^0)$ denotes the Hopf map.

In [Bak18], the first author showed all cases of Thm. 1.1 with the exception of the optimal realizability of $J(2)$ and $J(2)^*$. In this paper, we prove this case and give alternative proofs for $J(1)$ and $J(1)^*$ so that all cases fit a common pattern.

Conventions. We assume that all spaces and spectra are completed at the prime 2, although our arguments can be easily modified to work globally. We will often assume that we are working with CW complexes which have been given minimal cell structures. All cohomology is with coefficients in \mathbf{F}_2 , the field with two elements.

2. REALIZABILITY OF MODULES OVER THE STEENROD ALGEBRA

Recall that a \mathcal{A} -module is called *unstable* if

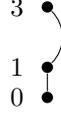
$$\text{Sq}^i(x) = 0 \text{ if } i > |x|.$$

Definition. Let M be an \mathcal{A} -module. The *unstable degree* $\sigma(M)$ of M is the minimal $s \in \mathbf{Z}$ such that $\Sigma^s M$ is an unstable \mathcal{A} -module.

Obviously, $\sigma(M)$ is a finite number if M is a finite module (but may be infinite otherwise), and if a finite module is stably realizable, it is realizable by a space after sufficiently high suspension by Freudenthal's theorem. If M is stably realizable by a spectrum X then M^* is stably realized by the Spanier–Whitehead dual $DX = F(X, \mathbf{S}^0)$. Only a finite number of iterated doubles of M can ever be stably realizable by the solution of the Hopf invariant 1 problem.

Example 2.1. Any \mathcal{A} -module M of dimension 1 over \mathbf{F}_2 is optimally realizable (by a point). Let M be cyclic of dimension 2 over \mathbf{F}_2 , thus $M \cong \mathbf{F}_2\langle \iota, \text{Sq}^{2^i} \iota \rangle$. By the solution of the Hopf invariant one problem, M is stably realizable if and only if $i = 0, 1, 2, 3$. In each case, M is optimally realizable by the projective plane over \mathbf{R} , \mathbf{C} , the quaternions, and the octonions, respectively.

Example 2.2. A simple example of a module that is not optimally realizable is the “question mark” complex



or $M = \mathcal{A}/(\text{Sq}^2, \text{Sq}^3, \dots)$. The unstable degree of this module is 1, but it is not optimally realizable because a hypothetical space X with $H^*(X) = \Sigma M$, $H^1(X) = \langle x \rangle$ would have $x^4 = (x^2)^2 \neq 0$ but $x^3 = 0$.

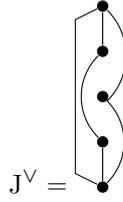
3. THE FAMILY OF JOKERS

The finite cyclic \mathcal{A} -module

$$J = \mathcal{A}/(\mathcal{A}\text{Sq}^3 + \mathcal{A}\text{Sq}^4\mathcal{A} + \mathcal{A}\text{Sq}^8\mathcal{A} + \dots)$$

is called the Joker.

Its linear dual $J^\vee = \text{Hom}(J, \mathbf{F}_2)$ is also a cyclic left module by the antipode χ of \mathcal{A} . It is not isomorphic to J , even by a shift, since $\chi(\text{Sq}^4) = \text{Sq}^4 + \text{Sq}^1\text{Sq}^2\text{Sq}^1$, which means that $\text{Sq}^4 \neq 0$ on J^\vee . Pictorially,



Here and in what follows, the slanted square brackets denote nontrivial operations Sq^4 .

The k -fold iterated doubles of these will be denoted by $J(k)$ and $J(k)^\vee$, so that $J(1)$ has cells in even dimensions 0, 2, 4, 8, 10, $J(2)$ cells in dimension divisible by 4, and so on.

Clearly, the unstable degrees are given by $\sigma(J(k)) = 2 \cdot 2^k$ (the bottom cohomology class supports a nontrivial operation $\text{Sq}^{2 \cdot 2^k}$), and $\sigma(J(k)^\vee) = 4 \cdot 2^k$ (the bottom cohomology class supports also a nontrivial operation $\text{Sq}^{4 \cdot 2^k}$).

Note that if $J(k)$ is optimally realized by a space, then that space is weakly equivalent to a CW complex X with cells in dimensions $i \cdot 2^k$, where $i = 2, \dots, 6$, hence X has dimension $6 \cdot 2^k$. The ring structure of the cohomology is implied by the unstability condition for \mathcal{A} -algebras, namely, $\text{Sq}^i(x) = x^2$ when $|x| = i$:

$$H^*(X) = \mathbf{F}_2[x_2, x_3]/(x_2, x_3)^3 \quad |x_i| = i \cdot 2^k.$$

If $J(k)^\vee$ is optimally realizable by a space, then that space is weakly equivalent to a CW complex Y with cells in dimensions $j \cdot 2^k$, where $j = 4, 5, 6, 7, 8$. For dimensional reasons, the ring structure of the cohomology has to be

$$H^*(Y) = \mathbf{F}_2[x_4, x_5, x_6, x_7]/(x_4^3) + x_4(x_5, x_6, x_7) + (x_5, x_6, x_7)^2; \quad |x_i| = i \cdot 2^k.$$

The first author showed:

Theorem 3.1 ([Bak18]). *The module $J(k)$ is stably realizable iff $k \leq 2$ iff $J(k)^\vee$ is stably realizable.*

For $k \leq 1$, the modules $J(k)$ and $J(k)^\vee$ are optimally realizable.

The main result of this note is the case $k = 2$. We will, however, also give an alternative proof of the cases where $k < 2$, which may aid as an illustration of how the more complicated case of $k = 2$ works.

4. DUAL JOKERS

If X is a spectrum with $H^*(X) \cong J(k)$ then it is obvious that the Spanier-Whitehead dual DX realizes $J(k)^\vee$, up to a shift:

$$\Sigma^{4 \cdot 2^k} H^*(DX) \cong J(k)^\vee.$$

Unstably, the situation is a bit more complicated, but we have the following result:

Proposition 4.1. *If $J(k)$ is stably realizable for any k then $J(k)^\vee$ is optimally realizable.*

Proof. Let X be a spectrum such that $H^*(X) \cong J(k)$. Consider the mapping spectrum $F = F(X, \mathbf{S}^{8 \cdot 2^k})$ and let $Y = \Omega^\infty F$ be the underlying mapping space. Since for any n -connected spectrum E , the augmentation $\Sigma^\infty \Omega^\infty E \rightarrow E$ is $(2n+2)$ -connected and F is $4 \cdot 2^k - 1$ -connected, the map $\Sigma^\infty Y \rightarrow F$ is $2(4 \cdot 2^k - 1) + 2 = 8 \cdot 2^k$ -connected. Hence the induced map $H^i(F) \rightarrow H^i(Y)$ is an isomorphism for $i < 8 \cdot 2^k$ and injective for $i = 8 \cdot 2^k$. Thus the $8 \cdot 2^k$ -skeleton of Y has the correct cohomology for optimally realizing $J(k)^\vee$ with the exception that it might have extra cohomology in the top degree. Let V be a complement of $H^{8 \cdot 2^k}(F)$ in $H^{8 \cdot 2^k}(Y)$ and $\alpha: Y \rightarrow K(V, 8 \cdot 2^k)$ its representing map. Then the $8 \cdot 2^k$ -skeleton of the homotopy fiber of α optimally realizes $J(k)^\vee$. \square

Unsurprisingly, the same argument does not work for showing that the stable realizability of $J(k)^\vee$ implies the optimal realizability of $J(k)$. This is nevertheless true, but for the deeper reason that any realizable $J(k)$ is optimally realizable, which is the main result of this paper.

5. DICKSON ALGEBRAS AND THEIR REALIZATIONS

The rank- n algebra of Dickson invariants $DI(n)$ is the ring of invariants of $\text{Sym}(\mathbf{F}_2^n) = \mathbf{F}_2[t_1, \dots, t_n]$ under the action of the general linear group $GL_n(\mathbf{F}_2)$. We think of $\text{Sym}(\mathbf{F}_2^n)$ as a graded commutative ring with t_i in degree 1. Dickson [Dic11] showed that

$$DI(n) \cong \mathbf{F}_2[c_0, \dots, c_{n-1}],$$

with $|c_i| = 2^n - 2^i$. The polynomials c_i are given by the formula

$$\prod_{v \in \mathbf{F}_2^n} (X + v) = X^{2^n} + c_{n-1} X^{2^n-1} + \dots + c_0 X \in \text{Sym}(\mathbf{F}_2^n)[X].$$

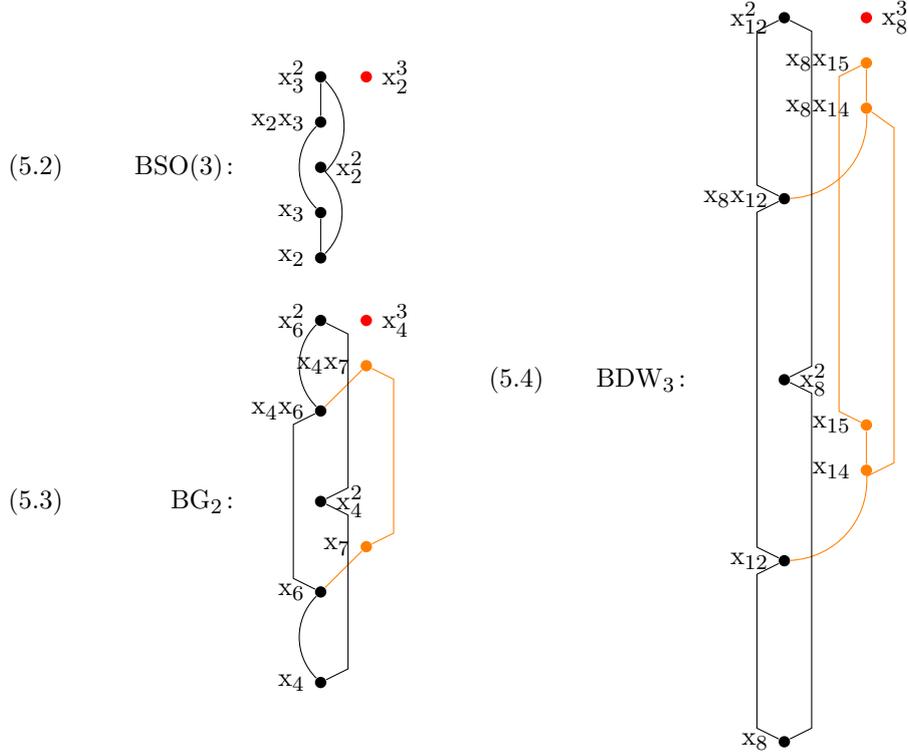
If we give $\text{Sym}(\mathbf{F}_2^n)$ the structure of an \mathcal{A} -algebra with $\text{Sq}(t_i) = t_i + t_i^2$ (i.e., using the isomorphism $\text{Sym}(\mathbf{F}_2^n) \cong H^*(B\mathbf{F}_2^n)$) then $DI(n)$ is an \mathcal{A} -subalgebra with

$$\text{Sq}^{2^i} c_{i-1} = c_i.$$

Theorem 5.1 (Smith-Switzer, Lin-Williams, Dwyer-Wilkerson). *The Dickson algebra $DI(n)$ is optimally realizable iff $n \leq 4$.*

Proof. Clearly, $DI(1) \cong \mathbf{F}_2[t]$ is optimally realized by \mathbf{RP}^∞ and $DI(2) \cong \mathbf{F}_2[x_2, x_3]$ is optimally realized by $BSO(3)$. Further investigations of Lie groups shows that $DI(3) \cong \mathbf{F}_2[x_4, x_6, x_7] \cong H^*(BG_2)$, the cohomology of the classifying space of the exceptional Lie group G_2 . The case $n = 4$ was settled in [DW93], where Dwyer and Wilkerson constructed a 2-complete space, the exceptional 2-compact group BDW_3 , with the required cohomology. For $n \geq 6$, $DI(n)$ is not realizable by [SS83, Cor. 3.2]. According to [DW93], J. Lannes first showed that $DI(5)$ is not optimally realizable, with a stronger nonexistence result being proved in [LW89, JS92]. \square

A graphical representation of a skeleton of the spaces realizing the Dickson algebras is given below. One observes that the Jokers $J(i)$ occur as quotients of skeleta of these spaces; the kernel consists of the classes on the right of each diagram. However, realizing these quotients as fibers of certain maps is non-obvious and the purpose of the following section.



6. THE JOKER J AND ITS DOUBLE $J(1)$

The cohomology picture (5.2) shows that the 6-skeleton of $BSO(3)$ is almost a realization of $J(0)$, its only defect lying in an additional class x_2^3 in the top cohomology group $H^6(BSO(3))$. Let $\alpha: BSO(3) \rightarrow K(\mathbf{F}_2, 6)$ represent this class and $X = \text{hofib}(\alpha)^{(6)}$, the 6-skeleton of its homotopy fiber. Then X realizes $J(0)$ optimally.

For the double Joker $J(1)$, as seen in the cohomology picture (5.3), it does not suffice any longer to take a skeleton of BG_2 and kill off a top-dimensional class.

We recall some standard results on the exceptional Lie group G_2 and its relationship with $\text{Spin}(7)$.

One definition of G_2 is as the group of automorphisms of the alternative division ring of Cayley numbers (octonions) \mathbf{O} . Since G_2 fixes the real Cayley numbers, it is a closed subgroup of $SO(7) \leq SO(8)$.

A different point of view is to consider the spinor representation of $\text{Spin}(7)$. Recall that the Clifford algebra $Cl_6 \cong \text{Mat}_8(\mathbf{R})$ is isomorphic to the even subalgebra of $Cl_7 \cong \text{Mat}_8(\mathbf{R}) \times \text{Mat}_8(\mathbf{R})$, so $\text{Spin}(7)$ is naturally identified with a subgroup of $SO(8) \subseteq \text{Mat}_8(\mathbf{R})$, and thus acts on \mathbf{R}^8 with its spinor representation. Then on identifying \mathbf{R}^8 with \mathbf{O} , we find that the stabilizer subgroup in $\text{Spin}(7)$ of a non-zero vector is isomorphic to G_2 . It follows that the natural fibration

$$\text{Spin}(7)/G_2 \rightarrow BG_2 \rightarrow B\text{Spin}(7)$$

is the unit sphere bundle of the associated spinor vector bundle $\sigma \rightarrow B\text{Spin}(7)$.

The mod-2 cohomologies of these space are related as follows. By considering the natural fibration

$$K(\mathbf{F}_2, 1) \rightarrow B\text{Spin}(7) \rightarrow BSO(7)$$

we find that

$$H^*(B\text{Spin}(7)) = \mathbf{F}_2[w_4, w_6, w_7, u_8]$$

where the w_i are the images of the universal Stiefel–Whitney classes in

$$H^*(BSO(7)) = \mathbf{F}_2[w_2, w_3, w_4, w_5, w_6, w_7],$$

and $u_8 \in H^8(B\text{Spin}(7))$ is detected by $z_1^8 \in H^8(K(\mathbf{F}_2, 1))$. It is known that

$$H^*(BG_2) = \mathbf{F}_2[v_4, v_6, v_7]$$

and it is easy to see that the generators can be taken to be the images of w_4, w_6, w_7 under the induced homomorphism $H^*(B\text{Spin}(7)) \rightarrow H^*(BG_2)$. As a consequence, these v_i are Stiefel–Whitney classes of the pullback $\rho_7 \rightarrow BG_2$ of the natural 7-dimensional bundle $\rho \rightarrow BSO(7)$ and since this lifts to a Spin bundle, it admits an orientation in real connective K-theory. This leads to the following observation.

Lemma 6.1. *There is a factorisation*

$$BG_2 \rightarrow \underline{kO}_7 \rightarrow K(\mathbf{F}_2, 7)$$

of a map representing $v_7 \in H^7(BG_2)$.

Here

$$\underline{kO}_7 = \Omega^\infty \Sigma^7 kO \sim \Omega BO\langle 8 \rangle.$$

and $\underline{kO}_7 \rightarrow K(\mathbf{F}_2, 7)$ is an infinite loop map induced from the unit morphism $kO \rightarrow H\mathbf{F}_2$.

The cohomology of $BO\langle 8 \rangle$ is a quotient of that of BO :

$$(6.2) \quad H^*(BO\langle 8 \rangle) = \mathbf{F}_2[w_{2^r} : r \geq 3] \otimes \mathbf{F}_2[w_{2^r+2^{r+s}} : r \geq 2, s \geq 1] \\ \otimes \mathbf{F}_2[w_{2^r+2^{r+s}+2^{r+s+t}} : r \geq 1, s, t \geq 1] \\ \otimes \mathbf{F}_2[w_{2^r+2^{r+s}+2^{r+s+t}} : r \geq 0, s, t \geq 1],$$

where the w_i are images of universal Stiefel–Whitney classes in $H^*(BO)$. Here

$$Sq^4 w_8 \equiv w_{12} \pmod{\text{decomposables}}.$$

A routine calculation shows that $H^*(\underline{kO}_7) \cong H^*(\Omega BO\langle 8 \rangle)$ is the exterior algebra on certain elements $e_i \in H^1(\underline{kO}_7)$ where e_i suspends to the generator w_{i+1} of (6.2).

In particular, up to degree 13,

$$H^*(\underline{kO}_7) = \mathbf{F}_2\{1, e_7, e_{11}\}$$

and

$$(6.3) \quad Sq^4 e_7 = e_{11}.$$

As well as v_7 , we also need to kill $v_4^3 \in H^{12}(BG_2)$ so we let

$$f: BG_2 \rightarrow \underline{kO}_7 \times K(\mathbf{F}_2, 12)$$

be the product of the map mentioned earlier with a map representing v_4^3 .

Lemma 6.4. *Up to degree 12, the cohomology ring of the fibre of f agrees with $\mathbf{F}_2[w_4, w_6]/(w_4^3)$ and with $J(1)$ as an \mathcal{A} -module. Thus $J(1)$ is optimally realizable.*

Proof. Let $\alpha: \mathrm{BG}_2 \rightarrow \underline{\mathbf{kO}}_7$ be the factorization of Lemma 6.4. By the above computations, $H^*(\mathrm{BG}_2)$, as an algebra over $H^*(\underline{\mathbf{kO}}_7)$, is isomorphic with

$$H^*(\mathrm{BG}_2) \cong H^*(\underline{\mathbf{kO}}_7)[w_4, w_6, \text{generators in degree greater than } 12]/R$$

where the module R of relations is at least 13-connected. This means that in the Eilenberg-Moore spectral sequence for the cohomology of the fiber of α , $E_2^{s,t} = 0$ for $s+t \leq 12$ and $s < 0$. Thus up to degree 12,

$$H^*(\mathrm{hofib}(\alpha)) \cong \mathbf{F}_2[w_4, w_6],$$

and the remaining top class w_4^3 is killed by the second component of f . \square

7. THE QUADRUPLE JOKER $J(2)$

The strategy to construct an optimal realization of $J(2)$ is to realize it as a skeleton of the homotopy fiber of a suitable map

$$\alpha: (\mathrm{BDW}_3)^{(24)} \rightarrow \underline{\mathrm{tmf}}/2_{14}$$

into the 14th space of the spectrum of topological modular forms modulo 2. The spectrum tmf is an analog of connective real K-theory, ko , but of chromatic level 2 [HM98, DFHH14, Beh07, Goe10] with well-known homotopy [Bau08].

A basic property is that $H^*(\mathrm{tmf}) \cong \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbf{F}_2$ and thus there is a non-split extension of \mathcal{A} -modules

$$0 \rightarrow H^*(\mathrm{tmf}) \rightarrow H^*(\mathrm{tmf}/2) \rightarrow \Sigma H^*(\mathrm{tmf}) \rightarrow 0$$

where Sq^1 acts non-trivially on the generator of $\Sigma H^0(\mathrm{tmf})$. This map should satisfy that under $\alpha^*: H^*(\underline{\mathrm{tmf}}/2_{14}) \rightarrow H^*(\mathrm{BDW}_3)$, the unit $\iota \in H^{14}(\underline{\mathrm{tmf}}/2_{14})$ is mapped to x_{14} . This implies that $\alpha^*(\mathrm{Sq}^1 \iota) = x_{15}$, $\alpha^*(\mathrm{Sq}^8 \iota) = x_8 x_{14}$, and $\alpha^*(\mathrm{Sq}^8 \mathrm{Sq}^1 \iota) = x_8 x_{15}$. Hence as in the case of $J(1)$,

$$H^*(\mathrm{BDW}_3) \cong H^*(\underline{\mathrm{tmf}}/2_{14})[x_8, x_{12}, \text{generators in degree greater than } 24]/R,$$

where the module R of relations is at least 25-connected. Thus the Eilenberg-Moore spectral sequence converging to $\mathrm{hofib}(\alpha)$ shows that up to degree 24,

$$H^*(\mathrm{hofib}(\alpha)) \cong \mathbf{F}_2[x_8, x_{12}],$$

and the remaining top class x_8^3 can be killed by taking the 24-skeleton of the fiber of its classifying map to $K(\mathbf{F}_2, 24)$.

The rest of this section is devoted to constructing the map α . Unfortunately, as of this writing, the homotopy representation theory of DW_3 has not been sufficiently developed to construct α geometrically. With a faithful 15-dimensional homotopy representation of DW_3 , one might be able to mimick the $J(1)$ -construction, using the topological Witten genus (a string orientation of tmf) instead of the spin orientation of \mathbf{kO} .

Instead of constructing a class in $(\mathrm{tmf}/2)^{14}(\mathrm{BDW}_3)$, we will construct a 2-torsion class $\beta \in \mathrm{tmf}^{15}(Y)$, where Y is the 24-skeleton of the homotopy fiber of the map $\mathrm{BDW}_3 \rightarrow K(\mathbf{F}_2, 24)$ classifying the cohomology class $x_8^3 \in H^{24}(\mathrm{BDW}_3)$ – that is, picture (5.4) without the topmost unattached class. By the Bockstein sequence, this class has to pull back to a class in $(\mathrm{tmf}/2)^{14}(Y)$ which is sufficient for constructing a space which optimally realizes $J(2)$.

We are looking for a class with a nontrivial Hurewicz image of the unit $\iota \in H^{15}(\underline{\mathrm{tmf}}_{15})$ in $H^{15}(Y)$; any such class will work. The question thus boils down to showing that in the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}(H^*(\mathrm{tmf}), H^*(Y)) \cong \mathrm{Ext}_{\mathcal{A}(2)}(\mathbf{F}_2, H^*(Y)) \implies \mathrm{tmf}^*(Y),$$

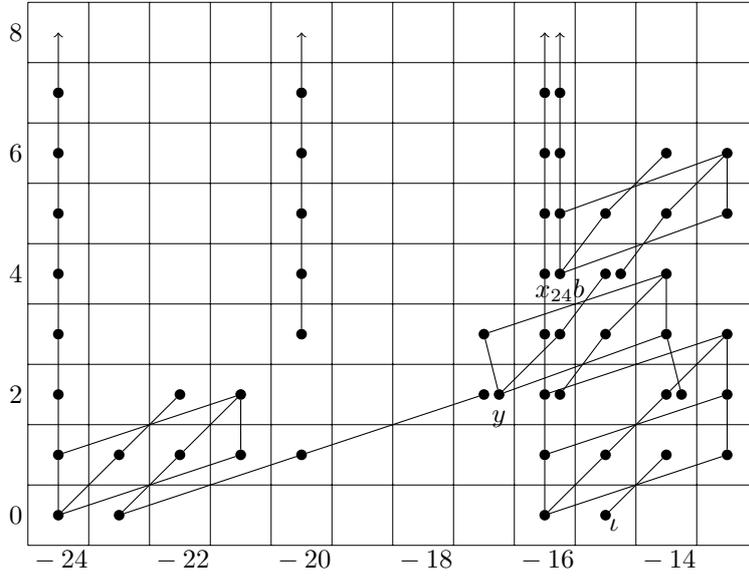
the unique nontrivial class in $E_2^{0,-15} = \text{Hom}_{\mathcal{A}}(H^0(\text{tmf}), H^{15}(\text{BDW}_3))$ survives. This in turn does not depend on the cells of Y in dimension less than 15. Thus let Z be the space $\text{cofib}(Y^{(14)} \rightarrow Y)$, a 6-cell complex with cells in dimension 15, 16, 20, 22, 23, and 24. The construction is completed by the following proposition.

Proposition 7.1. *In the Adams spectral sequence*

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}(2)}(\mathbf{F}_2, H^*(Z)) \implies \text{tmf}^*(Z),$$

the class $\iota \in E_2^{0,-15}$ is an infinite cycle.

Proof. The following is the E_2 -term of this spectral sequence computing $\text{tmf}^*(Y)$, determined with Bob Bruner's program [Bru]:



We will identify the possible targets of differentials on ι . Since $h_0\iota = 0$ and $h_1^2\iota = 0$, only classes in the kernel of h_0 and the kernel of h_1^2 can be targets. This means that there is no possible target for a d^2 in bidegree $(-16, 2)$ (in the displayed Adams grading).

The possible targets of a d^3 in bidegree $(-16, 3)$ are $x_{-16}h_0^3$ and the class yh_1 . However, the first target would require that $x_{-16}h_0^4$ has died in E_3 , thus is the target of a d_2 -differential. This is impossible for dimensional reasons.

The class yh_1 is equal to $x_{-24}c$, where $c \in \text{Ext}^3$ is the class representing the homotopy class ϵ in the 8-stem. For $d \in \text{Ext}^4$ the class representing κ in the 14-stem, note that $x_{-24}d \neq 0$ in the E_2 -term of the Adams spectral sequence, and it cannot be in the image of a d^2 -differential (or, in fact, any differential) because it is not divisible by h_2 . Suppose $d^3(\iota) = x_{-24}c$. Then by juggling, the Massey product $d = \langle c, h_0, h_1, h_2 \rangle$ implies that

$$0 \neq x_{-24}d = x_{-24}\langle c, h_0, h_1, h_2 \rangle = \langle x_{-24}, c, h_0, h_1 \rangle h_2.$$

But $\langle x_{-24}, c, h_1, h_2 \rangle = 0$, so $d^3(\iota) = 0$.

There are no longer differentials possible on ι either because no classes in filtration 4 or higher are h_0 -torsion in any E_n -term. \square

Remark 7.2. One might interpret the survival of this class as (albeit weak) evidence that a faithful 15-dimensional real homotopy representation of DW_3 exists,

a question we hope to address in a later paper. The lowest known faithful homotopy representation of DW_3 at the time of this writing has complex dimension 2^{46} [Zie09].

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