ELLIPTIC COHOMOLOGY AND PROJECTIVE SPACES —A COMPUTATION—

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ABSTRACT. In this paper, I compute the tmf-homology and cohomology of projective spaces as tmf-modules by means of skeletal filtrations. The multiplicative structure is not computed.

1. INTRODUCTION

In [HM], Hopkins and Miller define an E_∞ ring spectrum tmf, which is p-locally a connective version of the spectrum $EO_2 = E_2^{hG}$, where E_2 is the Hopkins-Miller spectrum [Rez98], and G is a maximal finite subgroup of the height 2 Morava stabilizer group acting on E_2. This spectrum has generated much interest because its homotopy groups are a topological variation of the ring on modular forms, and indeed tmf can be defined using diagrams of elliptic curves. From a stable homotopy point of view, it is interesting because it detects a surprisingly large subset of the ring of stable homotopy groups of spheres. For instance, at the prime $p = 2$, all of the nonzero classes usually named $\eta^i, \nu^j, \epsilon, \kappa, \bar{\kappa}$ go to nonzero classes in π_* tmf under the Hurewicz homomorphism.

The aim of this paper is to compute the tmf homology and cohomology of complex projective spaces. There is a famous map $\mathbf{C}P^\infty_+ \to \mathbf{S}^{-1}$ known as the $\mathbf{S}^1$-transfer, and there are some partial results on the image of this map in stable homotopy, e.g. [Mil82]. We are able to analyze the effect of the $\mathbf{S}^1$-transfer in tmf completely, and surprisingly it turns out that it is almost surjective. There are probably other applications where this computation might be valuable.

To compute $\mathrm{tmf}^*(\mathbf{C}P^\infty)$, it is useful to note that for any elliptic spectrum E, $E^*(\mathbf{C}P^\infty)$ can be though of as the abelian group of functions on the completion of E that vanish at the identity element. Similarly, $E^*(\mathbf{C}P^n)$ are such functions $f(z)$ modulo z^{n+1}.

Therefore, by the construction of tmf, there is a spectral sequence converging to $\mathrm{tmf}^*(\mathbf{C}P^\infty)$ whose E_2-term is the cohomology of the Hopf algebroid $(A[[z]], \Gamma[[z]])$ representing the following data:

- The objects are elliptic curves in Weierstrass form together with a function on the formal completion that vanishes at the identity element;
- The isomorphisms are isomorphisms of elliptic curves that are compatible with the function data

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The structure of this Hopf algebroid is easy to determine: (A, Γ) is the elliptic curve Hopf algebroid

$$A = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$$
$$\Gamma = A[r, s, t]$$

with the usual structure maps, z is primitive, and

$$\eta_R(z) = -\frac{x+r}{y+sx+t}.$$

In the present paper, we use the filtration of $(A[[z]], \Gamma[[z]])$ by ideals (z^n) and the resulting spectral sequence to compute the cohomology of this Hopf algebroid, and derive the differentials in the corresponding Adams spectral sequence from the differentials of $(A, \Gamma) \Rightarrow \pi_* \mathrm{tmf}$. This approach is equivalent to running the tmf-based Atiyah-Hirzebruch spectral simultaneously with the Adams spectral sequence converging to $\pi_* \mathrm{tmf}$. For computations, we break the spectral sequence up into a series of long exact sequences.

To ensure nice convergence, we prove along the way:

Proposition 1.1. *The tower $\{\mathrm{tmf}^*(\mathbf{C}P^n)\}_{n \geq 0}$ is Mittag-Leffler. In particular,*

$$\mathrm{tmf}^*(\mathbf{C}P^\infty) = \varprojlim_n \mathrm{tmf}^*(\mathbf{C}P^n).$$

For $p > 5$, we know that $\mathrm{tmf}_* \simeq \mathbf{Z}_{(p)}[c_4, c_6]$ is the classical ring of modular forms. Since it is concentrated in even degrees, it is complex orientable and

Lemma 1.2.

$$\mathrm{tmf}^*_{(p)}(\mathbf{C}P^\infty) \cong (\mathrm{tmf}_{(p)})_{-*} \otimes \mathbf{Z}_{(p)}[z],$$

where z has degree 2.

□

At the primes 2 and 3, much more interesting things happen.

1.1. **Some tmf_*-modules.** Let R denote the ring of classical (integral) modular forms:

Definition. Let $R = \mathrm{Ext}^{0,*}(A, \Gamma) \cong \mathbf{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ be the filtration zero line of the Adams-Novikov spectral sequence converging to $\pi_* \mathrm{tmf}$. By the boundary map h, R becomes a tmf_*-module.

Definition. Define the following elements in $\pi_* \mathrm{tmf}$:

- For $x = c_4$, c_6, Δ^i, let $\overline{x}$ be such that $h(\overline{x}) = nx$ for $n \geq 1$ minimal.
- Let P be a generator of the torsion in $\pi_{20} \mathrm{tmf}$. Thus at $p = 2$, $P = \bar{\kappa}$ and at $p = 3$, $P = \beta_1^2$, up to units.

Note that $c_4 = h(\overline{c_4})$, $2c_6 = h(\overline{c_6})$, $24\Delta = h(\overline{\Delta})$.

We define certain ideals in R for later use:

Definition. Define $I_n \triangleleft R$ to be the ideal generated by (n, c_4, c_6), and $I'_n \subseteq I_n$ the ideal in R generated by $(n, 2c_4, 2c_6)$.

We will also need some modules related to tmf_*:

Definition. Define

$$\mathrm{tmf}_*^- = \mathrm{tmf}_* /(24, \overline{c_4}, \overline{c_6}, 12P, i\overline{\Delta^i} \mid i \geq 1)$$

Theorem 1.3. *The* $\mathbf{S}^1$*-transfer map* $\mathbf{C}\mathrm{P}^\infty \to \mathbf{S}^{-1}$ *induces a map of* tmf_**-modules*

$$\Sigma^{-1}\,\mathrm{tmf}^* \to \Sigma^{-1}\,\mathrm{tmf}_*^- \hookrightarrow \mathrm{tmf}^*(\mathbf{C}\mathrm{P}^\infty).$$

The exact evaluation of $\mathrm{tmf}^*(\mathbf{C}\mathrm{P}^\infty)$ is given in Theorems 3.5 and 7.1, respectively.

Similarly, we have the following corollary for tmf homology of Theorems 3.6 and 8.2:

Theorem 1.4. *The* $\mathbf{S}^1$*-transfer map* $\mathbf{C}\mathrm{P}^\infty_+ \to \mathbf{S}^{-1}$ *induces a map of* tmf_**-modules*

$$\mathrm{tmf}^*(\mathbf{C}\mathrm{P}^\infty_+) \twoheadrightarrow \Sigma^{-1}\ker h \hookrightarrow \Sigma^{-1}\,\mathrm{tmf}_*$$

This implies that the $\mathbf{S}^1$-transfer is surjective on stable homotopy classes that are detected by elliptic cohomology.

Notation. In the Adams-Novikov spectral sequence, an element $x \in \mathrm{Ext}^{s,t}$ is said to have bidegree $(s, t-s)$, and we index spectral sequence charts accordingly (Adams indexing).

All rings are graded commutative and augmented with augmentation ideal $\mathcal{I}(-)$. For two such rings R, S, let $R \times S$ denote the product in this category, i.e. $(R \otimes S)/(\mathcal{I}(R) \otimes S + R \otimes \mathcal{I}(S))$. We write $R[x]$ for the free commutative algebra over R on one generator x in degree $|x|$; thus, if $|x|$ is odd and $2 \neq 0$, this is an exterior algebra, and a polynomial algebra otherwise.

In a differential graded ring, we denote n-fold Massey products by $\langle x_1, \dots, x_n\rangle$, and saying $x = \langle \dots \rangle$ implicitly claims that the indeterminacy is zero.

2. A tmf Thom isomorphism

Recall [AHS01] that tmf has an $MU\langle 6\rangle$ orientation, refining Witten's $MO\langle 8\rangle$ genus. That is, for every 0-dimensional virtual vector bundle $E \to X$ whose classifying map $X \to BU$ lifts to $BU\langle 6\rangle$, there is an isomorphism $\mathrm{tmf}^*(X_+) \xrightarrow{\cong} \mathrm{tmf}^*(X^E)$.

In this section, every theorem has a 2-local version, a 3-local version, and an integral version. Let $r = 8$, 3, or 24, accordingly.

Lemma 2.1. *Let* L *be the inverse of the tautological line bundle on* $\mathbf{C}\mathrm{P}^\infty$*. (*$L = \mathcal{O}(1)$*, the generator of the Picard group that has a section.) Then there is a Thom isomorphism*

$$\mathrm{tmf}^*(\mathbf{C}\mathrm{P}^\infty_+) \xrightarrow{\cong} \mathrm{tmf}^{*+2r}(\mathbf{C}\mathrm{P}^\infty)^{rL}.$$

Proof: Let $R(\mathbf{S}^1) = \mathbf{Z}[z, z^{-1}]$ be the representation ring of $\mathbf{S}^1$, where z denotes the standard one-dimensional representation of $\mathbf{S}^1$. The subsets of elements $V \in R(\mathbf{S}^1)$ such that tmf admits a Thom isomorphism for the associated bundle over $B\mathbf{S}^1 = \mathbf{C}\mathrm{P}^\infty$ form an ideal $I \triangleleft R(\mathbf{S}^1)$. Furthermore, orientability of bundles only depends on their J-equivalence class, i.e., the equivalence class of their associated spherical fibration. Let $A = J^{-1}R(\mathbf{S}^1)/I$ be the abelian group that is the quotient of $R(\mathbf{S}^1)/I$ modulo J-equivalence, i.e., modulo the Adams relations

$$k^\infty(\psi^k(x) - x) = 0 \quad \text{for all } k \in \mathbf{Z} \text{ and } x \in R(\mathbf{S}^1)/I.$$

This shorthand notation means that for any element $x \in R(\mathbf{S}^1)/I$ and integer k, there is an integer N such that $k^N(\psi^k(x) - x) = 0$ in A. ψ^k is the Adams operation, given as a ring homomorphism by $\psi^k(z) = z^k$.

The classifying map of the bundle $(L-1)^n \to \mathbf{C}P^\infty$ lifts to $BU\langle 2n\rangle$. This shows that $(L-1)^3$ is tmf-orientable, and hence $(x-1)^3 \in I$. Therefore, we have a surjection

$$B = J^{-1}\mathbf{Z}[z, z^{-1}]/(z^3 - 3z^2 + 3z - 1) \twoheadrightarrow A.$$

Let $z_i = [(z-1)^i] \in B$. We will now show that the additive order of z_1 is 24, which proves the lemma. This is a classical J-calculation.

- Let $k = -1$. Since k is invertible, we have $\psi^{-1}x - x = 0$ for all $x \in B$. Applying this to $x = z_1$, we get
$$0 = \psi^{-1}(z_1) - z_1 = z^{-1} - z = (z^2 - 3z + 3) - z = z_2 - 2z_1.$$
- Let $k = 2$. We compute
$$0 = 2^\infty(\psi^2(z_1) - z_1) = 2^\infty(z_2 + z_1) = 2^\infty 3z_1.$$
- Let $k = 3$. We have
$$0 = 3^\infty(\psi^3(z_1) - z_1) = 3^\infty(3z^2 - 4z + 1) = 3^\infty(3z_2 + 2z_1) = 3^\infty 8z_1$$

□

Definition. For $n \in \mathbf{N}$ and $m \in \mathbf{Z}$, let $\mathbf{C}P^n_m$ denote the Thom space of mL over $\mathbf{C}P^{n-1}$.

• This notation conflicts with the more classical definition that $\mathbf{C}P^n_m$ is the truncated projective space with cells in dimension m through n. But we have that $\mathbf{C}P^n_1 = \mathbf{C}P^n$, $\mathbf{C}P^n_0 = \mathbf{C}P^{n-1}_+$, and $\mathbf{C}P^n_m$ has n cells.

Corollary 2.2. *There are Thom isomorphisms*

$$\mathrm{tmf}^*(\mathbf{C}P^n_m) \xrightarrow{\cong} \mathrm{tmf}^{*+2rk}(\mathbf{C}P^n_{m+rk})$$

for all $n \in \mathbf{N}$, $m, k \in \mathbf{Z}$.

Proof: This is an immediate consequence of the previous lemma. □

Corollary 2.3. *If* $k = rn - 2$ *then*

$$\mathrm{tmf}_*(\mathbf{C}P^k) \cong \mathrm{tmf}^{2k+2-*}(\mathbf{C}P^k).$$

Proof: The tangent bundle $T(\mathbf{C}P^n)$ is isomorphic to $(n+1)L - 1$. By Atiyah duality, $D(\mathbf{C}P^n) = (\mathbf{C}P^n)^{-T} = \Sigma^2\mathbf{C}P^n_{-(n+1)}$. Hence,

$$\begin{aligned}\mathrm{tmf}_*(\mathbf{C}P^k) &= \mathrm{tmf}^{-*}(D(\mathbf{C}P^k)) \cong \mathrm{tmf}^{-*}(\Sigma^2\mathbf{C}P^k_{-k-1}) \\ &\xrightarrow{\text{Thom}} \mathrm{tmf}^{-*}(\Sigma^{-2k-2}\mathbf{C}P^k) = \mathrm{tmf}^{2k+2-*}(\mathbf{C}P^k)\end{aligned}$$

□

Corollary 2.4. *We have*

$$\mathrm{tmf}_*(\mathbf{C}P^{rn}) \cong \mathrm{tmf}^{rn-2-*}(\mathbf{C}P^{rn}_{-1}).$$

□

3. Cohomology and homology at the prime 3

In this section, let every ring, module, every space, and every spectrum be 3-local.

We remind the reader of the structure of $\pi_*(\mathrm{tmf})$ at $p = 3$:

Theorem 3.1. *At* $p = 3$,

(1)
$$\mathrm{Ext}^{*,*}(A, \Gamma) = \frac{R \otimes \mathbf{Z}[\alpha_1, \beta_1]}{(3, c_4, c_6)(\alpha_1, \beta_1)}$$
where $|\alpha_1| = (1, 3)$ *and* $|\beta_1| = (2, 10)$ *map to the classes of the same names under the Hurewicz homomorphism. Of course, for degree reasons,* α_1 *is an exterior generator, while* β_1 *is polynomial.*

(2) *The differentials are generated multiplicatively by*
$$d_5(\Delta) = \beta_1^2\alpha_1 \quad \textit{and} \quad d_9(\Delta^2\alpha_1) = \beta_1^5.$$
Thus,
$$\pi_*(\mathrm{tmf}) = \frac{\mathbf{Z}[\overline{c_4}, \overline{c_6}, \overline{\Delta}]}{(4\overline{c_4}^3 - \overline{c_6}^2 - 288\overline{\Delta})} \times \frac{\mathbf{Z}[\alpha_1, \beta_1, \tau]}{(\beta_1^5, \beta_1^2\alpha_1, \tau\alpha_1, \tau\beta_1^2, 3\alpha_1, 3\beta_1, 3\tau)}$$
where $\beta_1 = \langle \alpha_1, \alpha_1, \alpha_1 \rangle$ *and* $\tau = \langle \alpha_1, \beta_1^2, \alpha_1 \rangle$.

Complex conjugation of a line bundle induces an involution $c : \mathbf{C}P^\infty \to \mathbf{C}P^\infty$. Since 2 is invertible, $\tau = \frac{1+c}{2}$ is an idempotent endomorphism of $\mathbf{C}P^\infty$, and induces a (stable) splitting

$$\mathbf{C}P^\infty_{(3)} \simeq P^+ \vee P^-, \tag{3.2}$$

where $H^*(P^+) = \mathbf{Z}\{x^{2n}\}$ and $H^*(P^-) = \mathbf{Z}\{x^{2n-1}\}$.

Remark 3.3. *The spectral sequences for* $\mathrm{tmf}^*(P^\pm)$ *have as* E_2 *term the cohomology of the Hopf algebroid classifying elliptic curves with* even *(*odd*) functions on the formal completion. The splitting can also be seen from the structure of* η_R *of the Hopf algebroid associated with* $\mathrm{tmf}^*(\mathbf{C}P^\infty)$*, which at the prime* $p = 3$ *becomes equivalent to* $(A[[z]], \Gamma[[z]])$*, where*
$$A = \mathbf{Z}_{(3)}[a_2, a_4]$$
$$\Gamma = A[r]/(r^3 + a_2r^2 + a_4r)$$
and
$$\eta_R(z) = -\frac{x + r}{y} = z + rw = z + r(z^3 + a_2z^5 + (a_2^2 + a_4)z^7 + \dots).$$

3.1. The cohomology of the cone on ν.

Remember that ν is the generator of π_3^s, represented by the Hopf map $\mathbf{S}^7 \to \mathbf{S}^4$. This element is represented in the E^2-term of the tmf spectral sequence by α_1.

Note that by the splitting (3.2), 3-locally, $\mathbf{C}P^3 \simeq \mathbf{S}^4 \vee \Sigma^2 C(\nu)$, and we have a spectral sequence converging to $\mathrm{tmf}^* C(\nu)$ whose E_2-term is the Hopf algebroid classifying elliptic curves together with an odd function at the origin, modulo degree 5. Let e_i denote the generator corresponding to the $2i$-cell of $\mathbf{C}P^\infty$ in $E^{0,-2i}$.

In the chart in Figure 3.1, and all others, a box symbol stands for a generator of a rank 1 free module over R (cf. Section 1.1). More precisely, if the module displayed in the chart with $\square$ read as a copy of $\mathbf{Z}$ is M, then the chart shows

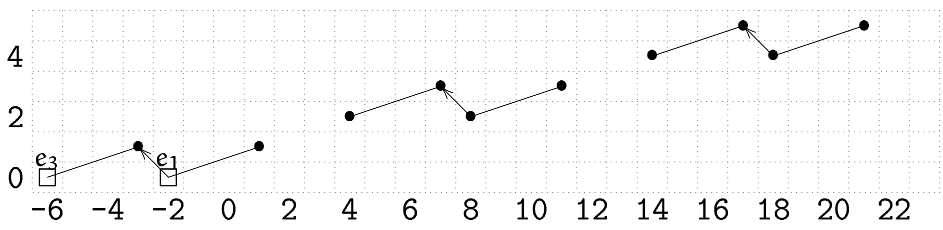


FIGURE 3.1. The 3-local long exact sequence for $\Sigma^2 C(\nu)$

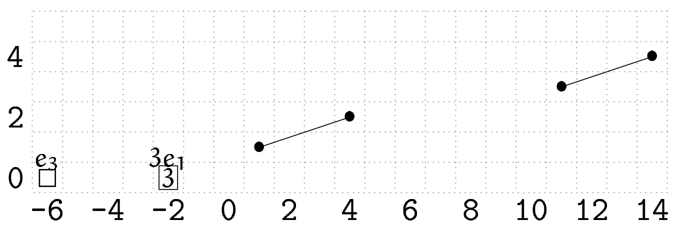


FIGURE 3.2. The 3-local Ext term for $\Sigma^2 C(\nu)$.

$(M \otimes R)/(\mathrm{Ext}^{\geq 1,*}(c_4, c_6))$. The indexing is reversed on the horizontal axis, i.e. the coordinates are $(s, s-t)$.

Because of the Massey products in Theorem 3.1, we have multiplicative extensions which result in a chart as given in Figure 3.1.

A symbol $\boxed{n}$ denotes a generator for a copy of the module I_n with the same conventions as for $\square$ in Figure 3.1.

Inspection of the differentials, which follow from the differentials for tmf, imply

Lemma 3.4.

$$\mathrm{tmf}^*(C(\nu)) = \Sigma^{-6} R \oplus \Sigma^{-2} I \oplus \Sigma\, \mathrm{tmf}_*^-$$
$$\mathrm{tmf}_*(C(\nu)) = R \oplus \Sigma^4 I \oplus \Sigma^7\, \mathrm{tmf}_*^-$$

3.2. **The E_2 term for $C(2\nu, \nu)$.** We will now compute the E_2-term for the space $C(2\nu, \nu)$ obtained by lifting the map $2\nu : \mathbf{S}^2 \to \mathbf{S}^{-1}$ to $\Sigma^2 C(\nu)t$ and taking its homotopy fiber. This is a three-cell complex with cells in dimensions -2, 2, and 6.

The Ext term resulting from the long exact sequence in Figure 3.2 has no elements in positive filtration, and

$$\mathrm{tmf}^*(C(2\nu, \nu)) \cong \Sigma^2 I \oplus \Sigma^{-2} I \oplus \Sigma^{-6} R.$$

3.3. **The E_2 term for $P^\pm$.** We can now compute the E_2-term for P^- from this by running a spectral sequence coming from filtering P^- by $12n-6$-skeleta. Note that the filtration quotients are

$$(P^-)^{[12n-6]}/(P^-)^{[12(n-1)-6]} \cong \left(\mathbf{C}P^6_{6(n-1)}\right)^{\frac{1+\epsilon}{2}}$$

end hence their tmf-cohomology is always isomorphic to

$$\Sigma^{12(n-1)+2} C(2\nu, \nu).$$

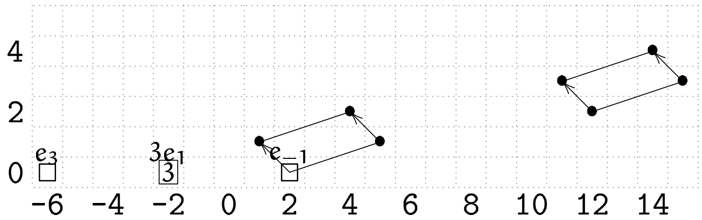


FIGURE 3.3. The long exact sequence associated to $\mathbf{S}^{-2} \to C(2\nu.\nu) \to \Sigma^2 C(\nu)$

Therefore the spectral sequence collapses at the E_1 term and yields

$$\mathrm{tmf}^*(P^-) = \left(\bigoplus_{n\equiv 1(3)} \Sigma^{-4n-2}R\right) \oplus \left(\bigoplus_{n\equiv 0,2(3)} \Sigma^{-4n-2}I\right) \oplus \Sigma\,\mathrm{tmf}_*^-$$

Similarly, we can filter P^+ by $12n$-skeleta. Since $(P^+)^{[12]} \simeq \Sigma^6 C(2\nu,\nu)$, the spectral sequence collapses again and yields

$$\mathrm{tmf}^*(P^+) = \left(\bigoplus_{n\equiv 0(3)} \Sigma^{-4n}R\right) \oplus \left(\bigoplus_{n\equiv 1,2(3)} \Sigma^{-4n}I\right)$$

3.4. **The 3-local tmf-cohomology of $\mathbf{C}P^\infty$.** Putting the results of the previous section together, we readily obtain:

Theorem 3.5. *At $p = 3$, we have*

$$\mathrm{tmf}^*(\mathbf{C}P^\infty) = \left(\bigoplus_{n\equiv 0(3)} \Sigma^{-2n}R\right) \oplus \left(\bigoplus_{n\equiv 1,2(3)} \Sigma^{-2n}I\right) \oplus \Sigma\,\mathrm{tmf}_*^-$$

where $n \geq 1$.

3.5. **The 3-local tmf-homology of $\mathbf{C}P^\infty$.** Since both $C(\nu)$ and $C(2\nu,\nu)$ are Spanier-Whitehead self-dual, we get

Theorem 3.6. *At $p = 3$, we have*

$$\mathrm{tmf}_*(\mathbf{C}P^\infty) = \left(\bigoplus_{n\equiv 2(3)} \Sigma^{2n}R\right) \oplus \left(\bigoplus_{n\equiv 0,1(3)} \Sigma^{2n}I\right) \oplus \Sigma^9\,\mathrm{tmf}_*^-$$

where $n \geq 1$.

4. THE PRIME 2: $\mathrm{tmf}^*(\mathbf{C}P^2)$

The starting point is the E_2-term of the Adams-Novikov spectral sequence associated to the elliptic curve Hopf algebroid (A, Γ) converging to $\pi_*\,\mathrm{tmf}$ as displayed in Figure 4, where, without indeterminacies, the following Massey product identities

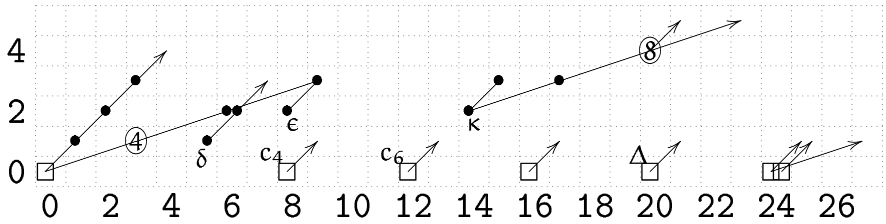


FIGURE 4.1. The Ext term of tmf at the prime 2.

hold:

$$\epsilon = \langle 2h_2, h_2, h_1\rangle = \langle 2, h_2^2, h_1\rangle = \langle h_2, h_1, h_2\rangle;$$
$$\delta = \langle 4, h_2, h_1\rangle;$$
$$\kappa = \langle 2, h_1, h_2, \epsilon\rangle;$$
$$\bar\kappa = \langle \begin{pmatrix} h_2^2 & h_1\end{pmatrix}, \begin{pmatrix} h_2 & \epsilon \\ \epsilon & h_2\end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2\end{pmatrix}, \begin{pmatrix} h_2 \\ \epsilon\end{pmatrix}\rangle.$$

The first equality is a consequence of a very useful equality proved by Toda in [Tod62, 3.10]:

Theorem 4.1. *For every $\alpha \in \pi_*^s$ of odd degree,*

$$\{\alpha, \beta, \alpha\} \cap \{2\alpha, \alpha, \beta\} \neq \emptyset$$

By the convergence of Massey products to Toda brackets, this implies that the same relations hold for Massey products of representing cycles in the Ext term.

Note that in tmf, the classes $\bar\nu, \epsilon \in \pi_8^s$ both map to the class represented by what we call ϵ; since $pi_7(\mathrm{tmf}) = 0$, there is no indeterminacy.

We also record the following identities:

$$\langle h_1, 2, h_1\rangle = 2h_2 \tag{4.2}$$

$$\langle h_1, h_2, h_1\rangle = \{h_2^2, h_2^2 + \delta\, h_1\} \tag{4.3}$$

(4.4)
$$\langle h_2, \epsilon, h_2\rangle \underset{\text{Thm. 4.1}}{=} \langle 2h_2, h_2, \epsilon\rangle = \langle\langle h_1, 2, h_1\rangle, h_2, \epsilon\rangle = h_1\langle 2, h_1, h_2, \epsilon\rangle = \kappa\, h_1$$

We also have:

Lemma 4.5. $\kappa = \langle h_2, 2h_2, h_2, 2h_2\rangle$

Proof: We will show that this equality holds after multiplication with h_1. Since the latter is a bijective map from $(14, 2)$ to $(15, 3)$, this implies the claimed inequality. Indeed,

$$h_1\langle h_2, 2h_2, h_2, 2h_2\rangle = \langle\langle h_1, h_2, 2h_2\rangle, h_2, 2h_2\rangle = \langle \epsilon, h_2, 2h_2\rangle.$$

Now we use (4.2):

$$\langle \epsilon, h_2, 2h_2\rangle = \langle \epsilon, h_2, \langle h_1, 2, h_1\rangle\rangle = \langle \epsilon, h_2, h_1, 2\rangle h_1 = \kappa h_1. \quad \square$$

$\mathbf{C}P^2$ is the second suspension of the cone on η. The long exact sequence associated with the cofibre sequence

$$\mathbf{S}^3 \to \mathbf{S}^2 \to \mathbf{C}P^2$$

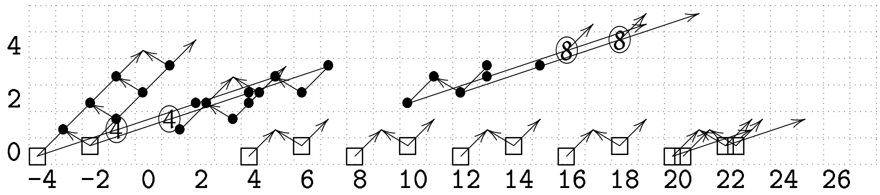


FIGURE 4.2. The long exact sequence for $\mathbf{C}P^2$

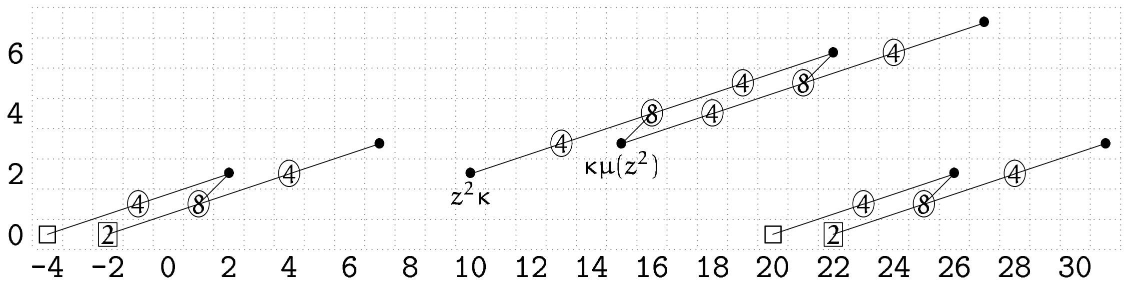


FIGURE 4.3. The Ext term for $\mathrm{C}P^2$

gives the connecting homomorphisms displayed in Figure 4.

Various multiplicative extensions happen at this point, as displayed in Figure 4. All of them come from basic identities for Massey products, namely:

$$\langle x_1\, x_2, x_3, x_4\rangle \subseteq (-1)^{|x_2|}\langle x_1, x_2\, x_3, x_4\rangle \tag{4.6}$$

$$\langle x_1, x_2, x_3\rangle x_4 = (-1)^{|x_1|+|x_2|+|x_3|+1} x_1\langle x_2, x_3, x_4\rangle \tag{4.7}$$

whenever both sides are defined.

We have:

$$4\langle z^2, h_1, h_2\rangle = z^2\langle h_1, h_2, 4\rangle = z^2\delta,$$

noting that $\langle z^2, h_1, h_2\rangle$ has a $\mathbf{Z}/2$ indeterminacy $(z^2\delta)$ killed by 2. Also,

$$2\langle z^2, h_1, h_2^2\rangle = z^2\langle h_1, h_2^2, 2\rangle = z^2\langle h_1, h_2, 2h_2\rangle = z^2\epsilon;$$

$$2\langle z^2, \kappa\, h_1, h_1\rangle = z^2\langle \kappa\, h_1, h_1, 2\rangle \underset{(4.4)}{=} z^2\langle\langle h_2, \epsilon, h_2\rangle, h_1, 2\rangle = z^2 h_2\langle \epsilon, h_2, h_1, 2\rangle = z^2\kappa\, h_2$$

$$h_1\langle z^2, h_1, h_2\rangle = z^2\langle h_1, h_2, h_1\rangle = z^2 h_2^2;$$

$$h_1\langle z^2\kappa, h_1, h_2\rangle = z^2\kappa\langle h_1, h_2, h_1\rangle\ z^2\kappa\, h_2^2.$$

The secondary operator μ is defined on classes x such that $x\,h_1 = 0$ as

$$\mu(x) = \langle x, h_1, h_2\rangle.$$

Similarly to the 3-local calculation, we use the following abbreviations in the charts: $\square$ stands for a copy of R, $\boxed{\mathbf{n}}$ for a copy of I_n, and $\boxed{\mathrm{n}}$ for a copy of I'_n.

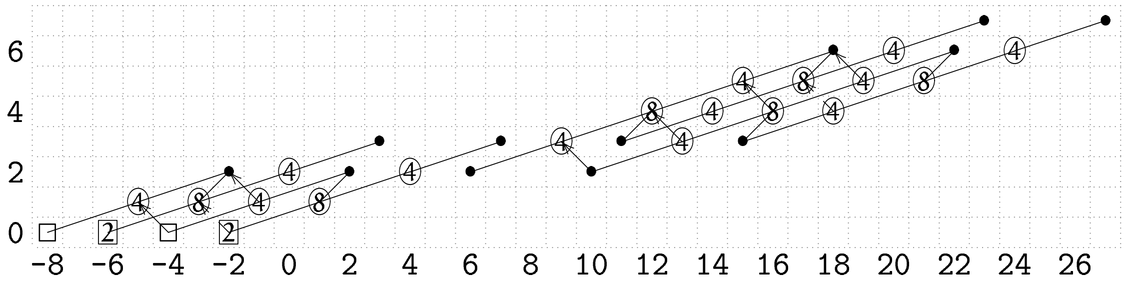


FIGURE 5.1. The long exact sequence associated to $\mathbf{C}P^2 \to \mathbf{C}P^4 \to \mathbf{C}P^2_3 \simeq \Sigma^4\mathbf{C}P^2$

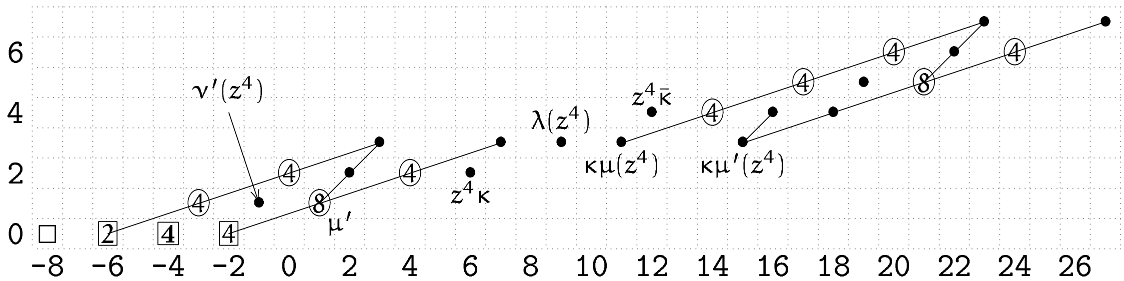


FIGURE 5.2. The Ext term for $\mathbf{C}P^4$

5. $\mathrm{tmf}^*(\mathbf{C}P^4)$ AT $p = 2$

We now run the long exact sequence for the cofibration sequence

$$\mathbf{C}P^2 \to \mathbf{C}P^4 \to \mathbf{C}P^2_3,$$

noting that $(\mathbf{C}P^2_3)_{(2)} \simeq (\Sigma^4\mathbf{C}P^2)_{(2)}$.

All differentials can be derived from the Steenrod module structure of $\mathbf{C}P^4$ except for the differential

$$d : \mathbf{Z}/8\ (1,1) \to \mathbf{Z}/4\ (0,2),$$

which could be 0 or 2 and still be compatible with the algebra structure. It is indeed 0, which follows from the fact that the generator $\mu'(z^4)$ in bidegree $(1,1)$ can be described as a Massey product only assuming the known differentials, and therefore has to be a cycle. Indeed, $\mu'(x)$ can be expressed as a matric Massey product:

$$\mu'(x) = \langle x, \begin{pmatrix} h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 2h_2 \end{pmatrix}, h_2\rangle \quad \text{for any } x \text{ such that } \langle x, h_2, h_1\rangle + \langle x, h_1, 2h_2\rangle = 0.$$

We compute μh_1^2 :

$$\mu(x)\, h_1^2 = \langle x, \begin{pmatrix} h_2 & h_1 \end{pmatrix}, \begin{pmatrix} \langle h_1, h_2, h_1^2\rangle \\ \langle 2h_2, h_2, h_1^2\rangle \end{pmatrix}\rangle = \langle x, \begin{pmatrix} h_2 & h_1 \end{pmatrix}, \begin{pmatrix} 0 \\ h_2^3 \end{pmatrix}\rangle = \langle x, h_1, h_2^3\rangle \neq 0.$$

The resulting Ext term is shown in Figure 5 where

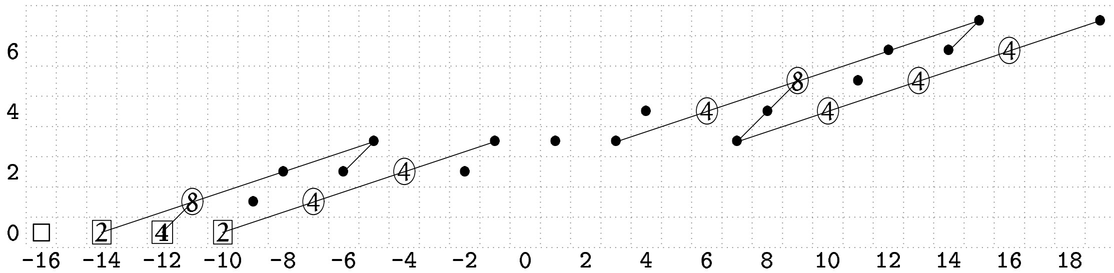


FIGURE 5.3. The Ext term for $\mathbf{CP}_5^4$

$$\nu'(x) = \langle x, h_2, 2h_2\rangle \quad \text{for any } x \text{ such that } x\, h_2 = 0,$$
$$\lambda(x) = \langle x, h_1, \kappa\, h_1\rangle \quad \text{for any } x \text{ such that } x\, h_1 = 0.$$

We note the following relation, identifying here and in the following an element of $\mathrm{Ext}(A_*)$ with the operator given by left multiplication with it:

$$(\nu')^2 = \kappa. \tag{5.1}$$

Proof:

$$\nu'(\nu'(x)) = \langle\langle x, h_2, 2h_2\rangle, h_2, 2h_2\rangle = x\langle h_2, 2h_2, h_2, 2h_2\rangle = x\kappa. \quad \square$$

For $\mathbf{CP}_5^4$, the spectral sequence looks very similar, but the differential

$$d : \mathbf{Z}\ (-10, 0) \to \mathbf{Z}/8\ (-11, 1)$$

becomes zero. This generates an extension

$$\langle z^8, h_2, 4\rangle h_1 = z^8\langle h_1, h_2, 4\rangle = z^8\delta.$$

and makes the analogue of the previous argument that

$$d : \mathbf{Z}/8\ (-7, 1) \to \mathbf{Z}/4\ (-8, 2)$$

is zero, invalid. In fact, this differential does not vanish. The result is shown in Figure 5.

6. $\mathrm{tmf}^*(\mathbf{CP}^8)$

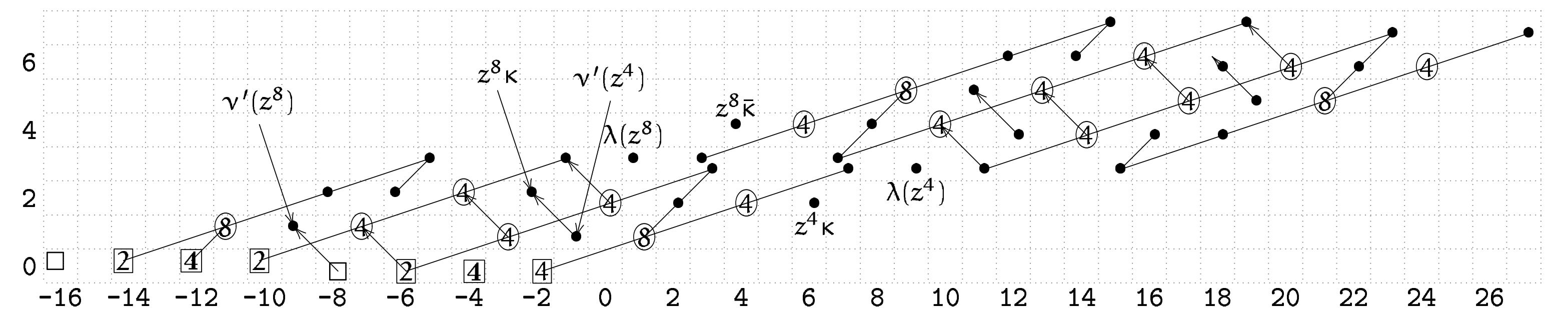
In the long exact sequence associated to $\mathbf{CP}^4 \to \mathbf{CP}^8 \to \mathbf{CP}_5^4$, the fundamental differentials are

$$d(z^4) = \nu'(z^8), d(z^3) = \mu'(z^8)$$

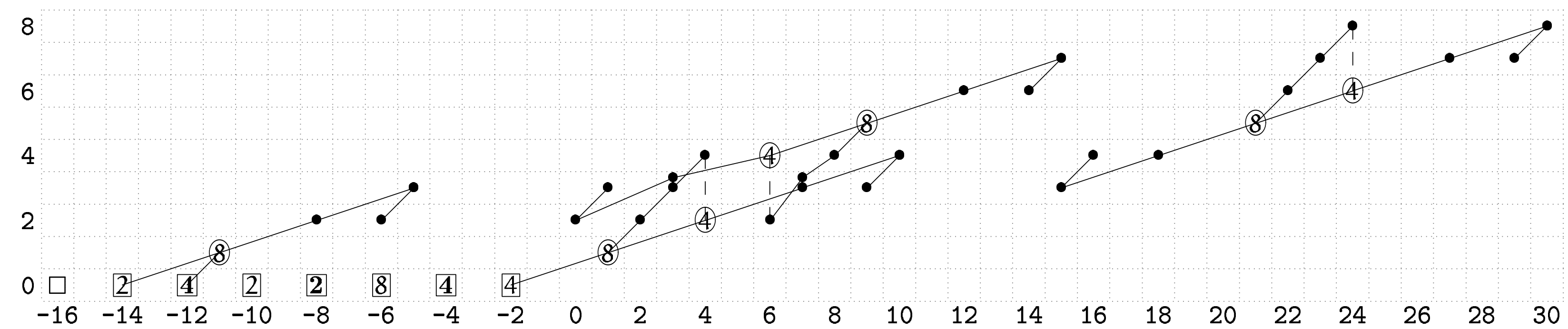
The first one implies that

$$d(\nu'(z^4)) = \nu'(d(z^4)) = \nu'(\nu'(z^8)) \underset{(5.1)}{=} z^8\kappa.$$

6.1. **Extensions on $\mathbf{CP}^8$.** We will now determine all h_0, h_1 and h_2 extensions on the Ext term for $\mathbf{CP}^8$, systematically ordered by dimension.



Differentials in the long exact sequence computing $\mathrm{Ext}(\mathbf{C}P^8)$.



$\mathrm{Ext}(\mathbf{C}P^8)$. Dashed vertical lines denote extensions that would happen in the E^5 term.

FIGURE 5.4. The computation for $\mathbf{C}P^8$

6.1.1. *Multiplication by h_1 on the class in bidegree $(0,2)$.*

Lemma 6.1. *The class in bidegree $(0,2)$ is given, without indeterminacy, by the Massey product*

$$\langle z^8, h_2, \begin{pmatrix} 2h_2 & \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon \\ h_2 \end{pmatrix} \rangle.$$

Using this expression, we compute:

$$\begin{aligned} \langle z^8, h_2, \begin{pmatrix} 2h_2 & \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon \\ h_2 \end{pmatrix} \rangle h_1 &= \\ = \langle z^8, h_2, \begin{pmatrix} 2h_2 & \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon\, h_1 \\ h_2\, h_1 \end{pmatrix} \rangle &= \\ = \langle z^8, h_2, 2h_2, \epsilon\, h_1 \rangle. \end{aligned}$$

On the other hand, the class $\lambda(z^8)$ in bidegree $(1,3)$ is given by

$$\lambda(z^8) = \langle z^8, \kappa\, h_1, h_1 \rangle = \langle z^8, \langle h_2, 2h_2, \epsilon \rangle, h_1 \rangle.$$

The claim now follows from the following general lemma about Massey products:

Lemma 6.2. *Let $a \in M$, b, c, d, $e \in A$ for a differential graded module M over a differential graded algebra A. Suppose $ab = bc = cd = 0$ and $\langle a, b, c \rangle = 0$. Then*

$$\langle a, \langle b, c, d \rangle, e \rangle \cap \langle a, b, c, d\, e \rangle \neq \emptyset.$$

Proof: We adopt the following notation: for a boundary x, we denote by $\underline{x}$ a chosen chain such that $d(\underline{x}) = x$, keeping in mind that it is not unique. Consider the following defining system for $\langle a, b, c, d\, e \rangle$:

$$\begin{array}{ccccccc} a & & b & & c & & d\, e \\ & \underline{ab} & & \underline{bc} & & \underline{cd}\, e & \\ & & \underline{\langle a, b, c \rangle} & & \underline{\langle b, c, d\, e \rangle} & & \end{array}$$

Note that this is not the most general defining system because we insist that the class bounding $c\, d\, e$ actually is a class $\underline{cd}$ bounding $c\, d$, multiplied with e.

On the other hand, a defining system for $\langle a, \langle b, c, d \rangle, e \rangle$ is given by

$$\begin{array}{ccccc} a & & \underline{bc}d + b\underline{cd} & & e \\ & \begin{array}{c} \underline{\langle a, b, c \rangle} d \\ + \underline{ab}\, \underline{cd} \end{array} & & \underline{\langle b, c, d\, e \rangle}. & \end{array}$$

If we compute the representatives of the Massey product for both defining systems, we get in both cases:

$$a \underline{\langle b, c, d\, e \rangle} + \underline{\langle a, b, c \rangle} d\, e + \underline{a\, b}\, \underline{c\, d}\, e.$$

□

6.1.2. *Multiplication by* h_2 *on the class in bidegree* $(0,2)$. A similar juggling lemma as in the previous section is needed to show that there is a nontrivial h_2 multiplication on the class in bidegree $(0,2)$:

Lemma 6.3. *Let* $a \in M$, b, c, d, $e \in A$ *for a differential graded module* M *over a differential graded algebra* A. *Suppose* $ab = bc = cd = 0$ *and* $\langle a,b,c\rangle = 0$. *Then*

$$\langle a, b, \langle c, d, e\rangle\rangle \cap \langle a, b\,c, d, e\rangle \neq \emptyset.$$

Proof: Pick a defining system $\underline{cd}$, $\underline{de}$ for $\langle c, d, e$. Then two defining systems can be chosen as follows:

$$\begin{array}{ccc|cccc}
\multicolumn{3}{c|}{\langle a, b, \langle c, d, e\rangle\rangle} & \multicolumn{4}{c}{\langle a, b\,c, d, e\rangle} \\
\hline
a & b & \underline{cd}e + c\underline{de} & a & bc & d & e \\
 & \underline{ab} \quad \underline{\langle b\,c, d, e\rangle} & & & \underline{ab}c \quad b\underline{cd} & \underline{de} & \\
 & & & & \underline{ab}\,\underline{cd} \quad \underline{\langle b\,c, d, e\rangle} & &
\end{array}$$

where $\underline{\langle b\,c, d, e\rangle}$ is defined on the right hand side. Now with these choices, both Massey product evaluate to

$$\underline{ab}\,\underline{cd}\,e + \underline{ab}\,c\,\underline{de} + a\underline{\langle b\,c, d, e\rangle}.$$

□

Lemma 6.4. *Let* $a \in M$, b, c, d, $x \in A$ *for a differential graded module* M *over a differential graded algebra* A. *Suppose* $ab = bc = cd = 0$ *and* $\langle a,b,c\rangle = 0$. *Then*

$$\langle a, b\,x, c, d\rangle \cap \langle a, b, c, x\,d\rangle \neq \emptyset.$$

Proof: Denote by $[cx]$ the commutator $cx - xc \in A$. Then we can choose two defining systems as follows:

$$\begin{array}{cccc|cccc}
\multicolumn{4}{c|}{\langle a, b\,x, c, d, \rangle} & \multicolumn{4}{c}{\langle a, b, c, x\,d\rangle} \\
\hline
a & b\,x & c & d & a & b & c & x\,d \\
 & \underline{ab}x \quad \begin{array}{c}\underline{bc}x \\ +b\underline{[cx]}\end{array} & \underline{cd} & & & \underline{ab} & \underline{bc} & \begin{array}{c}x\,\underline{cd} \\ +\underline{[xc]}\,d\end{array} \\
 & \begin{array}{c}\underline{\langle a,b,c\rangle}x \\ +\underline{ab}\underline{[cx]}\end{array} \quad \underline{\langle b\,x, c, d\rangle} & & & & \underline{\langle a,b,c\rangle} \quad \underline{\langle b, c, x\,d\rangle}. & &
\end{array}$$

Note that we can choose the representatives of the bottom right corner classes to be the same:

$$\underline{\langle b\,x, c, d\rangle} = \underline{\langle b, c, x\,d\rangle}.$$

The reason is that both classes are supposed to bound

$$b\,x\,\underline{cd} + \underline{bc}\,x\,d + b\underline{[cx]}d.$$

The Massey products on both sides now evaluate to:

$$a\underline{\langle b\,x, c, d\rangle} + \underline{ab}\,x\,\underline{cd} + \underline{ab}\underline{[cx]}d + \underline{\langle a, b, c\rangle}x\,d.$$

□

We are now ready to show that there is an h_2 extension on the class in bidegree $(0,2)$. Remember that that class is given by the Massey product

$$\langle z^8, h_2, \begin{pmatrix} 2h & \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon \\ h_2 \end{pmatrix} \rangle.$$

If we multiply this with h_2, the matric Massey product becomes an ordinary one, namely

$$\langle z^8, h_2, \epsilon, h_2^2 \rangle$$

since $\epsilon h_2 = 0$. On the other hand, the class in bidegree $(3,3)$ is given by

$$\begin{aligned} \mu(z^8)\kappa &= \langle z^8, h_1, h)2\rangle\kappa = \langle z^8, h_2, \kappa h_1\rangle \\ &= \langle z^8, h_2, \langle h_2, \epsilon, h_2\rangle\rangle \underset{\text{Lemma 6.3}}{=} \langle z^8, h_2^2, \epsilon, h_2\rangle \\ &\underset{\text{Lemma 6.4}}{=} \langle z^8, h_2, \epsilon, h_2^2\rangle. \end{aligned}$$

This shows the nontriviality of the extension.

6.1.3. *Multiplication by* h_1 *on the class in bidegree* $(6,2)$. The class in bidegree $(6,2)$ is given by the product

$$\langle z^8, h_2, 2h_2, \kappa\rangle.$$

On the other hand, the class in bidegree $(7,3)$ that has an h_1-multiplication is given by

$$\begin{aligned} \langle z^8, \begin{pmatrix} h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 2h_2 \end{pmatrix}, \kappa h_2\rangle &= \langle z^8, h_2, \kappa h_1, h_2\rangle \\ = \langle z^8, h_2, \langle 2h_2, h_2, \epsilon\rangle, h_2\rangle &= \langle z^8, h_2, 2h_2, \langle h_2, \epsilon, h_2\rangle\rangle \\ = \langle z^8, h_2, 2h_2, \kappa h_1\rangle &= \langle z^8, h_2, 2h_2, \kappa\rangle h_1. \end{aligned}$$

6.1.4. *Multiplication by* ϵ *on the class in bidegree* $(1,1)$. The lift of the class $\mu'(z^4)$ to $\mathrm{Ext}(\mathbf{CP}^8)$ can be described as a matric Massey product as follows:

$$\mu''(z^8) = \langle z^8, h_2, \begin{pmatrix} 2h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ 2h_2 \end{pmatrix}, h_2\rangle$$

On the other hand, the lift of the class $\lambda(z^4) \in \mathrm{Ext}(\mathbf{CP}^4)$ can be expressed as

$$\langle z^8, h_2, \begin{pmatrix} 2h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \kappa h_1\rangle. \tag{6.5}$$

We compute

$$\begin{aligned} \mu''(z^8)\epsilon &= \langle z^8, h_2, \begin{pmatrix} 2h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} \langle h_1, h_2, \epsilon\rangle \\ \langle 2h_2, h_2, \epsilon\rangle \end{pmatrix}\rangle \\ &= \langle z^8, h_2, \begin{pmatrix} 2h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1 \\ 0 & h_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \kappa h_1 \end{pmatrix}\rangle \\ &= \langle z^8, h_2, \begin{pmatrix} 2h_2 & h_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \kappa h_1\rangle. \end{aligned}$$

6.1.5. *Multiplication by* h_1 *on the class in bidegree* $(9,3)$. We want to show that there is a nontrivial h_1 multiplication on the class given in (6.5). The other hand, the surviving class in bidegree $(10,4)$ is

$$\langle z^8, h_2, h_1, 2\bar{\kappa}\rangle. \tag{6.6}$$

But

$$\langle z^8, h_2, \begin{pmatrix}2h_2 & h_1\end{pmatrix}, \begin{pmatrix}h_1\\ h_2\end{pmatrix}, \kappa h_1\rangle h_1 = \langle z^8, h_2, \begin{pmatrix}2h_2 & h_1\end{pmatrix}, \begin{pmatrix}\langle h_1, \kappa h_1, h_1\rangle\\ \langle h_2, \kappa h_1, h_1\rangle\end{pmatrix}\rangle$$
$$= \langle z^8, h_2, h_1, \langle h_2, \kappa h_1, h_1\rangle\rangle.$$

We are finished if we can show that $2\bar{\kappa} = \langle h_2, \kappa h_1, h_1\rangle$. But $2\bar{\kappa} = \langle \kappa h_1, h_1, h_2\rangle$, and it is straightforward from the defining systems that these two Massey products agree.

Note that the extensions in 6.1.4 and 6.1.5 together imply a nontrivial h_2 extension in bidegree $(7,3)$ because of $h_2^3 = \epsilon h_1$.

6.1.6. *The* E_5 *term for* $\mathbf{CP}^8$ *and* h_1 *multiplication on the class in bidegree* $(3,3)$. To compute the h_1 extension on the class $\mu''(z^8)h_1^2$, we make an indirect argument. From the module structure of $E_r(\mathbf{CP}^8)$ over $E_r(\mathbf{S}^0)$, we know that in $E_5(\mathbf{CP}^8)$, $4h_2 = h_1^3$. We will show that this extension is nonzero on $\mu''(z^8)h_2$. Indeed, we have in $E_5(\mathbf{CP}^8)$:

$$2\mu''(x) = \langle z^8, h_2, \begin{pmatrix}4h_2 & 0\end{pmatrix}, \begin{pmatrix}h_2 & h_1\\ 0 & h_2\end{pmatrix}, \begin{pmatrix}h_1\\ 2h_2\end{pmatrix}, h_2\rangle$$
$$= \langle z^8, h_2, 4h_2, \begin{pmatrix}h_2 & h_1\end{pmatrix}, \begin{pmatrix}h_1\\ 2h_2\end{pmatrix}, h_2\rangle$$

and hence

$$(2\mu''(x))(2x) = \langle x, h_2, 4h_2, \begin{pmatrix}h_2 & h_1\end{pmatrix}, \begin{pmatrix}\langle h_1, h_2, 2h_2\rangle\\ \langle 2h_2, h_2, 2h_2\rangle\end{pmatrix}\rangle$$
$$= \langle x, h_2, 4h_2, h_2, \langle 2h_2, h_2, h_1\rangle\rangle = \langle x, h_2, 4h_2, h_2, 2h_2, h_2\rangle h_1$$
$$= x\langle x, h_2, 4h_2, h_2, 2h_2, h_2, h_1\rangle.$$

But in E^5, that last Massey product is $\bar{\kappa}$. Hence we have a 4-extension of $\mu''(z^8)$ to the class $z^8\bar{\kappa}$ in bidegree $(4,4)$. This means that necessarily, h_1 on the class in bidegree $(3,3)$ must also be $z^8!$. For filtration reasons, this extension must already have happened in the E_2 term.

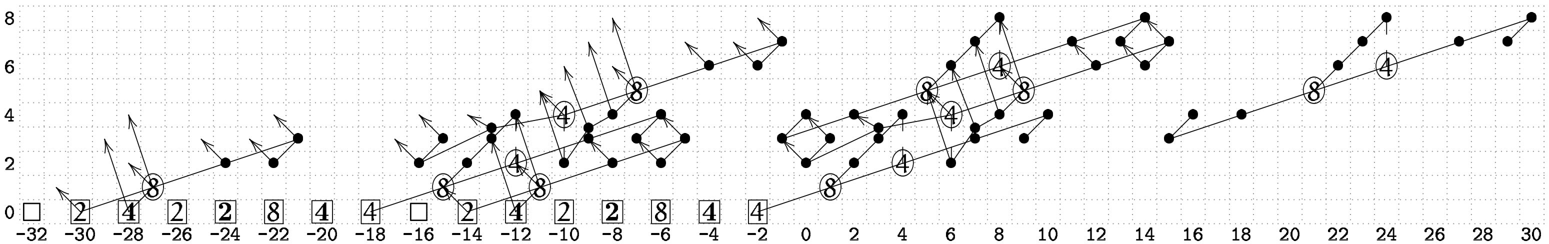
This concludes the computation of all 2, h_1, and h_2 extensions in $E_5(\mathbf{CP}^8)$.

7. The E_3 and E_5 term for $\mathbf{CP}^\infty$

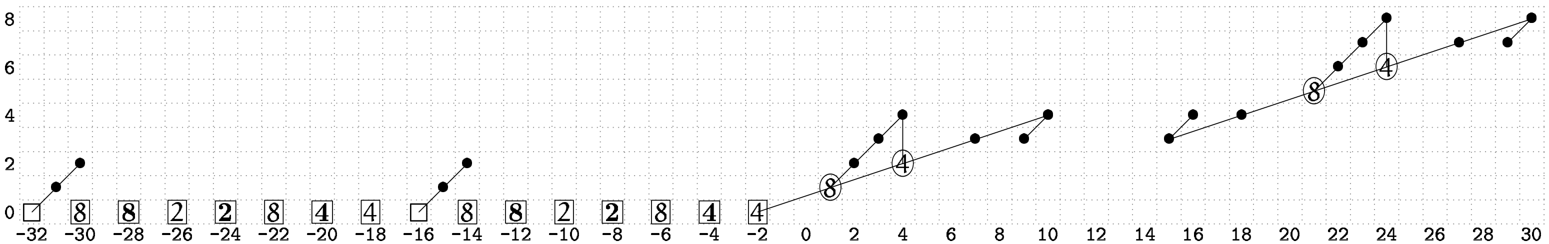
At this point, things become much easier. We now run a Bockstein spectral sequence for $\mathbf{CP}^\infty$ with E_2 term

$$E_1 = \bigoplus_{n\geq 0} \mathrm{Ext}(\mathbf{CP}^8_{8n+1}) \cong \mathrm{Ext}(\mathbf{CP}^8)[q_{-16}]$$

All d_1 differentials for this spectral sequence follow from the Steenrod algebra structure of $\mathbf{CP}^\infty$, cf. Figure 7. The Bockstein spectral sequence collapses at this



The spectral sequence associated to the filtration of $\mathbf{C}P^\infty$ by $8k$-skeleta.



End result: the E_5 term of the spectral sequence converging to $\mathrm{tmf}^*(\mathbf{C}P^\infty)$

FIGURE 7.1. The computation for $\mathrm{tmf}^*(\mathbf{C}P^\infty)$

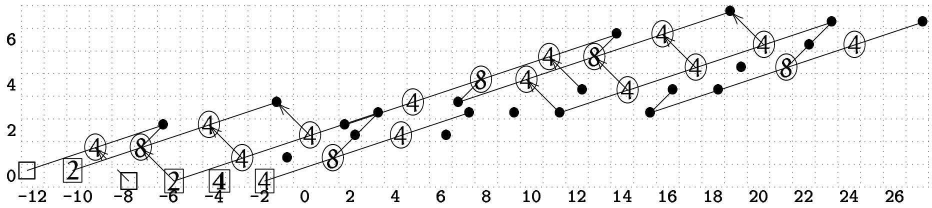


FIGURE 8.1. The long exact sequence induced by $\mathbf{C}P^4 \to \mathbf{C}P^6 \to \mathbf{C}P^2_5$

point and yields the beautiful E_5 picture of $\varprojlim_n \mathrm{tmf}^*(\mathbf{C}P^n)$ on the right of Figure 7.

Proof of Prop. 1.1: The chart in Figure 7 immediately shows that $\mathrm{im}(\mathrm{tmf}^*(\mathbf{C}P^{8n+i}) \to \mathrm{tmf}^*(\mathbf{C}P^{8n}))$ is constant for $i \geq 8$. □

By inspection, we have:

Theorem 7.1. *For* $p = 2$,

$$\mathrm{tmf}^*(\mathbf{C}P^\infty) = \left(\bigoplus_{n\equiv 1(8)} \Sigma^{-2n} I'_4\right) \oplus \left(\bigoplus_{n\equiv 3(4)} \Sigma^{-2n} I'_8\right) \oplus \left(\bigoplus_{n\equiv 5(8)} \Sigma^{-2n} I'_2\right)$$
$$\oplus \left(\bigoplus_{n\equiv 2(8)} \Sigma^{-2n} I_4\right) \oplus \left(\bigoplus_{n\equiv 6(8)} \Sigma^{-2n} I_8\right) \oplus \left(\bigoplus_{n\equiv 4(8)} \Sigma^{-2n} I_2\right)$$
$$\oplus \left(\bigoplus_{n\equiv 0(8)} \Sigma^{-2n} \tilde{R}\right) \oplus \Sigma^{-2} \widetilde{I'_4}$$

where $n \geq 2$, *and*

$$0 \to \Sigma(\mathbf{Z}/2[h_1]/(h_1^2) \to \tilde{R} \to R \to 0$$

and

$$0 \to \Sigma^3 \mathrm{tmf}_*^- \to \widetilde{I'_4} \to I''_4 \to 0$$

are the unique nontrivial extensions of tmf_**-modules, with* $I''_4 < I'_4$ *the submodule generated by* $(4, 2c_4, 2c_6, 8\Delta)$.

8. THE tmf-HOMOLOGY OF $\mathbf{C}P^\infty$

We will calculate the tmf-homology of $\mathbf{C}P^\infty$ using the spectral sequence associated to the following skeletal filtration:

$$* \hookrightarrow \mathbf{C}P^6 \hookrightarrow \mathbf{C}P^{14} \hookrightarrow \ldots \hookrightarrow \mathbf{C}P^{6+8k} \hookrightarrow \ldots \tag{8.1}$$

By Corollary 2.3, $\mathrm{tmf}_*(\mathbf{C}P^6) \cong \mathrm{tmf}^{14-*}(\mathbf{C}P^6)$. The cofiber sequence $\mathbf{C}P^4 \to \mathbf{C}P^6 \to \mathbf{C}P^2_5 \simeq \Sigma^8\mathbf{C}P^2$ induces the long exact sequence of Ext terms displayed in Figure 8, and the homology is shown in Figure 8. The extensions are direct consequences of extensions in the cohomological charts; a detailed check is left to the diligent reader.

Running the spectral sequence associated to (8.1) yields the differentials and extensions in Figure 8.

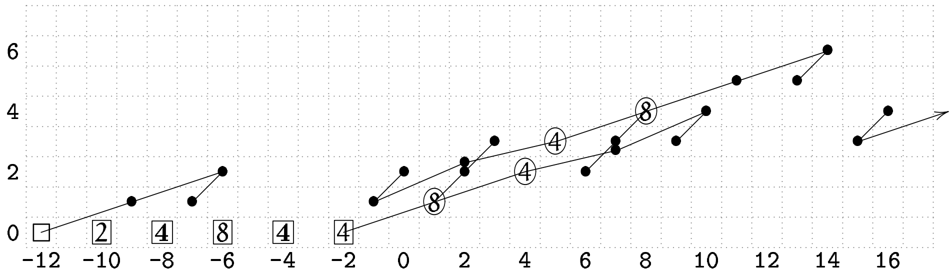


FIGURE 8.2. The Ext term for $\mathbf{C}P^6$

Definition. Let $A = (R/\operatorname{im}(h))/(I_8/\operatorname{im}(h))$, and let J be the unique nontrivial extension of tmf_*-modules

$$0 \to \Sigma^{-1}A \to J \to (h_2^2, \epsilon, \kappa, \bar{\kappa}) \to 0$$

Theorem 8.2.

$$\begin{aligned}\mathrm{tmf}_*(\mathbf{C}P^\infty) = &\left(\bigoplus_{n\equiv 7(8)} \Sigma^{2n} I_4'\right) \oplus \left(\bigoplus_{n\equiv 1(4)} \Sigma^{2n} I_8'\right) \oplus \left(\bigoplus_{n\equiv 3(8)} \Sigma^{2n} I_2'\right)\\ &\oplus \left(\bigoplus_{n\equiv 2(8)} \Sigma^{2n} I_8\right) \oplus \left(\bigoplus_{n\equiv 6(8)} \Sigma^{2n} I_4\right) \oplus \left(\bigoplus_{n\equiv 4(8)} \Sigma^{2n} I_2\right)\\ &\oplus \left(\bigoplus_{n\equiv 0(8)} \Sigma^{2n} \tilde{R}\right) \oplus \Sigma^2 R'\end{aligned}$$

where $n \geq 2$, and

$$0 \to \Sigma(\mathbf{Z}/2[h_1]/(h_1^2)) \to \tilde{R} \to R \to 0$$

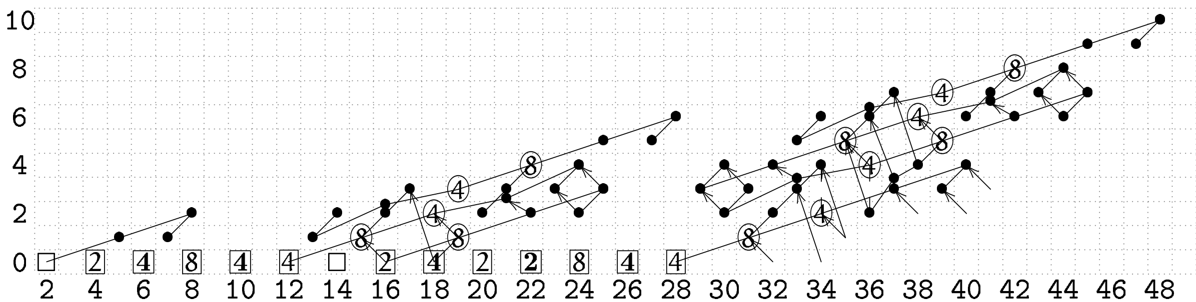
and

$$0 \to \Sigma^3 J \to R' \to R \to 0$$

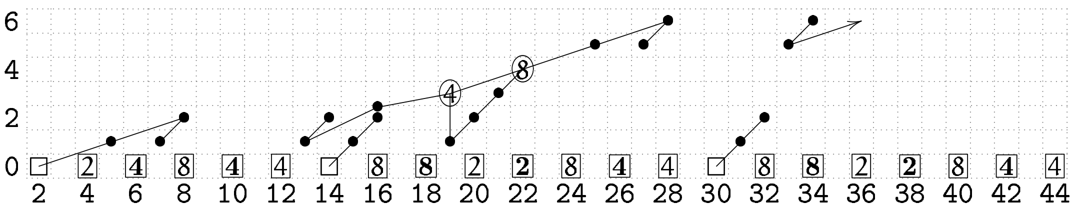
are the unique nontrivial extensions of tmf_**-modules.*

□

FIGURE 8.3. The computation of $\mathrm{tmf}_*(\mathbf{CP}^\infty)$



The homological spectral sequence associated to the filtration of $\mathbf{C}P^\infty$ by $6+8k$-skeleta



End result: the E_5 term of the spectral sequence converging to $\mathrm{tmf}_*(\mathbf{C}P^\infty)$.

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