Computation of the homotopy of the spectrum tmf

Tilman Bauer

This paper contains a complete computation of the homotopy ring of the spectrum of topological modular forms constructed by Hopkins and Miller. The computation is done away from 6, and at the (interesting) primes 2 and 3 separately, and in each of the latter two cases, a sequence of algebraic Bockstein spectral sequences is used to compute the E_2 term of the elliptic Adams–Novikov spectral sequence from the elliptic curve Hopf algebroid. In a further step, all the differentials in the latter spectral sequence are determined. The result of this computation is originally due to Hopkins and Mahowald (unpublished).

55N34; 55T15

1 Introduction

In [2; 3; 4], Hopkins, Mahowald, and Miller constructed a new homology theory called tmf, or topological modular forms (it was previously also called eo_2). This theory stands at the end of a development of several theories called elliptic cohomology (Landweber, Ravenel and Stong [9] Landweber [8], Franke[1] and others). The common wish was to produce a universal elliptic cohomology theory, ie, a theory Ell together with an elliptic curve C_u over π_0 Ell such that for any complex oriented spectrum E and any isomorphism of the formal group associated to E with the formal completion of a given elliptic curve C, there are unique compatible maps $\mathrm{Ell} \to E$ and $C_u \to C$. It was immediately realized that this is impossible due to the nontriviality of the automorphisms of the "universal" elliptic curve. Earlier theories remedied this by inverting some small primes (eg, by considering this moduli problem over $\mathbf{Z}[1/6]$), or by considering elliptic curves with level structures, or both. The Hopkins–Miller approach, however, is geared in particular towards studying the small prime phenomena that arise; as a trade-off, their theory tmf is not complex orientable. This, however, turned out to have positive aspects. Hopkins and Mahowald first computed the homotopy groups of tmf and realized that the Hurewicz map $\pi_* \mathbf{S}^0 \to \pi_* \mathrm{tmf}$ detects surprisingly many classes, at least at the primes 2 and 3; thus tmf was found to be a rather good approximation to the stable stems themselves at these primes. Their computation was never published, and the aim of this note is to give a complete calculation (in a different way than theirs) of the homotopy of tmf at the primes 2 and 3.

Published: 22 February 2008 DOI: 10.2140/gtm.2008.13.11

Along with the construction of $\texttt{tmf}$, which is constructed as an A_∞–ring spectrum and is shown to be E_∞, Hopkins and Miller set up an Adams–Novikov type spectral sequence that converges to $\pi_*(\texttt{tmf})$; its E_2 term is given as an Ext ring of a Hopf algebroid,

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{(A,\Gamma)}(A, A),$$

where all the structure of the Hopf algebroid (A, Γ) can be described completely explicitly. One particular feature of this Hopf algebroid is that both A and Γ are polynomial rings over $\mathbf{Z}$ on *finitely* many generators; it is this finiteness that generates a kind of periodicity in $\pi_*(\texttt{tmf})$ and makes it possible to compute the whole ring and not only the homotopy groups up to a certain dimension.

In Section 2 I will review some notions and tools from the homological algebra of Hopf algebroids; in Section 3, the elliptic curve Hopf algebroid is defined and studied. The short Section 4 contains the computation of $\pi_*\texttt{tmf}$ away from 6; Sections 5 and 6 contain the computation of the Ext term and the differentials, respectively, of the spectral sequence at the prime 3; in Sections 7 and 8, the same is done for the prime 2.

Acknowledgements I do not claim originality of any of the results of this paper. As already noted, the computation was first done by Hopkins and Mahowald. Rezk [11] also has some lecture notes with an outline of the computations that need to be done when computing the homotopy from the Hopf algebroid. I found it worthwhile to make this computation available anyway since it is, in my opinion, an interesting computation, and explicit knowledge of it is useful when using $\texttt{tmf}$ as an approximation to stable homotopy theory.

I am very grateful to Doug Ravenel, Andy Baker, John Rognes, and the anonymous referee for suggesting a number of improvements and corrections of this computation. Most importantly, I am deeply indepted to Mike Hopkins, my thesis advisor, from who I learned everything I know today about $\texttt{tmf}$.

The LaTeX package `sseq` which produced the numerous spectral sequence charts in this paper, is available from CTAN servers or from the author's home page.

2 Homological algebra of Hopf algebroids

In this section I review some important constructions that help in computing the cohomology of Hopf algebroids. All of this is well-known to the experts, but explicit references are a little hard to come by.

Definition The category of *Hopf algebroids* over a ring k is the category of cogroupoid objects (A, Γ) in the category of commutative k–algebras such that the left (equivalently, right) unit map $\eta_L\colon A \to \Gamma$ (resp. η_R) is flat.

The category of left *comodules* over a Hopf algebroid (A, Γ) is the category of (A, Γ)–left coaction objects in k–modules.

(For a more explicit description, consult Ravenel [10, Appendix A1].)

By convention, we will always consider Γ as an A–module using the left unit η_L unless otherwise stated. The flatness condition (which not all authors consider part of the definition of a Hopf algebroid) ensures that the category of comodules is an abelian category, which has enough injectives so that homological algebra is possible. We will abbreviate

$$H^n(A, \Gamma; M) = \mathrm{Ext}^n_{(A,\Gamma)-\mathrm{comod}}(A, M).$$

and $H^*(A, \Gamma) = H^*(A, \Gamma; A)$. In our situation, all objects are $\mathbf{Z}$-graded, and we write

$$H^{n,m}(A, \Gamma; M) = \mathrm{Ext}^n_{(A,\Gamma)-\mathrm{comod}}(A, M[-m])$$

where $(M[m])_n = M_{n-m}$.

Definition A *natural transformation* between two maps of Hopf algebroids

$$f = (f_0, f_1),\ g = (g_0, g_1)\colon (A, \Gamma) \to (A', \Gamma')$$

is a natural transformation of functors between cogroupoid objects; explicitly, it is an algebra map $c\colon \Gamma \to A'$ such that $c \circ \eta_L = f_0$ and $c \circ \eta_R = g_0$, and such that the following two composites are equal:

$$\begin{array}{ccccccc} & & & \xrightarrow{c\otimes \mathrm{id}} & A' \otimes_A \Gamma & \searrow & \\ \Gamma & \xrightarrow{\psi} & \Gamma \otimes_A \Gamma & & & & \Gamma', \\ & & & \xrightarrow[\mathrm{id}\otimes c]{} & \Gamma \otimes_A A' & \nearrow & \end{array}$$

where ψ is the Hopf algebroid diagonal. In the lower row, A' is an A–module by means of the map f_0, and the rightmost map is induced by $f_*\colon \Gamma \otimes_A A' \to \Gamma' \otimes_{A'} A'$. The upper row is defined analogously with g.

An *equivalence* of Hopf algebroids $(A, \Gamma) \simeq (A', \Gamma')$ consists of maps

$$f\colon (A, \Gamma) \leftrightarrows (A', \Gamma') :g$$

and natural transformations between the identity and $f \circ g$ and between the identity and $g \circ f$.

2.1 Remark The condition that $c \circ \eta_L = \mathrm{id}$ for a natural transformation from the identity to $f \circ g$ is saying that c is an A–algebra map.

An equivalence of Hopf algebroids induces an equivalence of comodule categories; in particular,

$$H^*(A, \Gamma; M) \cong H^*(A', \Gamma'; f_* M).$$

2.2 Remark In the language of stacks, this has the following interpretation. To every Hopf algebroid (A, Γ) there is associated a stack $\mathcal{M}_{(A,\Gamma)}$; this assignment is left adjoint to the inclusion of stacks into groupoid valued functors. Equivalent Hopf algebroids give rise to equivalent associated stacks. The Hopf algebroid cohomology with coefficients in the comodule M can be interpreted as the cohomology of the quasi-coherent sheaf $\widetilde{M}$ on the stack.

2.3 Base change

Let (A, Γ) be any Hopf algebroid, and let $A \to A'$ be a morphism of rings. Define $\Gamma' = A' \otimes_A \Gamma \otimes_A A'$, where the left tensor product is built using the map $\eta_L\colon A \to \Gamma$, and the right one using $\eta_R\colon A \to \Gamma$. Then (A', Γ') is also a Hopf algebroid, and there is a canonical map of Hopf algebroids $(A, \Gamma) \to (A', \Gamma')$.

The following theorem is proved by Hovey and Sadofsky [6, Theorem 3.3] and Hovey [5, Corollary 5.6]. The authors attribute it to Hopkins.

2.4 Theorem *If $(A, \Gamma) \to (A', \Gamma')$ is such a map of Hopf algebroids, and there exists a ring R and a morphism $A' \otimes_A \Gamma \to R$ such that the composite*

$$A \xrightarrow{1 \otimes \eta_R} A' \otimes_A \Gamma \to R$$

is faithfully flat, then it induces an equivalence of comodule categories, and in particular an isomorphism

$$H^*(A, \Gamma) \xrightarrow{\cong} H^*(A', \Gamma')$$

This theorem is reminiscent of the change of rings theorem [10, A1.3.12]. In fact, the change of rings theorem is enough to prove all the consequences of the above theorem used in this work, but I will use Theorem 2.4 for expository reasons.

2.5 The algebraic Bockstein spectral sequence

Let $I \lhd (A, \Gamma)$ be an invariant ideal of a Hopf algebroid, ie, $I \lhd A$ and $\eta_R(I) \subseteq I\Gamma$. Denote by $(A, \Gamma)/I$ the induced Hopf algebroid $(A/I, I\backslash\Gamma/I)$.

2.6 Theorem (Miller, Novikov) *There is a spectral sequence of algebras, called the algebraic Bockstein spectral sequence or the algebraic Novikov spectral sequence:*

$$E_2 = H^*\left((A, \Gamma)/I; I^n/I^{n+1}\right) \Longrightarrow H^*(A, \Gamma)$$

arising from the filtration of Hopf algebroids induced by the filtration

$$0 < A/I < A/I^2 < \cdots < A$$

If A is Cohen–Macaulay and $I \lhd A$ is a regular ideal, then $I^n/I^{n+1} \cong \mathrm{Sym}^n(I/I^2)$.

In the applications in this paper, $I = (x)$ will always be a principal ideal, and $\mathrm{Sym}^*(I/I^2) = A'[y]$ will be a polynomial algebra. Thus the E_2 term simplifies to

$$E_2 = H^*\left((A, \Gamma)/I\right)[y].$$

If $x = p$, then this is the ordinary (bigraded) Bockstein spectral sequence. In the topological context, Hopf algebroids are usually graded, and thus the spectral sequence is really tri-graded. In displaying charts, I will disregard the homological filtration degree in the above spectral sequence.

3 The elliptic curve Hopf algebroid

The Hopf algebroid of Weierstrass elliptic curves and their (strict) isomorphisms is given by:

$$A = \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]; \quad |a_i| = 2i;$$
$$\Gamma = A[r, s, t]; \quad |r| = 4;\ |s| = 2;\ |t| = 6$$

and the following structure maps:

The left units η_L are given by the standard inclusion $A \hookrightarrow \Gamma$, the right units are:

$$\eta_R(a_1) = a_1 + 2s$$
$$\eta_R(a_2) = a_2 - a_1 s + 3r - s^2$$
$$\eta_R(a_3) = a_3 + a_1 r + 2t$$
$$\eta_R(a_4) = a_4 - a_3 s + 2a_2 r - a_1 t - a_1 rs - 2st + 3r^2$$
$$\eta_R(a_6) = a_6 + a_4 r - a_3 t + a_2 r^2 - a_1 rt - t^2 + r^3.$$

The comultiplication map is given by:

$$\begin{aligned}\psi(s) &= s\otimes 1 + 1\otimes s\\ \psi(r) &= r\otimes 1 + 1\otimes r\\ \psi(t) &= t\otimes 1 + 1\otimes t + s\otimes r.\end{aligned}$$

This Hopf algebroid classifies plane cubic curves in Weierstrass form; the universal Weierstrass curve is defined over A in affine coordinates (x,y) as

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{3.1}$$

and the universal strict isomorphism classified by Γ is the coordinate change

$$\begin{aligned}x &\mapsto x + r\\ y &\mapsto y + sx + t.\end{aligned}$$

It is easy, and classical, to compute the ring of invariants of this Hopf algebroid:

$$H^{0,*}(A,\Gamma) = \mathbf{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

where the classes c_4, c_6, Δ are given in the universal case, and hence for every Weierstrass curve, by the equations (cf Silverman [12, III.1])

$$\begin{aligned}b_2 &= a_1^2 + 4a_2\\ b_4 &= 2a_4 + a_1a_3\\ b_6 &= a_3^2 + 4a_6\\ c_4 &= b_2^2 - 24b_4\\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6\\ \Delta &= \tfrac{1}{1728}(c_4^3 - c_6^2).\end{aligned}$$

A cubic of the form (**3.1**) is an elliptic curve if and only if Δ, the discriminant, is invertible. An elliptic curve is a 1–dimensional abelian group scheme, and thus the completion at the unit (which is the point at infinity) is a formal group. In fact, this construction even gives a formal group when the discriminant is not invertible because the point at infinity in a Weierstrass curve is always a smooth point. Thus we get a map of Hopf algebroids

$$H\colon (MU_*, MU_*MU) \to (A,\Gamma).$$

and a map from the MU–based Adams–Novikov spectral sequence to the elliptic spectral sequence, which converges to the Hurewicz map $h\colon \mathbf{S}^0 \to \mathtt{tmf}$. This last statement is due to the fact that there is a complex oriented ring spectrum $\underline{A}$ with

$\pi_* \underline{A} = A$; it can be constructed as $\underline{A} = \mathtt{tmf} \wedge Y(4)$, where $Y(4)$ is the Thom spectrum of $\Omega U(4) \to \mathbf{Z} \times BU$ (Rezk [11, Sections 12–14]).

Let $MU_* = \mathbf{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$, and $MU_*MU = MU_*[m_1, m_2, \dots]$ with $|m_i| = 2i$. By direct verification, we may choose the generators x_i, m_i such that

(3.2)
$$\begin{aligned} H(x_i) &= a_i \quad \text{for } i = 1, 2; \\ H(m_1) = s \quad &\text{and} \quad H(m_2) = r. \end{aligned}$$

This very low-dimensional observation enables us to compare the tmf spectral sequence with the usual MU–Adams–Novikov spectral sequence and will be the key to determining the differentials in sections 6 and 8.

4 The homotopy of tmf at primes $p > 3$

As was mentioned in the introduction, tmf becomes complex oriented at large primes, and thus the homotopy ring becomes very simple; it is isomorphic to the ring of classical modular forms, which is generated by the Eisenstein forms E_4 and E_6.

We first state a result valid whenever 2 is invertible in the ground ring.

Let $\widetilde{A} = \mathbf{Z}[\frac{1}{2}, a_2, a_4, a_6]$, $f\colon A[\frac{1}{2}] \to \widetilde{A}$ be the obvious projection map sending a_1 and a_3 to 0, and

$$\widetilde{\Gamma} = \widetilde{A} \otimes_A \Gamma \otimes_A \widetilde{A} = A[\tfrac{1}{2}, r, s, t]/(a_1, a_3, \eta_R(a_1), \eta_R(a_3)) = \widetilde{A}[r].$$

4.1 Lemma *The Hopf algebroids* $(A[\frac{1}{2}], \Gamma[\frac{1}{2}])$ *and* $(\widetilde{A}, \widetilde{\Gamma})$ *are equivalent.*

Proof Define a map $g\colon (\widetilde{A}, \widetilde{\Gamma}) \to (A[\frac{1}{2}], \Gamma[\frac{1}{2}])$ in the opposite direction of f by

$$\begin{aligned} g(a_2) &= a_2 + \tfrac{1}{4} a_1^2, \\ g(a_4) &= a_4 + \tfrac{1}{2} a_1 a_3, \\ g(a_6) &= a_6 + \tfrac{1}{4} a_3^2, \text{ and} \\ g(r) &= r. \end{aligned}$$

Then clearly $f \circ g = \mathrm{id}_{(\widetilde{A}, \widetilde{\Gamma})}$, and if we define $c\colon \Gamma[\frac{1}{2}] \to A[\frac{1}{2}]$ to be A–linear and

$$c(r) = 0; \quad c(s) = -\tfrac{1}{2} a_1; \quad c(t) = -\tfrac{1}{2} a_3,$$

we find that $c \circ \eta_R = g \circ f$. □

4.2 Remark This proof is the formal version of the following algebraic argument: As 2 becomes invertible, we can complete the squares in the Weierstrass equation (**3.1**) by the substitution

$$y \mapsto y - \tfrac{1}{2}a_1x - \tfrac{1}{2}a_3.$$

That is, we can find new parameters x and y such that the elliptic curve is given by

$$y^2 = x^3 + a_2x^2 + a_4x + a_6. \tag{4.3}$$

All the strict isomorphisms of such curves are given by substitutions $x \mapsto x + r$, hence the Hopf algebroid is equivalent to $(\widetilde{A}, \widetilde{\Gamma})$.

When 3 is also invertible in the ground ring, we can in a similar fashion complete the cube on the right hand side of (**4.3**) by sending $x \mapsto x - \frac{1}{3}a_2$ and obtain

4.4 Proposition

(1) *The Hopf algebroid* $\left(A[\tfrac{1}{6}], \Gamma[\tfrac{1}{6}]\right)$ *is equivalent to the discrete Hopf algebroid* $(A', \Gamma' = A')$, *where* $A' = \mathbf{Z}[\tfrac{1}{6}, a_4, a_6]$.

(2) $H^{n,*}(A[\tfrac{1}{6}], \Gamma[\tfrac{1}{6}]) = \begin{cases} \mathbf{Z}[\tfrac{1}{6}, a_4, a_6]; & n = 0 \\ 0; & \textit{otherwise} \end{cases}$

(3) $\pi_*\left(\mathtt{tmf}[\tfrac{1}{6}]\right) = \mathbf{Z}[\tfrac{1}{6}, a_4, a_6]$.

Proof Clearly, (1) implies (2), and the latter implies (3) because the elliptic spectral sequence collapses with E_∞ concentrated on the horizontal axis. □

5 The cohomology of the elliptic curve Hopf algebroid at $p = 3$

As a warm-up for the more difficult case of $p = 2$, we first compute $H^{**}(A_{(3)}, \Gamma_{(3)})$.

Lemma 4.1 gives us an equivalence of $(A_{(3)}, \Gamma_{(3)})$ with $(\widetilde{A}, \widetilde{\Gamma})$, where

$$\widetilde{A} = \mathbf{Z}_{(3)}[a_2, a_4, a_6] \quad \text{and} \quad \widetilde{\Gamma} = \widetilde{A}[r].$$

To simplify the Hopf algebroid further, we map $\widetilde{A}$ to $A' = \mathbf{Z}_{(3)}[a_2, a_4]$ by sending a_6 to 0. Now

$$g\colon \widetilde{A} \xrightarrow{1 \otimes \eta_R} A' \otimes_{\widetilde{A}} \widetilde{\Gamma} = R = A'[r]$$

is defined by

$$\begin{aligned} g(a_2) &= a_2 + 3r \\ g(a_4) &= a_4 + 2a_2 r + 3r^2 \\ g(a_6) &= a_4 r + a_2 r^2 + r^3 \end{aligned}$$

This is a faithfully flat extension since $g(a_6)$ is a monic polynomial in r, and we invoke Theorem 2.4 to conclude that $H^*(A,\Gamma) \cong H^*(A',\Gamma')$ with

$$(A',\Gamma') = \Big(\mathbf{Z}_{(3)}[a_2,a_4], A'[r]/\Big(r^3 + a_2 r^2 + a_4 r\Big)\Big).$$

Let $I_0 = (3)$, $I_1 = (3,a_2)$, and $I_2 = (3,a_2,a_4)$ be the invariant prime ideals of (A',Γ'), and denote by (A_n,Γ_n) the Hopf algebroid $(A',\Gamma')/I_n$ for $n = 0,\ 1,\ 2$.

As an A'–module, Γ' is free with basis $\{1,r,r^2\}$. We first compute the cohomology of $(A_2,\Gamma_2) = (\mathbf{Z}/3, \mathbf{Z}/3[r]/(r^3))$. Since A_2 is a field, Γ_2 is injective as an (A_2,Γ_2)–comodule, and we can take the minimal resolution

$$\mathbf{Z}/3 \xrightarrow{\eta_L} \Gamma_2 \xrightarrow{\frac{\partial}{\partial r}} \Gamma_2 \xrightarrow{\frac{\partial^2}{\partial r^2}} \Gamma_2 \xrightarrow{\frac{\partial}{\partial r}} \Gamma_2 \xrightarrow{\frac{\partial^2}{\partial r^2}} \cdots$$

and get the cohomology algebra

$$H^{**}(A_2,\Gamma_2) = E(\alpha) \otimes P(\beta),$$

where $\alpha \in H^{1,4}(A_2,\Gamma_2)$ and $\beta \in H^{2,12}(A_2,\Gamma_2)$.

In the cobar complex, α is represented by $[r]$ and β by $[r^2|r] - [r|r^2]$. Furthermore, β can be expressed as a Massey product:

$$\beta = \langle \alpha,\alpha,\alpha \rangle. \tag{5.1}$$

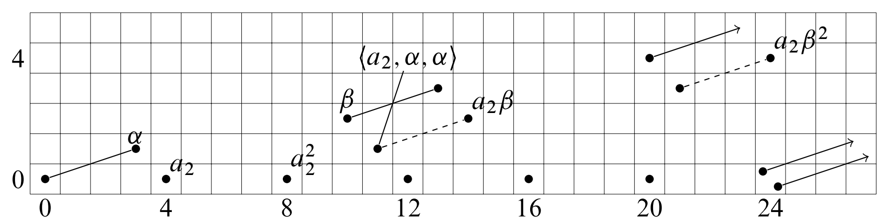
If we consider the algebraic Bockstein spectral sequence associated with the ideal $(a_4) \triangleleft A_1 = \mathbf{Z}/3[a_4] = A'/(3,a_2)$, we see that the polynomial generator corresponding to a_4 supports no differentials and hence

$$H^{**}(A_1,\Gamma_1) = H^{**}(A_2,\Gamma_2) \otimes P(a_4).$$

We now look at the Bockstein spectral sequence for $(a_2) \triangleleft A_0 = \mathbf{Z}/3[a_2, a_4] = A'/(3)$. We find that in the cobar complex,

$$\begin{aligned} a_4 a_2^n &\mapsto 2a_2^{n+1}[r] \\ a_4[r] + a_2[r^2] &\mapsto 0 \qquad\qquad \text{Massey product: } \langle a_2^{n+1}, \alpha, \alpha\rangle \\ a_4^2 a_2^n &\mapsto a_4 a_2^{n+1}[r] \\ a_4^2[r] + a_4 a_2[r^2] &\mapsto a_2^2\left([r^2|r] + [r|r^2]\right) = a_2^2\beta \\ a_4^3 &\mapsto 0. \end{aligned}$$

Hence we get the following picture for the E_∞ term:



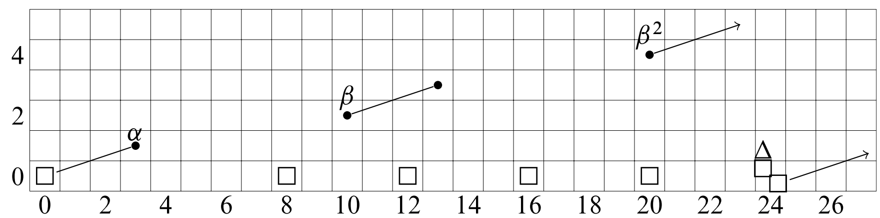
Here, and in all subsequent charts, we use Adams indexing, ie, the square at coordinate (x, y) represents the group $H^{y,x+y}$. We also use the following conventions:

- a dot '•' represents a generator of $\mathbf{Z}/p$ (here, $p = 3$);
- a circle around an element denotes a nontrivial $\mathbf{Z}/p$–extension of the group represented by that element;
- a square '□' denotes a generator of $\mathbf{Z}_{(p)}$;
- a line of slope ∞, 1 or $1/3$ denotes multiplication with p, the generator of $H^{1,2}$ (not present at $p = 3$), or of $H^{1,4}$;
- an arrow with positive slope denotes generically that the pattern is continued in that direction;
- an arrow with negative slope denotes a differential; the arrow, the source, and the target are then drawn in a gray color to denote that these classes do not survive.

The dashed line of slope $1/3$ denotes an exotic multiplicative extensions which follows from the Massey product shuffling lemma [10, Appendix 1] and (**5.1**):

$$\langle a_2, \alpha, \alpha\rangle\alpha = a_2\langle \alpha, \alpha, \alpha\rangle = a_2\beta.$$

Finally, we run the Bockstein spectral sequence to get the 3–local homology. The differentials derive from $d_1(a_2) = 3r$ and yield



where Δ is the discriminant.

6 The differentials

In this section, I compute the differentials in the elliptic spectral sequence whose E_2–term was computed in the previous section. The differential which implies all further differentials in the 3–local `tmf` spectral sequence is

$$d_5(\Delta) = \beta^2\alpha. \tag{6.1}$$

This follows from comparison with the Adams–Novikov spectral sequence: By (**3.2**), the class $[x_2]$ representing the Hopf map $\alpha_1 \in \pi_3(\mathbf{S}^0)$ in the cobar complex of the Adams–Novikov spectral sequence is mapped to α; because of the Massey product $\beta = \langle \alpha, \alpha, \alpha \rangle$, we also conclude that β represents the Hurewicz image of the class $\beta_1 \in \pi_{10}(\mathbf{S}^0)$.

Now in the stable stems, $\beta_1^3\alpha_1$ represents the trivial homotopy class although $\beta^3\alpha$ is nontrivial in the Hopf algebroid cohomology; thus it must be killed by a differential. The only possibility is $\beta^3\alpha = d_5(\Delta\beta)$, which implies the differential (**6.1**).

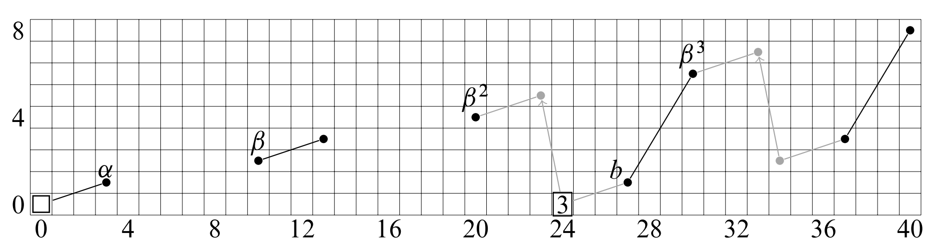


Figure 1: The elliptic spectral sequence at $p = 3$ up to dimension 40

It also shows that there is an element b representing a class in π_{27}`tmf` which is the unique element in the Toda bracket $\langle \beta_1^2, \alpha_1, \alpha_1 \rangle$. The Massey product $\langle \alpha, \alpha, \alpha \rangle$

converges to the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$. The Shuffling Lemma for Toda brackets [10, Appendix 1] implies that there is a multiplicative extension $b\alpha_1 = \beta_1^3$ in π_{27} as displayed in the chart. It implies that there has to be a differential $d_9(\Delta^2\alpha) = 2\beta^5$ since $\beta_1^5 = \beta_1^2 b\alpha_1$.

The complete E_∞ term has a free polynomial generator Δ^3 in degree 72 and is displayed in Figures 1 and 2, where most classes in filtration 0 have been omitted.

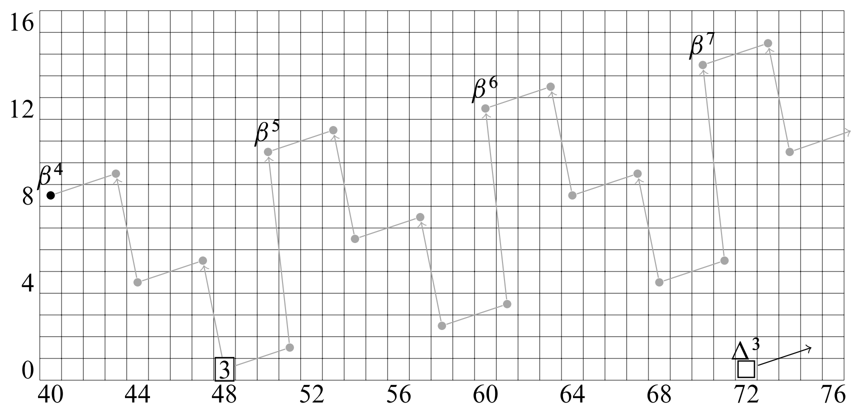


Figure 2: The elliptic spectral sequence at $p = 3$, dimensions 40–76

7 The cohomology of the elliptic curve Hopf algebroid at $p = 2$

In this section we are going to compute the cohomology of (A, Γ), localized at the prime 2.

Note that if 3 is invertible, we can proceed as in Remark 4.2 and complete the cube in x by a transformation $x \mapsto x + r$, thereby eliminating the coefficient a_2. This means that the resulting Hopf algebroid is

$$(\widetilde{A}, \widetilde{\Gamma}) = (\mathbf{Z}_{(2)}[a_1, a_3, a_4, a_6], \widetilde{A}[s, t]),$$

where the map $\Gamma \to \widetilde{\Gamma}$ maps r to $\frac{1}{3}(s^2 + a_1 s)$.

By two applications of Theorem 2.4, we can furthermore eliminate a_4 and a_6 (a_4 maps to a monic polynomial s^4+lower terms in s, a_6 to a monic polynomial t^2+lower terms in t). This introduces new relations in Γ:

$$s^4 - 6st + a_1 s^3 - 3a_1 t - 3a_3 s = 0 \quad \text{and}$$
$$s^6 - 27t^2 + 3a_1 s^5 - 9a_1 s^2 t + 3a_1^2 s^4 - 9a_1^2 st + a_1^3 s^3 - 27a_3 t = 0$$

Hence, at the prime 2, the Hopf algebroid

$$(A', \Gamma') = \big(\mathbf{Z}_{(2)}[a_1, a_3], A'[s,t]/(\text{the above relations})\big)$$

has the same cohomology as (A, Γ).

Note that as an A'–module, Γ' is free with basis $\{1, s, s^2, s^3, t, st, s^2t, s^3t\}$ and we have invariant prime ideals $I_0 = (2)$, $I_1 = (2, a_1)$, and $I_2 = (2, a_1, a_3)$. Define $(A_n, \Gamma_n) = (A', \Gamma')/I_n$ for $n = 0,\ 1,\ 2$.

We are going to compute the cohomology of this Hopf algebroid with a sequence of algebraic Bockstein spectral sequences. As in the 3–primary case, we first compute $H^{**}(A_2, \Gamma_2)$. Observe that this is isomorphic as a Hopf algebra to the double $DA(1)$ of $A(1)$, where $A(1)$ is the dual of the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2.

The cohomology of $DA(1)$ is well-known. I will compute it here nevertheless to determine cobar representatives of some classes.

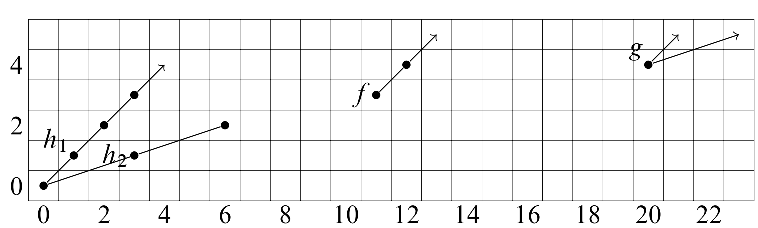
Note that in $DA(1)$, s^2 is primitive, hence we can divide out by it to get a Hopf algebra which is isomorphic to the Hopf algebra $E(s) \otimes E(t)$. The Cartan–Eilenberg spectral sequence associated to the ideal (s^2) coincides with the algebraic Bockstein spectral sequence of Theorem 2.6 in this situation and has

$$E_1 = H^{**}(DA(1)/(s^2)) \otimes P(s^2) \Longrightarrow H^{**}(DA(1)).$$

The cobar complex gives us the following differentials:

$$d_1[s^{2n}t] = [s^{2n+2}|s];$$
$$d_1\Big([t|t] + [s|s^2t] + [st|s^2]\Big) = [s^2|s^2|s^2].$$

No more differentials are possible for dimension reasons, yielding the following E_∞–term:



The elements labeled h_i are represented by s^i; the element f is represented by

$$f = [s|t|t] + [s|s|s^2 t] + [s|st|s^2 + t|s^2|s^2] = \langle h_2^2, h_2, h_1 \rangle,$$

and the class g is a polynomial generator in this algebra. It is represented in the cobar complex by the class $[t|t|t|t]$ plus elements of higher filtration in the above spectral sequence. It can be expressed as a Massey product

(7.1) $$g = \langle h_2^2, h_2, h_2^2, h_2 \rangle.$$

as can easily be seen from the differentials.

We record the following Massey product relation:

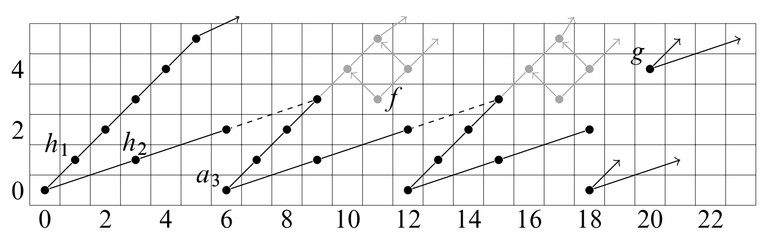
(7.2) $$h_2^2 = \langle h_1, h_2, h_1 \rangle$$

which follows from $d[t] = [s^2|s]$, $d\big([t]+[s^3]\big) = [s|s^2]$, and

$$d[ts] = [t|s] + [s|t+s^3] + [s^2|s^2].$$

We will now start an algebraic Bockstein spectral sequence to compute the cohomology of (A_1, Γ_1) from (A_2, Γ_2):

$$E_1 = H^{**}(DA(1)) \otimes P(a_3) \Longrightarrow H^{**}(A_1, \Gamma_1).$$



The elements h_1 and h_2 lift to cycles in the E_∞ term, but the cobar complex shows that

$$d_1(f) = a_3 h_1^4.$$

We have an extension $a_3 h_1^3 = h_2^3$ because in the cobar complex,

$$[t|t] + [s|s^2 t] + [st|s^2] \mapsto [s^2|s^2|s^2] + a_3[s|s|s],$$

as well as an exotic Massey product extension

(7.3) $$a_3 h_1^2 = \langle h_2, h_1, h_2 \rangle.$$

Indeed, the Massey product has the cobar representative

$$[t|s^2] + [s^2|t+s^3],$$

and

$$d[s^2(t+s^3)] = [t|s^2] + [s^2|t+s^3] + [s^4|s], \text{ and } s^4 = a_3 s \pmod{2, a_1}.$$

No more differentials are possible for dimension reasons.

We thus have computed

$$E_\infty = P(a_3, h_1, h_2, g)/(h_1 h_2, h_2^3 + a_3 h_1^3)$$

or, additively only,

$$E_\infty = P(a_3, g)\{h_2, h_2^2, h_1^i \mid i \geq 0\}/(a_3 h_1^3)$$

It will be necessary to keep track of a Massey product expression for g. Equation **(7.1)** becomes invalidated because $h_2^3 \neq 0$; instead, **(7.1)** lifts to a matric Massey product representation

(7.4) $$g = \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & a_3 h_1^2 \\ a_3 h_1^2 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ a_3 h_1^2 \end{pmatrix} \right\rangle.$$

The next step is to introduce a_1. The d_1–differential is linear over $P(a_3^2, g)$ and satisfies

$$d_1(a_3 h_2^i) = a_1 h_2^{i+1}$$

where h_2 is the lift of the cycle previously called h_2, represented by $[s^2 + a_1 s]$ in the cobar complex.

The resulting homology may be written additively as

$$P(g) \otimes (K \oplus P(a_1) \otimes L \oplus H_1)$$

where

$$K = P(a_3^2)\left\{h_2, h_2^2, h_2^3\right\}$$

is the a_1–torsion,

$$L = P(a_3^2)\left\{h_1^i, x, x h_1 \mid 0 \leq i \leq 3\right\},$$

and

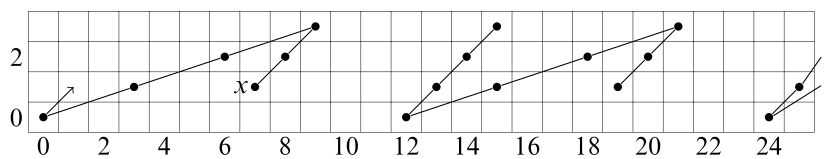
$$H_1 = \mathbf{F}_2\left\{h_1^i \mid i \geq 4\right\}$$

is the initial infinite h_1–tower. Here

(7.5) $$x = \langle a_1, h_2, h_1 \rangle$$

is cobar represented by $[a_3 s + a_1 t]$.

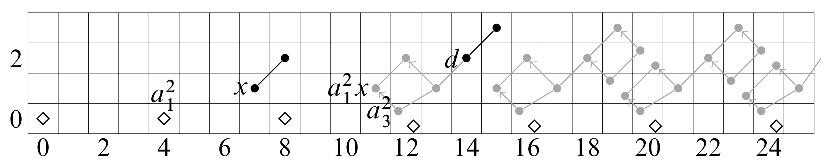
The submodule $K \oplus 1 \otimes L + H_1$ constitutes the filtration 0 part; this is displayed in the following diagram, along with the h_1– and h_2–multiplications:



The d_2–differentials all follow from $d_2(a_3^2) = a_1^2 x$; explicitly,

$$d_2(a_3^{2k} a_1^n) = \begin{cases} a_3^{2k-2} a_1^{n+2} x; & k \text{ odd}; \\ 0; & k \text{ even}. \end{cases}$$

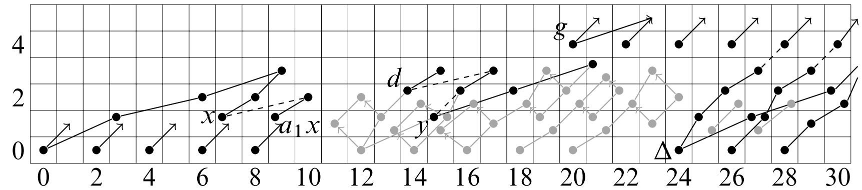
Since d_2 does not involve a_1–torsion and is (g, a_3^4)–linear, it is enough to understand what it does on $L/(a_3^4, g)$. This module decomposes into two d_2–invariant copies $L' \oplus a_1 L'$ of a module L', which is shown in the following chart:



Diamonds denote h_1–strings of $P(h_1)/(h_1^4)$ of length 4. Thus

$$H_*(L', d_2) = P(a_1^2, h_1)/(h_1^4) \oplus \mathbf{F}_2\{x, xh_1, d, dh_1\} \quad \text{where } d = x^2.$$

The next chart displays the full spectral sequence modulo Bockstein filtration 5 (the full chart would become too cluttered). Gray classes that do not seem to be hit by or to support a differential support a differential into higher filtration.



The new indecomposable class y has the Massey product representation:

(7.6) $$y = \langle x, a_1^2, h_2 \rangle = \langle a_1 x, a_1, h_2 \rangle,$$

and the class Δ can be expressed as

(7.7) $\Delta \in \langle a_1 x, a_1, a_1 x, a_1 \rangle$ with indeterminacy $\subseteq a_1 H^{0,22}(A_0, \Gamma_0)$

There are a number of exotic multiplicative extensions, which I will now verify.

7.8 Lemma *The following multiplicative extensions exist in the above algebraic Bockstein spectral sequence:*

(1) $xh_2 = a_1 x h_1$

(2) $dh_2 = a_1 d h_1$

(3) $yh_1 = a_1 d$

(4) $a_1^4 g = \Delta h_1^4$

Proof (1) This follows from the Massey product Shuffling Lemma (cf [10, Appendix 1]), $\langle a, b, c\rangle d = \pm a \langle b, c, d\rangle$:

$$xh_2 \underset{\textbf{(7.5)}}{=} \langle a_1, h_2, h_1\rangle h_2 = a_1 \langle h_2, h_1, h_2\rangle \underset{\textbf{(7.3)}}{=} a_1 x h_1.$$

(2) This follows from (1) by multiplication with x since $d = x^2$.

(3) $yh_1 \underset{\textbf{(7.6)}}{=} \langle x, a_1^2, h_2\rangle h_1 = x\langle a_1^2, h_2, h_1\rangle = a_1 x^2 = a_1 d.$

(4) This is the hardest extension to verify. We first claim:

(7.9) $$g a_1^2 = \langle h_1, dh_1, x\rangle.$$

This implies (4) since then,

$$g a_1^4 = \langle h_1, dh_1, x\rangle a_1^2 = h_1 \langle dh_1, x, a_1^2\rangle.$$

Thus $g a_1^4$ is divisible by h_1, and the only possibility is $g a_1^4 = \Delta h_1^4$.

To prove **(7.9)**, we start from the Massey product expression for g and compute

$$\begin{aligned} g a_1^2 &\underset{\textbf{(7.4)}}{=} \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & xh_1 \\ xh_1 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ xh_1 \end{pmatrix} \right\rangle a_1^2 \\ &\subseteq \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} 0 & a_1 x h_1 \\ a_1 x h_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_1 h_1 \\ a_1 h_1 & 0 \end{pmatrix}, \begin{pmatrix} h_2 \\ xh_1 \end{pmatrix} \right\rangle \\ &= \langle h_1, a_1 x h_1, a_1 h_1, x h_1\rangle + \langle h_2^2, a_1 x h_1, a_1 h_1, h_2\rangle = L + R. \end{aligned}$$

The second summand, R, is not strictly defined in the technical sense that the sub-Massey products have indeterminacy, as

$$\langle a_1 x h_1, a_1 h_1, h_2\rangle = \{0, dh_2\}.$$

Despite this, the four-fold product R has no indeterminacy and contains only 0. To see this, we calculate

$$a_1 R \subseteq \langle a_1, h_2^2, a_1 x h_1, a_1 h_1\rangle h_2,$$

where the Massey product is now strictly defined, showing that $a_1 R$ is divisible by h_2, which implies that it is 0. Multiplication with a_1 in that degree is injective, hence $R = 0$.

As for the first summand, we find that

$$L = \langle h_1, xh_2, h_1, xh_2\rangle \underset{\text{Lemma 7.11}}{=} \langle h_1, \langle xh_2, h_1, h_2\rangle, x\rangle = \langle h_1, dh_1, x\rangle.$$

Equality holds because both left and right hand side have zero indeterminacy. □

7.10 Remark One might be tempted to think that $\Delta h_1^4 = x^4$, but this cannot hold here since $\Delta h_1^8 \neq 0$ whereas $(xh_1)^4 = 0$ because $xh_1^3 = 0$.

For proving 7.8(4), we made use of the following lemma:

7.11 Lemma *Let $a \in M$, b, c, d, $e \in A$ for a differential graded module M over a differential graded $\mathbf{F}_2$–algebra A. Suppose $ab = bc = cd = 0$ and $\langle a, b, c\rangle = 0$. Then*

$$\langle a, \langle b, c, d\rangle, e\rangle \cap \langle a, b, c, d\, e\rangle \neq \varnothing.$$

Proof We adopt the following notation: for a boundary x, we denote by $\underline{x}$ a chosen chain such that $d(\underline{x}) = x$, keeping in mind that it is not unique. Consider the following defining system for $\langle a, b, c, d\, e\rangle$:

$$\begin{array}{ccccccc} a & & b & & c & & d\,e \\ & \underline{ab} & & \underline{bc} & & \underline{cd}\,e & \\ & & \underline{\langle a,b,c\rangle} & & \underline{\langle b,c,d\,e\rangle} & & \end{array}$$

Note that this is not the most general defining system because we insist that the class bounding $c\,d\,e$ actually is a class $\underline{cd}$ bounding $c\,d$, multiplied with e.

On the other hand, a defining system for $\langle a, \langle b, c, d\rangle, e\rangle$ is given by

$$\begin{array}{ccc} a & \underline{bc}d + b\underline{cd} & e \\ & \begin{array}{c}\underline{\langle a,b,c\rangle}d \\ +\underline{ab}\,\underline{cd}\end{array} \ \underline{\langle b,c,d\,e\rangle}. & \end{array}$$

If we compute the representatives of the Massey product for both defining systems, we get in both cases:

$$a\underline{\langle b,c,d\,e\rangle} + \underline{\langle a,b,c\rangle}d\,e + \underline{a\,b}\,\underline{c\,d}\,e. \qquad \square$$

The last step is to run the Bockstein spectral sequence to compute the integral cohomology. We have that

$$d_1(a_1^{2k+1}) = 2a_1^{2k}h_1$$
$$d_2(a_1^2) = 4h_2$$
$$d_4(yh_2^2) = 8g$$

The first two are immediate, and the last one follows once again from a Massey product expression for g in the E_4 term of the BSS.

Setting $c = \langle h_2, h_1, h_2\rangle$ (cf (**7.3**)), we have

$$g = \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & c \\ c & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ c \end{pmatrix} \right\rangle.$$

Note that

$$d_1(x) = d_1\langle a_1, h_2, h_1\rangle = 2\langle h_1, h_2, h_1\rangle \underset{\textbf{(7.2)}}{=} 2h_2^2$$

Thus, when multiplying g by 2 from the left,

$$\left\langle 2, \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 \\ c \end{pmatrix} \right\rangle = 0 \quad \text{and}$$
$$\left\langle 2, \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} c \\ h_2 \end{pmatrix} \right\rangle = \langle 2, h_2^2, c\rangle = dh_1.$$

Hence

(7.12) $$2g = \left\langle dh_1, \begin{pmatrix} h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ c \end{pmatrix} \right\rangle = \langle dh_1, h_1, h_2\rangle.$$

From this it follows that

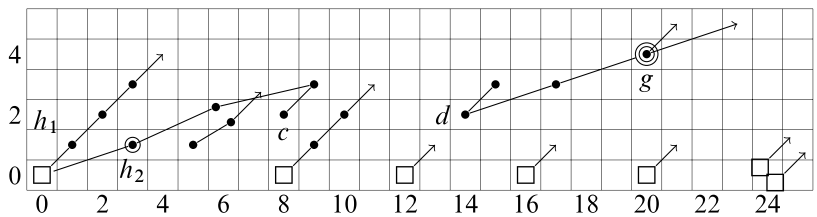
(7.13) $$4g = 2\langle dh_1, h_1, h_2\rangle = \langle 2, dh_1, h_1\rangle h_2 = dh_2^2.$$

Therefore, the differential follows from this exotic h_2 extension and $d_1(y) = 2d$. This in turn follows by multiplication with h_1: $d_1(yh_1) = d_1(a_1 d) = 2dh_1$.

Let us also record the identity

(7.14) $$d = \langle h_2, 2h_2, h_2, 2h_2 \rangle.$$

The cohomology of (A', Γ') is given by the page below:



8 The differentials

In this section, I compute the differentials in the spectral sequence whose E_2–term was computed in the previous section. The first couple of differentials can be derived by comparing this spectral sequence to the Adams–Novikov spectral sequence; all the further differentials follow from Toda bracket relations; in fact, all those differentials are determined from knowing that tmf is an A_∞–ring spectrum.

Under the Hurewicz map H of Section 3, the classes I called h_1 and h_2 in the previous section are the images of the classes of the same name in the ANSS by (**3.2**); in particular, they will represent the Hurewicz images of $\eta \in \pi_1$ and $\nu \in \pi_3$. Furthermore, as the map $\mathbf{S}^0 \to \mathtt{tmf}$ is an A_∞–map, Toda brackets are mapped to Toda brackets, and Massey products in the ANSS are mapped to Massey products in the elliptic spectral sequence. By the computations of Massey products summarized in Appendix A, it follows that the classes c and d are the images of the classes called c_0 and d_0 in the ANSS and represent the homotopy classes $\epsilon \in \pi_8$ and $\kappa \in \pi_{14}$, respectively.

Furthermore, the generator $g \in H^{4,24}(A', \Gamma')$ represents

(8.1) $$\bar{\kappa} = \langle \kappa, 2, \eta, \nu \rangle \in \pi_{20}\mathtt{tmf},$$

the image of the class of the same name in $\pi_{20}\mathbf{S}^0$. This point requires a bit of explanation since the Toda bracket is not reflected by a Massey product in the elliptic spectral sequence, and because the corresponding class in the ANSS has filtration 2 instead of 4. The given Toda bracket is computed by Kochman [7, 5.3.8] for the sphere. Since both in the sphere (by an exotic extension in the Adams–Novikov E_∞–term) and in tmf (by (**7.13**)), we have that $4\bar{\kappa} = \kappa\nu^2$, we have that $\bar{\kappa} \in \pi_{20}\mathtt{tmf}$ must be an odd multiple (mod 8) of $\bar{\kappa} \in \pi_{20}\mathbf{S}^0$. We can alter g by an odd multiple to make sure they agree on the nose.

8.2 d_3–differentials

It is easy to see that there is no room in the spectral sequence for d_2–differentials (indeed, the checkerboard phenomenon implies that there are no d_i for any even i). We want to describe all d_3 differentials.

Since η^4 is zero in π_4^s, h_1^4 has to be hit by a differential at some point. The only possibility for this is

$$d_3(a_1^2 h_1) = h_1^4.$$

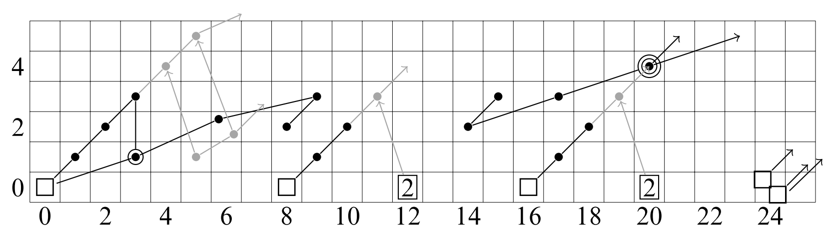
This implies

$$d_3(a_1^{4n+2}) = a_1^{4n} h_1^3.$$

There is no more room in the spectral sequence for any differentials up to dimension 15. We may thus study the multiplicative extensions in $E_5 = E_\infty$ in this range. The Hurewicz map dictates that there must be a multiplicative extension $4h_2 = h_1^3$.

By multiplicativity, there are no more possibilities for a nontrivial d_3.

The following chart shows the d_3 differentials; black classes survive.



8.3 Higher differentials and multiplicative extensions

For the determination of the higher differentials, we will only need to appeal once more to an argument involving the Hurewicz map from the sphere; the remaining differentials and exotic extension all follow algebraically using Massey products and Toda brackets. The crucial differential is

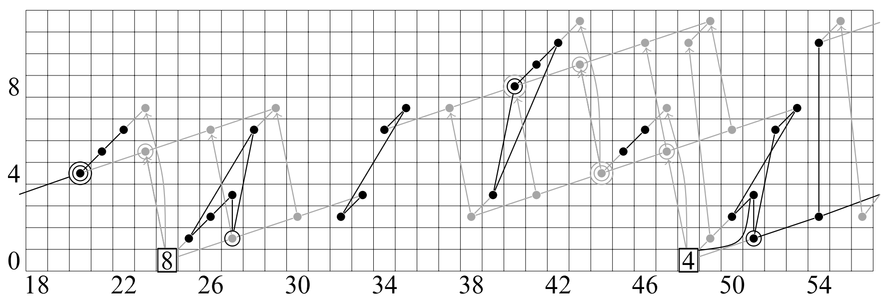
$$d_5(\Delta) = gh_2.$$

There are several ways to see this. Maybe the easiest is to notice that

$$\bar{\kappa}\nu^3 = 0 \in \pi_{29}(\mathbf{S}^0)$$

thus the class gh_2^3 representing its Hurewicz image has to be killed by a differential. There is only one class in $E_5^{30+*,*}$, namely, $\Delta h_2^2 \in E_5^{32,2}$. But if $d_5(\Delta h_2^2) = gh_2^3$, then necessarily $d_5(\Delta) = gh_2$. Because of the multiplicative extension $4\nu = \eta^3$, there also has to be a $d_7(4\Delta) = gh_1^3$.

The following chart displays the resulting d_5 and d_7 differentials up to dimension 52 along with the multiplicative extensions they incur. From here on I omit the classes in filtration ≤ 2 which look like the Adams–Novikov chart of bo; it is a straightforward verification that they can never support a differential from E_5 on.



Since $d_5(\Delta) = gh_2$, we have by multiplicativity that $d_5(\Delta^i) = i\,\Delta^{i-1}gh_2$. Thus the following powers are cycles for d_5: $\{\Delta^{4i}, 2\Delta^{4i+2}, 4\Delta^{2i+1}\}$. Since $d_7(4\Delta) = gh_1^3$, we furthermore have that the following classes survive to E_9:

$$\{\Delta^{8i}, 2\Delta^{8i+4}, 4\Delta^{4i+2}, 8\Delta^{2i+1}\}.$$

It will turn out that these are actually also the minimal multiples of the powers of Δ that survive to E_∞. Thus, Δ^8 is a polynomial generator in $\pi_{192}\mathtt{tmf}$.

I will now verify the extensions displayed in the above chart.

8.4 Proposition *The following exotic multiplicative extensions exist in $\pi_*\mathtt{tmf}$. We denote a class in $\pi_t\mathtt{tmf}$ represented by a cycle $c \in E_5^{t+s,s}$ by $e[s,t]$ if c is the only class in that bidegree that is not divisible by 2.*

(1) $e[25,1] = \langle \bar\kappa, \nu, \eta\rangle$, *thus* $e[25,1]\nu = \bar\kappa\epsilon$
(2) $e[27,1] = \langle \bar\kappa, \nu, 2\nu\rangle$, *thus* $e[27,1]\eta = \bar\kappa\epsilon$
(3) $e[32,2] = \langle \bar\kappa, \nu, \epsilon\rangle$, *thus* $e[32,2]\nu = \bar\kappa\kappa\eta$
(4) $e[39,3] = \kappa e[25,1]$, *thus* $e[39,3]\eta = 2\bar\kappa^2$
(5) $\bar\kappa\eta^2 = \kappa\epsilon$, *thus* $e[39,3]\nu = \bar\kappa^2\eta^2$

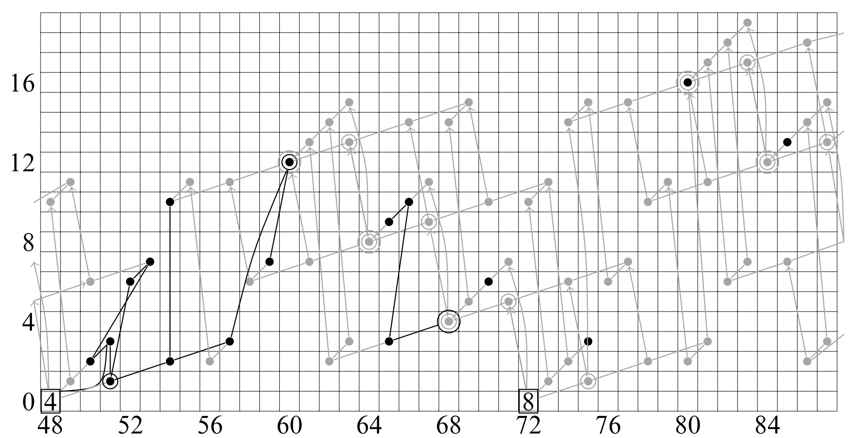
Proof The representations as Toda brackets follow immediately from Massey products in the dga (E_5, d_5). For all multiplicative extensions, use the shuffling lemma for Toda brackets together with the identities from Appendix A.

I will verify the interesting ϵ extension on κ. Since $\kappa\nu^2 = 4\bar{\kappa}$ and $4\nu = \eta^3$, we find that $\kappa\epsilon\eta = \bar{\kappa}\eta^3$. Since the group in this degree is uniquely divisible by η, the result follows. □

This concludes the computation of the E_9 page. For degree reasons, we have that $E_9^{s,t} = E_\infty^{s,t}$ for $t - s \leq 48$; the first possible d_9 differential is

$$d_9(e[49,1]) \stackrel{?}{=} g^2 c$$

This differential is in fact present. In fact, if it were not, we would have an η extension analogous to 8.4(2) from $2e[47,5]$ to $\bar{\kappa}^2\epsilon$; but the representative of the former class is hit by the differential d_5 on Δ^2. This propagates to the following picture of differentials:



I will first explain the induced d_9–differentials. The class $e[54,2]$ is in the Massey product $\langle h_2, e[49,1], h_2\rangle$, and thus

$$d_9(e[56,2]) = d_9(\langle h_2, e[49,1], h_2\rangle) = \langle h_2, g^2c, h_2\rangle = g^2dh_1.$$

Secondly, the class $e[73,1]$ can be written as $\langle g, h_2, e[49,1]\rangle$; thus,

$$d_9(e[73,1]) = \langle g, h_2, g^2c\rangle = e[72,10].$$

The remaining d_9–differentials follow easily from the multiplicative structure.

We now turn to the multiplicative extensions.

8.5 Proposition

(1) $e[50,2] = \left\langle \bar{\kappa}^2, \begin{pmatrix}\epsilon & \nu\end{pmatrix}, \begin{pmatrix}\eta \\ \nu^2\end{pmatrix}\right\rangle$, *thus* $e[50,2]\nu = e[53,7]$.

(2) $e[51,1]\eta = e[52,6]$.

(3) $e[54,2] = \langle \bar\kappa^2, \nu, 2\nu, \nu\rangle \nu$, *thus* $2e[54,2] = e[54,10]$.

(4) $e[57,3] = \langle \bar\kappa^2, \kappa\eta, \eta\rangle$, *thus* $e[57,3]\nu = 2\bar\kappa^3$.

(5) $e[65,3]\eta = e[64,10]$.

Proof

(1) The extension follows from the Toda bracket expression as follows:

$$\left\langle \bar\kappa^2, \begin{pmatrix}\epsilon & \nu\end{pmatrix}, \begin{pmatrix}\eta \\ \nu^2\end{pmatrix}\right\rangle \nu = \langle \bar\kappa^2, \nu, \nu^3\rangle = e[53,7].$$

(2) This is the same as 8.4(2).

(3) This follows from

$$\langle \bar\kappa^2, \nu, 2\nu, \nu\rangle \cdot 2\nu = \bar\kappa^2 \langle \nu, 2\nu, \nu, 2\nu\rangle = \bar\kappa^2\kappa.$$

(4) This follows from

$$2\bar\kappa^3 = \bar\kappa^2\langle \kappa\eta, \eta, \nu\rangle = \langle \bar\kappa^2, \kappa\eta, \eta\rangle \nu.$$

(5) This is the extension (2) multiplied with κ, using 8.4(5). □

The first higher differential occurs in dimension 62, where

$$d_{11}(e[62,2]) = g^3 h_1.$$

It suffices to show that $d_{11}(e[63,3]) = g^3 h_1^2$. The latter class has to be hit by some differential since it is gh_2 times $e[39,3]$ by Proposition 8.4(5), and the claimed d_{11} is the only possibility.

The 2–extension in the 54–stem also forces a

$$d_{11}(e[75,2]) = g^3 d$$

since the target class would have to be $2e[74,6]$, which is a d_5–boundary.

In a similar fashion, the η–extension from 8.4(4) forces

$$d_{13}(e[81,3]) = 2g^4.$$

We proceed to the next chunk, starting in dimension 84 and displayed in Figure 3.

Up to total dimension 119, $E_{13} = E_\infty$ because there is no room for longer differentials, and the remaining d_{11} and d_{13} differentials follow from the ones already computed by multiplicativity. There are some multiplicative extensions which I collect in the following proposition. The crucial part here is to express $\bar\kappa$ as a Toda bracket in a different fashion.

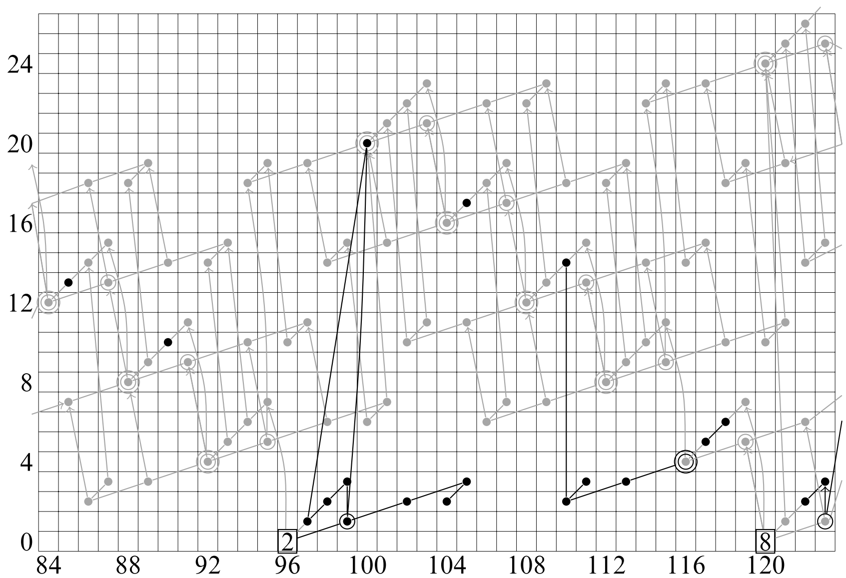


Figure 3: The elliptic spectral sequence — dimensions 84–123

8.6 Lemma *We have*

$$\bar{\kappa} = \langle \nu, 2\nu, \nu, 4\nu, \nu, \eta \rangle = \langle \nu, 2\nu, \nu, 4\nu, \eta, \nu \rangle.$$

Proof We start with the representation $\bar{\kappa} = \langle \kappa, 2, \eta, \nu \rangle$ (cf Appendix A). Then

$$\begin{aligned} \eta\bar{\kappa} = \langle \kappa\eta, 2, \eta, \nu \rangle &= \langle \langle \epsilon, \nu, 2\nu \rangle, 2, \eta, \nu \rangle \\ &= \langle \epsilon, \nu, 4\nu, \eta, \nu \rangle = \langle \langle \eta, \nu, 2\nu \rangle, \nu, 4\nu, \eta, \nu \rangle = \eta \langle \nu, 2\nu, \nu, 4\nu, \eta, \nu \rangle. \end{aligned}$$

Since both classes are uniquely divisible by η, the first part follows. For the second equality, the right hand side can be split up as

$$\langle \langle \nu, 2\nu, \nu, 2\nu \rangle, 2, \eta, \nu \rangle = \langle \kappa, 2, \eta, \nu \rangle. \qquad \square$$

8.7 Corollary

(1) $e[97,1] = \langle \bar{\kappa}^4, \nu, 2\nu, 3\nu, 4\nu, \eta \rangle$*, thus* $e[97,1]\nu = \bar{\kappa}^5$.

(2) $e[99,1] = \langle \bar{\kappa}^4, \nu, 2\nu, 3\nu, 4\nu, \nu \rangle$*, thus* $e[99,1]\eta = \bar{\kappa}^5$.

(3) $2e[110,2] = e[110,14]$

Proof The first two parts are immediate using the above lemma.

The last part requires further explanation. We have

$$e[110,2] = \langle \bar\kappa^4, \nu, 2\nu, \nu, 4\nu, \kappa\rangle$$

and thus

$$\begin{aligned} 2e[110,2] &= \langle \bar\kappa^4, \nu, 2\nu, \langle \nu, 4\nu, \kappa, 2\rangle\rangle = \langle \bar\kappa^4, \nu, 2\nu, \langle 2, \bar\kappa, 2\rangle\rangle \\ &= \langle \bar\kappa^4, \nu, 2\nu, \bar\kappa\eta\rangle = \langle \bar\kappa^5, \nu, 2\nu, \eta^2\rangle = e[110,14]. \end{aligned}$$ □

There is one long differential

$$d_{23}(e[121,1]) = g^6$$

which follows immediately from the η–extension 8.7(2).

Far from running out of steam, we press on to the next 40 dimensions, as displayed in Figure 4.

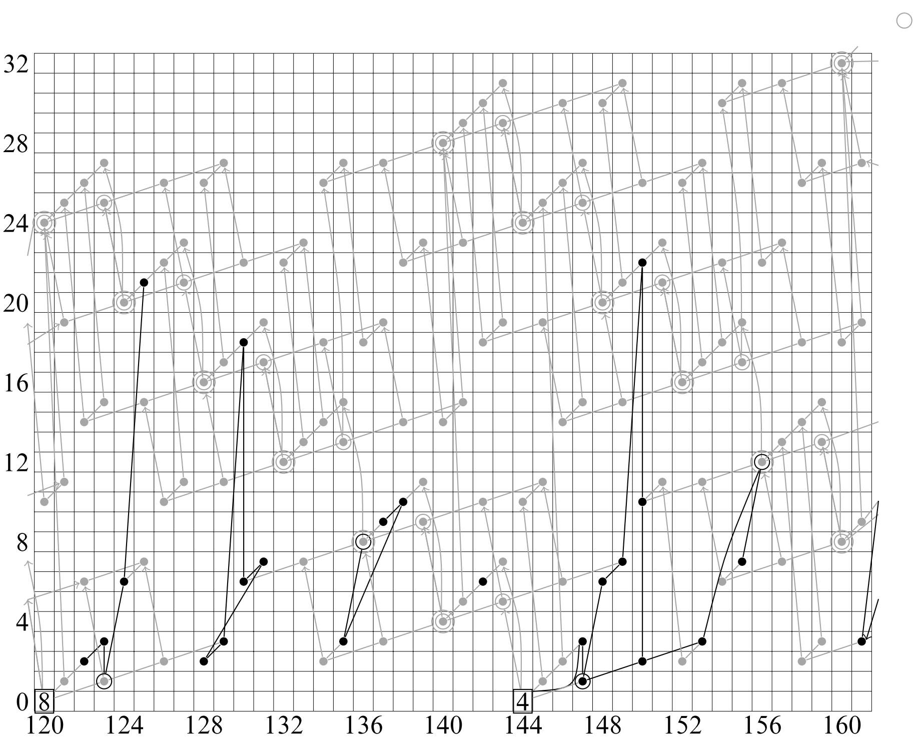


Figure 4: The elliptic spectral sequence — dimensions 120–161

Again, we address the multiplicative extensions indicated in the chart. The extensions that raise filtration by at most nine are exactly the same as in dimensions 24 to 64, with the same arguments. There are a few longer extensions:

8.8 Proposition

(1) $e[124,6]\eta = e[125,21]$

(2) $e[129,3]\eta = e[130,18]$

(3) $e[149,7]\eta = 2e[150,10] = e[150,22]$

Proof

(1) Due to the new d_{23}–differential and the ν–extension 8.4(1), we have that

$$e[124,6] = \langle \bar{\kappa}^5, \bar{\kappa}, \nu \rangle,$$

and multiplication with η followed by Toda shuffling yields the extension to $e[125,21]$.

(2) We have that $e[129,3] = \langle e[124,6], \eta, \nu \rangle$ and $e[130,18] = \langle e[125,21], \eta, \nu \rangle$ (the latter using 8.4(5)), thus this extensions follows from (1).

(3) This follows from (2) by multiplying with $\bar{\kappa}$. □

There is one new differential, namely $d_{23}(e[146,2]) = e[145,25]$ which immediately follows from the η extension 8.8(1).

The rest of the chart, up to the periodicity dimension of 192, is displayed in Figure 5.

Since almost all classes die, there are only extensions that already occured in the respective dimensions lowered by $96 = |\Delta^4|$.

The new differential $d_{23}(e[171,3]) = e[172,26]$ follows from 8.8(3). From here on, all classes die until dimension 192. This implies that Δ^8 is a polynomial generator for the whole spectral sequence as well as in $\pi_*\mathtt{tmf}$.

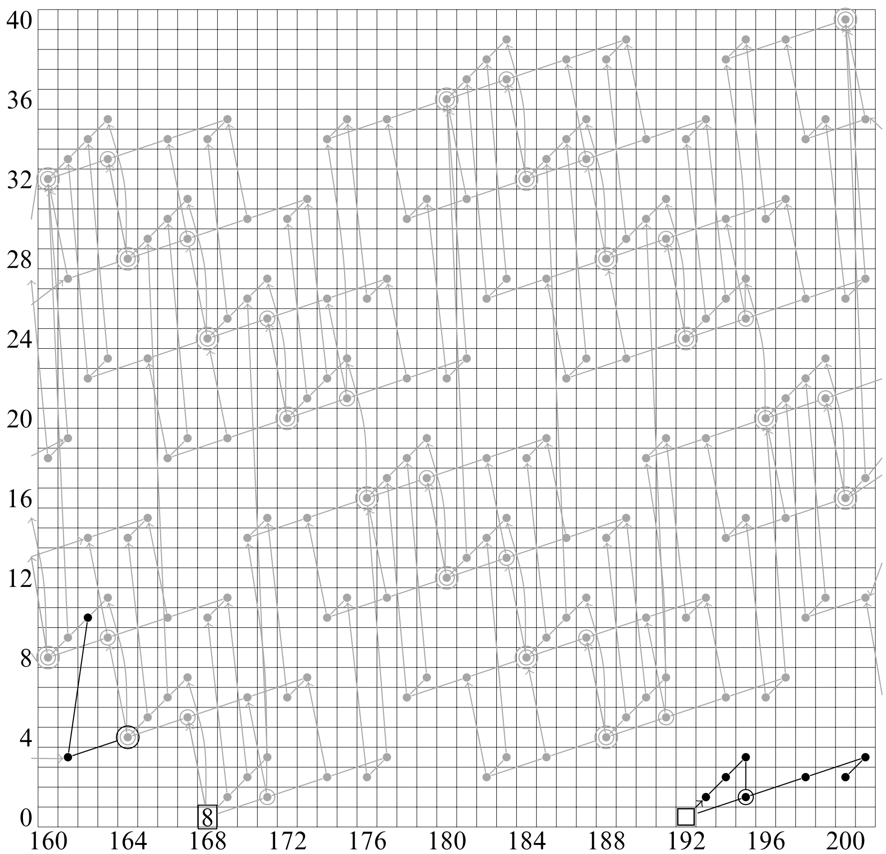


Figure 5: The elliptic spectral sequence — dimensions 160–201

Appendix A Some Toda brackets in the homotopy of tmf

In this appendix, I will assemble some 2–primary Toda bracket relations in tmf for quick reference. These are shown in Table 1. Many of them are proved somewhere in the text, others follow from the relation

$$\langle y, x, y\rangle \cap \langle x, y, 2y\rangle \neq \varnothing \qquad \textbf{(A.1)}$$

valid in any E_∞ ring spectrum for odd-dimensional classes y (Toda [13]).

Dim	Class	E_2 repr.	Toda bracket	Massey product	Ref.
6	ν^2	h_2^2	$\langle \eta, \nu, \eta\rangle$	$\langle h_1, h_2, h_1\rangle$	**(7.2)**
8	ϵ	c	$\langle \nu, \eta, \nu\rangle$	$\langle h_2, h_1, h_2\rangle$	**(7.3)**
			$\langle 2\nu, \nu, \eta\rangle$	$\langle 2h_2, h_2, h_1\rangle$	**(A.1)**
			$\langle \nu, 2\nu, \eta\rangle$	$\langle h_2, 2h_2, h_1\rangle$	juggling
			$\langle 2, \nu^2, \eta\rangle$	$\langle 2, h_2^2, h_1\rangle$	juggling
14	κ	d	$\langle \nu, 2\nu, \nu, 2\nu\rangle$	$\langle h_2, 2h_2, h_2, 2h_2\rangle$	**(7.14)**
15	$\kappa\eta$	dh_1	$\langle \nu, \epsilon, \nu\rangle$	$\langle h_2, c, h_2\rangle$	**(A.2)**, **(A.1)**
			$\langle \epsilon, \nu, 2\nu\rangle$	$\langle c, h_2, 2h_2\rangle$	**(A.2)**
20	$\bar\kappa$	g	$\langle \kappa, 2, \eta, \nu\rangle$	—	**(8.1)**
			$\langle \nu, 2\nu, \nu, 4\nu, \nu, \eta\rangle$	—	Lemma 8.6
			$\langle \nu, 2\nu, \nu, 4\nu, \eta, \nu\rangle$	—	Lemma 8.6
20	$2\bar\kappa$	$2g$	$\langle \kappa\eta, \eta, \nu\rangle$	$\langle dh_1, h_1, h_2\rangle$	**(7.12)**
21	$\bar\kappa\eta$	gh_1	$\langle \kappa, 2, \nu^2\rangle$	—	**(A.3)**

Table 1: Some Toda brackets and Massey products

We verify the remaining relations not explained in the main text:

$$dh_1 = h_1\langle h_2, 2h_2, h_2, 2h_2\rangle = \langle\langle h_1, h_2, 2h_2\rangle, h_2, 2h_2\rangle = \langle c, h_2, 2h_2\rangle \qquad \textbf{(A.2)}$$

$$\bar\kappa\eta = \langle \kappa, 2, \eta, \nu\rangle\eta = \langle \kappa, 2, \langle \eta, \nu, \eta\rangle\rangle = \langle \kappa, 2, \nu^2\rangle \qquad \textbf{(A.3)}$$

References

[1] **J Franke**, *On the construction of elliptic cohomology*, Math. Nachr. 158 (1992) 43–65 MR1235295

[2] **M J Hopkins**, *Topological modular forms, the Witten genus, and the theorem of the cube*, from: “Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)”, Birkhäuser, Basel (1995) 554–565 MR1403956

[3] **M Hopkins**, **M Mahowald**, *From elliptic curves to homotopy theory*, preprint available on the Hopf archive (1998)

[4] **M J Hopkins**, **H R Miller**, *Elliptic curves and stable homotopy I*, Draft (1999)

[5] **M Hovey**, *Morita theory for Hopf algebroids and presheaves of groupoids*, Amer. J. Math. 124 (2002) 1289–1318 MR1939787

[6] **M Hovey**, **H Sadofsky**, *Invertible spectra in the $E(n)$–local stable homotopy category*, J. London Math. Soc. (2) 60 (1999) 284–302 MR1722151

[7] **S O Kochman**, *Stable homotopy groups of spheres, a computer-assisted approach*, Lecture Notes in Mathematics 1423, Springer, Berlin (1990) MR1052407

[8] **P S Landweber**, Editor, *Elliptic curves and modular forms in algebraic topology (Princeton 1986)*, Lecture Notes in Mathematics 1326, Springer, Berlin (1988) MR970278

[9] **P S Landweber**, **D C Ravenel**, **R E Stong**, *Periodic cohomology theories defined by elliptic curves*, from: "The Čech centennial (Boston, 1993)", Contemp. Math. 181, Amer. Math. Soc., Providence, RI (1995) 317–337 MR1320998

[10] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic Press, Orlando, FL (1986) MR860042

[11] **C Rezk**, *Supplementary notes for math 512 (v. 0.17)* (2002) Available at `http://www.math.uiuc.edu/~rezk/512-spr2001-notes.pdf`

[12] **J H Silverman**, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics 106, Springer, New York (1986) MR817210

[13] **H Toda**, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies 49, Princeton University Press (1962) MR0143217

Mathematisches Institut der Universität Münster
Einsteinstr. 62, 48149 Münster, Germany

`tbauer@uni-muenster.de`

`http://wwwmath.uni-muenster.de/u/tbauer/`

Received: 30 September 2005 Revised: 1 September 2006