AN INFINITE LOOP SPACE STRUCTURE ON THE NERVE OF SPIN BORDISM CATEGORIES

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Abstract

In this paper, we exhibit an infinite loop space structure on the nerve of certain spin bordism 2-categories and compare it with the classifying space of suitably stabilized spin mapping class groups. We show that the stable spin mapping class group has the homology of an infinite loop space. In order to do this, we adapt Harer's homology stabilization results for spin mapping class groups to a setting compatible with the methods Tillmann used to prove that the classifying space of (non-spin) mapping class groups has the homology of an infinite loop space.

We also study a variant of the spin mapping class groups, due to Masbaum, and show that its homology also stabilizes as the genus tends to infinity.

1. Introduction

Let F be a connected, compact, oriented surface of genus g with n boundary circles. The mapping class group $\Gamma(F) = \Gamma_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving self-diffeomorphisms of F, fixing the boundary pointwise.

A spin structure $\mathfrak{s}$ on F is by definition a choice of a square root of the tangent complex line bundle of F. This can be given by either

(a) a homomorphism $\sigma\colon \pi_1(\mathbf{S}F) \longrightarrow \mathbb{Z}/2$ which is non-zero on the fibres of the tangent sphere bundle $\mathbf{S}F$ of F, or

(b) a quadratic form Q on $H_1(F;\mathbb{Z}/2)$ whose associated bilinear form is the skew symmetric intersection form of F [**1**, **9**].

DEFINITION For a spin surface $(F, \mathfrak{s})$, the spin mapping class group $G_{\mathfrak{s}}(F) < \Gamma(F)$ is the subgroup of all mapping classes $f\colon F \longrightarrow F$ such that $f^*\mathfrak{s} = \mathfrak{s}$.

In [**7**], Harer studied the groups $G_{\mathfrak{s}}(F)$ and showed that the inclusion of stabilizers of certain arc systems on a spin surface into the full group induces a homology isomorphism in degrees which are less than approximately one-third of the genus of F.

The aim of this paper is to carry over the following result of Ulrike Tillmann [**17**] to the world of spin surfaces and spin mapping class groups.

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THEOREM 1 (Tillmann) *Let Γ_∞ be the direct limit of the groups $\Gamma_{g,1}$ formed by iterated attachment of a torus with two boundary components. Then $\mathbb{Z} \times B\Gamma_\infty^+$ (and hence also $B\Gamma_\infty^+$) is an infinite loop space, where $(-)^+$ denotes Quillen's plus-construction.*

In exciting new work [**11**], Madsen and Weiss are able to identify the homotopy type of this infinite loop space as $\Omega^\infty \mathbb{C}P^\infty_{-1}$, the 0-space of the Thom spectrum of the negative of the universal bundle over $\mathbb{C}P^\infty$.

To prove Theorem 1, Tillmann constructs a 2-category whose objects are compact closed 1-manifolds, whose morphisms are bordisms, and whose 2-morphisms are mapping classes between diffeomorphic surfaces. She then compares the nerve of this category with $\mathbb{Z} \times B\Gamma_\infty^+$.

In section 5, a bordism 2-category $\mathcal{S}$ of spin surfaces and spin mapping class groups similar to the ones of [**16–18**] is constructed and studied. The following theorem allows us to define a stabilized spin mapping class group G_∞ by iterated attachment of spin surfaces, and shows that the choice of spin structure becomes immaterial when passing to the limit.

THEOREM 2 *Let $(F_1, \mathfrak{s}_1)$ and $(F_2, \mathfrak{s}_2)$ be two spin surfaces with a common set of boundaries S on which the two spin structures agree. Let F denote the union $F_1 \cup_S F_2$, and let $\mathfrak{s}$ be a spin structure on F which restricts to $\mathfrak{s}_i$ on F_i (such an $\mathfrak{s}$ always exists but need not be unique). Then the inclusion $F_1 \hookrightarrow F$ induces an isomorphism $H_k(G_{\mathfrak{s}_1}(F_1)) \cong H_k(G_{\mathfrak{s}}(F))$ as long as* $\operatorname{genus}(F_1) \geqslant 4k + 7$.

(This formulation follows from the slightly stronger Theorem 10 and Corollary 11 in section 3.)

Tillmann's proof of Theorem 1, using the methods of categorical group completion of [**14**, **17**], now carries over to the spin setting.

MAIN THEOREM *There is a homology equivalence*

$$\mathbb{Z} \times \mathbb{Z}/2 \times BG_\infty \longrightarrow \Omega(\mathcal{N}\mathcal{S}),$$

where $\Omega(\mathcal{N}\mathcal{S})$ is the loops on the nerve of the spin bordism category $\mathcal{S}$. In particular, the homology localization $L_H BG_\infty$ of BG_∞ is an infinite loop space.

Theorem 2 is proved in section 3. The proof is built upon Harer's stabilization results for inclusions of stabilizers of certain arc systems in surfaces; the translation of this setting to the current context requires some care and is done in section 2.

In [**12**], Masbaum considers a different but related kind of spin mapping class groups of *closed* surfaces. In geometric terms, they can be defined as follows. Let F be any spin surface. An (ordinary) spin diffeomorphism of F can then be regarded as an endomorphism of F in a $2+1$-bordism category of spin manifolds, namely its mapping cylinder. Composition in the category corresponds to the group multiplication. It is possible to extend the spin structure to this 3-manifold, but not uniquely; there are always two choices. A spin mapping class in the sense of Masbaum is defined to be an element of the ordinary spin mapping class group together with a choice of extension of the spin structure to the cylinder; these can certainly be composed and therefore yield a group which is a (non-trivial) extension $\tilde{G}_{\mathfrak{s}}$ of $G_{\mathfrak{s}}(F)$ by $\mathbb{Z}/2$. In this formulation, it is not entirely clear how this definition could extend to surfaces with boundaries, but in [**12**], Masbaum gives an equivalent definition which does generalize.

In section 4, we show that the homology of Masbaum's groups also stabilizes when the genus tends to infinity.

THEOREM 3 *With the same notation as in Theorem* 2, *the inclusion* $F_1 \hookrightarrow F$ *induces an isomorphism* $H_k(\tilde{G}_{\mathfrak{s}_1}(F_1)) \cong H_k(\tilde{G}_{\mathfrak{s}}(F))$ *for* $\mathrm{genus}(F_1) \geqslant 2k^2 + 6k - 2$ *in the case where* F_2 *has a boundary component that is not involved in gluing, and for* $\mathrm{genus}(F_1) \geqslant 2k^2 + 6k + 3$ *otherwise.*

An immediate corollary of this stabilization result is the analogue of the main theorem in Masbaum's setting (Corollary 15).

It would be most interesting to identify the occurring infinite loop spaces with zero-spaces of some Thom spectra, analogous to the unspun identification of $\mathbb{Z} \times B\Gamma_\infty^+$ with $\Omega^\infty \mathbb{C}P_{-1}^\infty$ in [**11**]. In fact, there is a map of infinite loop spaces $\alpha_\infty \colon \mathbb{Z} \times \mathbb{Z}/2 \times L_H BG_\infty \longrightarrow \Omega^\infty(\mathbb{C}P^\infty)^{-L^2}$, where L is the tautological line bundle over $\mathbb{C}P^\infty$, and L^2 denotes the tensor square of L. In the somewhat speculative section 6, we define this map and give some evidence that it might be a homotopy equivalence.

This paper was part of my Oxford M.Sc. thesis [**2**] and my Bonn Diplomarbeit [**3**]. I hesitated for a few years to make this more publicly available, but in light of recent results of Madsen and Weiss and the question posed above it may now be of more interest.

2. Surfaces and mapping class groups

Harer's approach to studying the stabilization of the homology of spin mapping class groups is not to consider gluing operations but inclusions of arc systems and their stabilizers. The goal of this slightly technical section is to prove that these two notions are in a certain sense equivalent.

DEFINITION [**7**] A *simple arc* in a surface F is an immersed interval with endpoints which is an embedding away from the endpoints. An *arc system* on a surface F is a finite collection of such simple arcs $\{\gamma_i\}$ in F such that

(1) two arcs intersect at most in their endpoints;

(2) all endpoints lie in ∂F;

(3) in every component of ∂F, there is at most one intersection point with the whole arc system;

(4) no arc is trivial, that is, homotopic to a point by a homotopy fixing the endpoints, and no two arcs are isotopic to each other.

PROPOSITION 4 *Let* F_1, F_2 *be two surfaces with a common set of boundaries* S, *and let* F *denote the union* $F_1 \cup_S F_2$. *Assume that every component of* F_2 *contains a component of* ∂F. *Then there exists an arc system* $\{\gamma_j\} \subseteq F$ *and a commutative diagram*

$$\begin{array}{ccc} \Gamma(F_1) & \xrightarrow{\ \sim\ } & \mathrm{Stab}_{\Gamma(F)}\{\gamma_j\} \\ & \searrow \quad \swarrow & \\ & \Gamma(F) & \end{array}$$

where the diagonal arrows are the obvious inclusions.

Moreover, if $F_1 = F_{11} \cup_{S_1} F_{12}$ *is a further decomposition as above, and* $\{\delta_j\}$ *is the arc system* *associated to* $F_{11} \hookrightarrow F$*, then* $\{\gamma_j\}$ *can be chosen to be a sub-arc system of* $\{\delta_j\}$*, making*

$$\begin{array}{ccc} \Gamma(F_{11}) & \xrightarrow{\sim} & \mathrm{Stab}_{\Gamma(F)}\{\delta_j\} \\ \downarrow & & \downarrow \\ \Gamma(F_1) & \xrightarrow{\sim} & \mathrm{Stab}_{\Gamma(F)}\{\gamma_j\} \end{array}$$

commute.

This means that on the level of mapping class groups, the inclusion of a surface into a bigger one can always be realized by the inclusion of the stabilizer of a suitable arc system into the full group.

Consider maps of the following kind between surfaces.

DEFINITION Let F, F' be surfaces, $f\colon F \longrightarrow F'$. Call f a *weak embedding* if the restriction to the interior of F is an embedding. An *isotopy* of weak embeddings is a map $f\colon \mathbb{I} \times F \longrightarrow F'$ where each f_t is a weak embedding.

For any subset $X \subseteq F$, the *relative mapping class group* is $\Gamma(F; X) = \pi_0 \operatorname{Diff}^+(F; X \cup \partial F)$.

REMARK When X is a neighbourhood retract of some open neighbourhood $V \supseteq X$, let $(F - X)\check{}$ denote the 'closure' $F - V$. If this is again a surface then we have: $\Gamma(F - X)\check{} \cong \Gamma(F; X)$.

LEMMA 5 *A weak embedding* $f : F \longrightarrow F'$ *induces a map*

$$f_*\colon \Gamma(F) \longrightarrow \Gamma(F')$$

making Γ *a functor from the category of surfaces and isotopy classes of weak embeddings into the category of groups.*

The proof is straightforward.

DEFINITION Let $X \subseteq F$ be a subset of a surface F. Define $\mathrm{Stab}_{\Gamma(F)} X < \Gamma(F)$ to be the components of $\operatorname{Diff}^+(F; \partial F)$ that intersect $\mathrm{Stab}_{\operatorname{Diff}^+(F;\partial F)} X$ non-trivially:

$$\mathrm{Stab}_{\Gamma(F)} X = \frac{\operatorname{Diff}_0^+(F; \partial F) \cdot \mathrm{Stab}_{\operatorname{Diff}^+(F;\partial F)} X}{\operatorname{Diff}_0^+(F; \partial F)}.$$

Here, Diff_0^+ is the component of the identity in Diff^+.

The groups $\mathrm{Stab}_{\Gamma(F)} X$ and $\Gamma(F; X)$ are in general distinct, but there is a canonical surjection

$$\Gamma(F; X) \twoheadrightarrow \mathrm{Stab}_{\Gamma(F)} X. \tag{6}$$

We will now establish a link between stabilizers of certain sets and mapping class groups of weakly embedded surfaces. Let us first restrict to *surjective* weak embeddings.

LEMMA 7 *If* $f\colon F \twoheadrightarrow F'$ *is a surjective weak embedding then*

$$f_*\colon \Gamma(F) \longrightarrow \mathrm{Stab}_{\Gamma(F')} f(\partial F)$$

is also surjective.

Proof. We have an isomorphism

$$\begin{aligned}\mathrm{Diff}^+(F', \partial F' \cup f(\partial F)) &\cong \mathrm{Diff}^+((F' - f(\partial F))\check{}, \partial)\\ &\underset{\text{f surj}}{\cong} \mathrm{Diff}^+((F - \partial F)\check{}, \partial) = \mathrm{Diff}^+(F).\end{aligned}$$

In the same way, we get an isomorphism between the 1-components. So we get

$$\Gamma(F) \cong \Gamma(F',\ f(\partial F)) \twoheadrightarrow \mathrm{Stab}_{\Gamma(F')}\, f(\partial F) \quad \text{(by (6))}$$

It is now natural to ask under which conditions an embedding $f: F \hookrightarrow F'$ induces an inclusion of the mapping class groups.

The inclusion of an annulus into a disc certainly cannot induce an inclusion of mapping class groups since the mapping class group of the latter is trivial whereas the one of the former is not. This is in a certain sense the only counterexample.

PROPOSITION 8 *Let F_1, F_2 be two surfaces, $F = F_1 \cup_\alpha F_2$ for some diffeomorphism $\alpha: \partial_{\text{in}} F_1 \longrightarrow \partial_{\text{in}} F_2$, where $\emptyset \neq \partial_{\text{in}} F_i \subseteq \partial F_i$ is a subset of boundary components. Let $\partial_{\text{out}} F_i = \partial F - \partial_{\text{in}} F_i$. Denote by f_i the inclusion of F_i into F. Assume furthermore that*

(a) *F is connected, and*

(b) *each component of F_2 contains at least one component of $\partial_{\text{out}} F_2$ (hence we exclude the above example).*

Then $f_{1*}: \Gamma(F_1) \longrightarrow \Gamma(F)$ is injective.

Proof. By induction on the components and a decomposition of F_2, we can assume that F_2 is a pair of pants such that $\partial_{\text{in}} F_2$ has either one or two components.

In the first case, when sewing along one component, the statement is trivial: if $g: F \longrightarrow F_1$ is the map which identifies one of the two remaining boundary components of P to a point, then $g \circ f_1 \simeq \mathrm{id}_{F_1}$, showing that f_{1*} is injective.

The second case is slightly more difficult. Proposition 9 below shows that we can isotop f_1 into a surjective weak embedding g_1 such that $A := g_1(\partial F_1) - \partial F$ is a single arc, indeed we can assume a closed curve, and we know that $\Gamma(F_1) = \Gamma(F; A)$.

Now consider the fibration

$$\mathrm{Diff}^+(F; A) \hookrightarrow \mathrm{Diff}^+(F) \twoheadrightarrow \frac{\mathrm{Diff}^+(F)}{\mathrm{Diff}^+(F; A)}.$$

Let $\mathfrak{I}(F)$ be the space of all embeddings of circles with a fixed endpoint p (we take $p = A \cap \partial F'$). Let $a: (\mathbb{S}^1, *) \hookrightarrow (F, p)$ be a basepoint for this space with $\mathrm{im}(a) = A$. Consider the map

$$\begin{aligned}\mathrm{Diff}^+(F) &\to \mathfrak{I}(F)\\ \phi &\mapsto \phi \circ a.\end{aligned}$$

The kernel of this map is $\mathrm{Diff}^+(F; A)$. Therefore, we have an embedding

$$\frac{\mathrm{Diff}^+(F)}{\mathrm{Diff}^+(F; A)} \hookrightarrow \mathfrak{I}(F).$$

This embedding is extremely well behaved in the sense that if a point of $\mathfrak{I}(F)$ is in the image then the whole component is. To see this, take two embeddings in the same component—without loss of generality a and another one, a'. Then they are linked by a path in $\mathfrak{I}(F)$, that is, an isotopy of embeddings. Any such isotopy can be extended to the ambient surface (cf. [**5**, Theorem 4.1]), so the preimage of a' we are looking for comes with a path in $\mathrm{Diff}^+(F)$ connecting it with id_F.

This tells us that $\pi_1\left(\dfrac{\mathrm{Diff}^+(F)}{\mathrm{Diff}^+(F;\,A)}\right) = \pi_1(\mathfrak{I}(F))$, and it is known (for example, [**6**]) that the latter group is trivial.

Hence the last bit of the exact fibre sequence looks like

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_1(\mathfrak{I}(F)) & \longrightarrow & \pi_0(\mathrm{Diff}^+(F;\,A)) & \longrightarrow & \pi_0(\mathrm{Diff}^+(F)) & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ & 1 & & \Gamma(F,A) & & \Gamma(F). & \end{array}$$

Therefore the map

$$\Gamma(F_1) \cong \Gamma(F;\,A) \longrightarrow \mathrm{Stab}_{\Gamma(F)}(A) \subseteq \Gamma(F)$$

is injective.

The following result links the notion of stabilizers with mapping classes of subsurfaces and shows that we did not treat too special a case in Lemma 7.

PROPOSITION 9 *Assume a situation as in Proposition* 8*. Then* f_1 *is isotopic to a* surjective *weak embedding* $g_1\colon F_1 \twoheadrightarrow F$*, and the following diagram commutes, where* $A_1 = g_1(\partial F_1) - \partial F$*.*

$$\begin{array}{ccc} \Gamma(F_1) & \xrightarrow{g_{1*}} & \mathrm{Stab}_{\Gamma(F)}\,A_1 \\ & {\scriptstyle f_{1*}}\searrow \quad \swarrow{\scriptstyle \text{incl.}} & \\ & \Gamma(F) & \end{array}$$

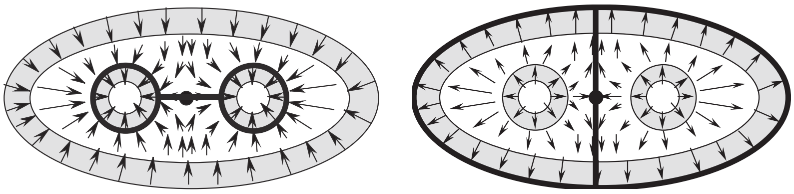
Moreover, if $F_1 = F_{11} \cup_{S_1} F_{12}$ *is decomposed in the same fashion, and* f_{11} *denotes the inclusion* $F_{11} \hookrightarrow F$*, then the associated surjective weak embeddings* g_1, g_{11} *can be chosen such that* $A_1 \subseteq A_{11} = g_{11}(\partial F_{11}) - \partial F$ *and*

$$\begin{array}{ccc} \Gamma(F_{11}) & \longrightarrow & \mathrm{Stab}_{\Gamma(F)}\,A_{11} \\ \downarrow & & \downarrow \\ \Gamma(F_1) & \longrightarrow & \mathrm{Stab}_{\Gamma(F)}\,A_1 \end{array}$$

commutes.

Proof. As in Proposition 8, we can assume that F_2 is a pair of pants because we can decompose F_2 in such.

We construct a flow $(\phi_t)_{0\leqslant t\leqslant 1}$ on F_2 such that $\phi_0(x) = x$ and $\partial_{\mathrm{out}} F_2 \subseteq \phi_1(\partial_{\mathrm{in}} F_2)$. This flow induces an isotopy between the inclusion $f_1\colon F_1 \hookrightarrow F$ and a surjective map in the following way.



(a) glued along one component (b) glued along two components

Fig. 1 The flow ϕ on a pair of pants

Let $\coprod_{j=1}^{k} \mathbb{S}^1 \times \mathbb{I} \hookrightarrow F_1$ be a collar around $\partial_{\text{in}} F_1$. Extend the flow ϕ on F_2 to this collar by defining on each component of the collar:

$$\begin{array}{ccc} \mathbb{I} \times \mathbb{S}^1 \times \mathbb{I} & \longrightarrow & F \\ (t, (\theta, s)) & \mapsto & \begin{cases} \phi_{s+t-1}(\theta) & \text{if } s + t > 1, \\ (\theta, s + t) & \text{if } s + t \leqslant 1. \end{cases} \end{array}$$

Extend it by the identity further on the whole of F.

The flow on a pair of pants is shown in Fig. 1, where thick lines in these pictures denote the image of the inner boundary of F_1 at the end.

The resulting isotopy agrees with $f_1 \colon F_1 \hookrightarrow F$ at $t = 0$ and with $g_1 \colon F_1 \twoheadrightarrow F$ at $t = 1$. Therefore $f_{1*} = g_{1*}$ and the commutativity of the first diagram is shown. For the second diagram, note that since $\partial_{\text{out}} F_2 \subseteq g_1(\partial_{\text{in}} F_1)$, this procedure guarantees that $A_1 \subseteq A_{11}$, and the commutativity is immediate.

The set of pictures in Fig. 2 illustrates the above procedure. They show the surface F_2, composed of three pairs of pants, and the image of the inner boundary of F_1 at each step (thick lines).

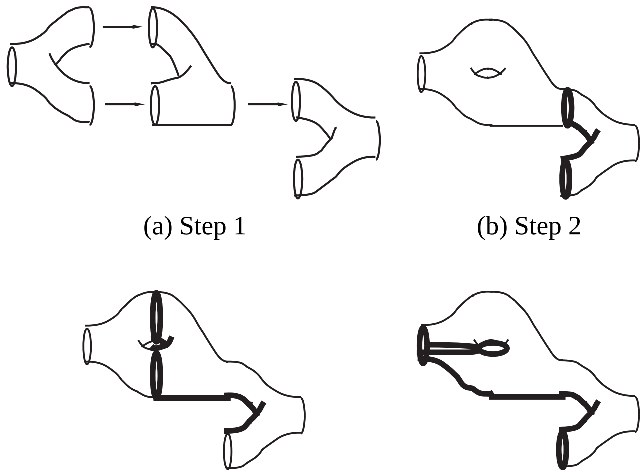
SUPPLEMENT *The construction in Proposition* 9 *reveals that the set A_1 is a connected graph on F which contains every outer boundary component of F_2. If ϕ is a diffeomorphism that fixes A_1 pointwise then we can find a small neighbourhood U_1 of A_1 in F such that ϕ can be deformed isotopically into a diffeomorphism ϕ' which fixes all of U_1 pointwise. Having done this, we can deform A_1 into an arc system $\{\gamma_j\}$ with endpoints in ∂F_2, and this deformation can be chosen to be the identity outside U_1. Then we have*

(1) $\operatorname{Stab}_{\Gamma(F)} A_1 = \operatorname{Stab}_{\Gamma(F)}\{\gamma_j\}$,

(2) *every γ_j is a simple arc with endpoints in ∂F_2,*

(3) *$F - \{\gamma_j\}$ is still connected.*

This deformation may be constructed as follows. Choose a spanning forest F for the graph A_1 with roots r_j in $\partial_{\text{out}} F_2$. Since F is homotopy equivalent to the discrete set $\{r_j\}$, we may choose a homotopy $F \times \mathbb{I} \to F$ from the identity to the projection to the roots. If we choose this homotopy



(a) Step 1 (b) Step 2

(c) Step 3 (d) Step 4

Fig. 2 Four steps in the construction of the flow ϕ

such that it is a diffeomorphism for every (a, t), $t < 1$, we can extend it to a homotopy of F that leaves $F - U_1$ fixed for any chosen open neighbourhood U_1 of A_1 at all times, and is a diffeomorphism for $t < 1$.

Lemma 7, Proposition 8 and Proposition 9 together imply Proposition 4.

3. Homology stabilization for spin mapping class groups

Let F be an oriented surface (with boundary or without). As noted in the Introduction, Atiyah [**1**] and Johnson [**9**] showed that spin structures $\mathfrak{s}$ on F correspond bijectively to quadratic forms Q on $H_1(F; \mathbb{Z}/2)$ satisfying

$$Q(x + y) = Q(x) + Q(y) + \langle x, y\rangle,$$

where $\langle \cdot, \cdot\rangle$ is the intersection form on $H_1(F; \mathbb{Z}/2)$.

REMARK We do not require that F is closed. Indeed, Q can take arbitrary values on the homology classes of the boundary components, subject only to the necessary condition that $Q(\partial_1 + \cdots + \partial_r) = 0$ if $\{\partial_i\}$ are all r boundary components of F (since $\partial_1 + \cdots + \partial_r \sim 0$).

We can now reformulate Harer's stabilization theorem in the following way.

THEOREM 10 *If* $(F_1, \mathfrak{s}_1)$ *is a connected embedded sub-spin-surface of the connected spin surface* $(F, \mathfrak{s})$ *such that every component of* $F - \mathring{F}_1$ *contains at least one boundary component of* F*, then the inclusion* $f_1\colon (F_1, \mathfrak{s}_1) \hookrightarrow (F, \mathfrak{s})$ *induces an isomorphism*

$$f_*\colon H_k(G_{\mathfrak{s}_1}(F_1)) \xrightarrow{\sim} H_k(G_{\mathfrak{s}}(F)) \text{ \textit{for} } \mathrm{genus}(F_1) \geqslant 4k + 2.$$

Proof. In [**7**, Theorem 3.1], Harer showed that if we have an arc system γ in a surface F with

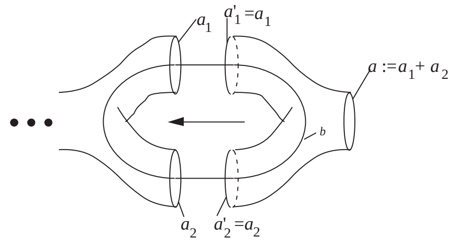


Fig. 3 Attaching a spin pair of pants

exactly one boundary component and we add another loop to obtain a system γ' then the inclusion $\mathrm{Stab}_{G_{\mathfrak{s}}(F)}\,\gamma' \hookrightarrow \mathrm{Stab}_{G_{\mathfrak{s}}(F)}\,\gamma$ gives an isomorphism

$$i\colon H_k(\mathrm{Stab}_{G_{\mathfrak{s}}(F)}\,\gamma') \xrightarrow{\sim} H_k(\mathrm{Stab}_{G_{\mathfrak{s}}(F)}\,\gamma)$$

for genus$(F-\gamma) \geqslant 4k+2$. So, if $\gamma'-\gamma$ contains more than one arc, we can apply this theorem repeatedly and we get as a sufficient condition for the induced map being an isomorphism in the kth homology: genus$(F-\gamma') \geqslant 4k+2$.

Let F have genus g and r boundary components. Then F can be included into the surface $\overline{F} = F_{g+r-1,1}$ of genus $g+r-1$ and with only one boundary component by repeated attachment of a pair of pants to a pair of boundary components of F. We can give $\overline{F}$ a spin structure such that this inclusion is an embedding of spin surfaces. To see this, represent $\mathfrak{s}$ by a quadratic form Q and take two boundary components b_1F and b_2F of F with $Q(b_iF) = a_i$. When attaching a pair of pants, we have to define two new Q-values in a compatible way: the Q-value a of the new boundary $b\overline{F}$ has to be $a_1 + a_2$ (mod 2) because b_1 is homologous to $b + b_2$, and the Q-value of the created new longitude β that transverses b_1F and b_2F can be chosen arbitrarily (Fig. 3).

By Proposition 4, there are arc systems $\gamma \subseteq \gamma_1$ such that $\Gamma(F_1) \xrightarrow{\sim} \mathrm{Stab}_{\Gamma(\overline{F})}\,\gamma_1$ and $\Gamma(F) \xrightarrow{\sim} \mathrm{Stab}_{\Gamma(\overline{F})}\,\gamma$. Of course, these isomorphisms still hold if we intersect everything with the stabilizer of the quadratic form. Thus we have a commutative diagram

$$\begin{array}{ccc} G_{\mathfrak{s}_1}(F_1) & \xrightarrow{\sim} & \mathrm{Stab}_{G_{\overline{\mathfrak{s}}}(\overline{F})}\,\gamma_1 \\ \downarrow f & & \downarrow \text{incl.} \\ G_{\mathfrak{s}}(F) & \xrightarrow{\sim} & \mathrm{Stab}_{G_{\overline{\mathfrak{s}}}(\overline{F})}\,\gamma \end{array}$$

which yields the stability of the kth homology for genus$(\overline{F}-\gamma) = $ genus$(F_1) \geqslant 4k+2$.

Harer also showed in [**7**] that the attachment of a disc to a surface with exactly one boundary induces an isomorphism in H_k for $g \geqslant 4k+7$. This is also true for surfaces with initially more than one boundary component.

COROLLARY 11 *Let* $(F,\mathfrak{s})$ *be a connected spin surface with associated quadratic form* Q, ∂ *any boundary component of* F *with* $Q(\partial) = 0$, *and* D *a* 2*-disc. Then the attachment of* D *to* F *along* ∂

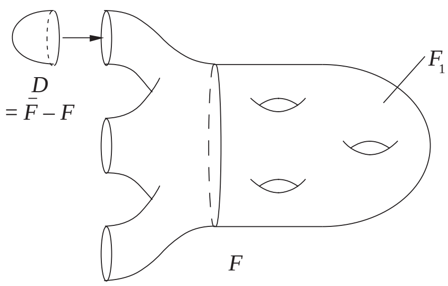


Fig. 4 Attaching a disc to a spin surface

induces a homomorphism of spin mapping class groups and an isomorphism in their kth homology if genus$(F) \geqslant 4k+7$.

Proof. The proof for one boundary component was done in [**7**]. So let F have at least two boundary components, $\overline{F}$ be the surface with a cap attached to one of the boundary components, and F_1 be a subsurface of F with only one boundary component but of the same genus (Fig. 4).

Then the inclusions are compatible:

$$\begin{array}{ccc} F_1 & \longrightarrow & F \\ & \searrow & \Big\downarrow i \\ & & \overline{F} \end{array}$$

where i is the map induced by attachment of the disc. Theorem 10 applies to the other two inclusions, and so, $i_*\colon H_k(G(F)) \longrightarrow H_k(G(\overline{F}))$ is an isomorphism for genus $\geqslant 4k+2$. This result, although better than that stated above, is of course only true in the case where $\overline{F}$ is not closed.

4. Masbaum's extended spin mapping class groups

Recall Masbaum's construction [**12**] of his extended spin mapping class groups. A spin structure on an oriented surface F can be represented by a homomorphism

$$\sigma\colon \pi_1(\mathbf{S}F) \longrightarrow \mathbb{Z}/2$$

which does not vanish on the fibres of the tangent sphere bundle $\mathbf{S}F$ of F. Let $\Gamma^1(F)$ be the group of all mapping classes of F that keep a chosen tangent direction (p_0, v_0) fixed. If F has a boundary or genus at least 2, then $\mathrm{Diff}^+(F)$ is homotopy discrete [**4**], hence there is a short exact sequence

$$1 \longrightarrow \pi_1(\mathbf{S}F) \xrightarrow{\Psi} \Gamma^1(F) \longrightarrow \Gamma(F) \longrightarrow 1,$$

where the map Ψ can be described geometrically as follows (cf. [**3**]).

Given a differentiable simple closed curve c in $\mathbf{S}F$ with $Tc(0) = Tc(1) = (p_0, v_0)$, which represents a homotopy class $\alpha \in \pi_1(\mathbf{S}F)$. Let c^+, c^- be boundary curves of a cylindrical neighbourhood of c in F, where c^+ is 'on the right' of c, defined by means of the orientation of F. Then $\Psi(\alpha)$ is defined to be $\mathrm{Tw}_{c^-} \circ \mathrm{Tw}_{c^+}^{-1}$, the product of the Dehn twists. Any homotopy class in $\pi_1(\mathbf{S}F)$ can be represented as a product of such α, and so Ψ is defined by multiplicativity. It is not hard to see that this map is well defined and fits into the sequence above.

Now consider $K_\sigma =_{\mathrm{def}} \Psi(\ker \sigma) < \Gamma^1$. Although $K_\sigma \lhd \operatorname{im} \Psi \lhd \Gamma^1$, K_σ is not normal in Γ^1; instead the following holds.

LEMMA 12 [**12**] *Let $\gamma \in \Gamma^1$ and σ be a spin structure as above. Then*

$$\gamma^{-1} K_\sigma \gamma = K_{\gamma^* \sigma}.$$

Hence $N(K_\sigma)$, the normalizer inside Γ^1, equals

$$G_\sigma^1 =_{\mathrm{def}} \left\{ f \in \Gamma^1 \;\middle|\; f^* \sigma = \sigma \right\}.$$

DEFINITION Masbaum's spin mapping class group of $(F, \mathfrak{s})$, where $\mathfrak{s}$ is defined by a homomorphism $\sigma\colon \pi_1(\mathbf{S}F) \longrightarrow \mathbb{Z}/2$, is

$$\tilde{G}_{\mathfrak{s}}(F) = G_\sigma^1 / K_\sigma.$$

Note that one does not need F to be closed, but if F is closed then its genus must be at least 2.

REMARK In this definition of $\tilde{G}_{\mathfrak{s}}(F)$, the group seems to depend on the particular choice of a base point. Change of base points along a path c certainly induces an isomorphism, but moreover, this isomorphism is independent of c: take a closed curve $c \in \pi_1(\mathbf{S}F, (p_0, v_0))$; without loss of generality we may assume it is embedded. The map induced by changing the basepoint along c is conjugation with $\Psi(c)$. But this map becomes trivial in $\tilde{G}_{\mathfrak{s}}(F)$. This is clear if $\sigma(c) = 0$ since then $\Psi(c) \in K_\sigma$. But if $\sigma(c) = 1$ then $[K : K_\sigma] = 2$ implies that G_σ^1 also conjugates the other component $K' = K - K_\sigma$ into itself. Hence $\Psi(c)\gamma\Psi(c)^{-1}\gamma^{-1} \in K'\gamma K'\gamma^{-1} = (K')^2 = K_\sigma$.

The definition of the (ordinary) spin mapping class groups can be extended to the *spin mapping class groupoid*, whose objects are all oriented compact spin surfaces with a parametrization of the boundary, and whose morphisms $(F, \mathfrak{s}) \to (F', \mathfrak{s}')$ are the isotopy classes of diffeomorphisms that pull the parametrization of $\partial F'$ back to the parametrization of ∂F, and $\mathfrak{s}'$ to $\mathfrak{s}$. We will denote this set by $G_{\mathfrak{s},\mathfrak{s}'}(F, F')$. It has a free and transitive right $G_{\mathfrak{s}}(F)$ and left $G_{\mathfrak{s}'}(F')$ action.

For Masbaum's groups, this generalization is not quite as immediate. Assume that F, F' are two oriented surfaces with fixed tangent directions (p_0, v_0) and (p_0', v_0'). The set $\Gamma^1(F, F')$ of isotopy classes of diffeomorphisms that map (p_0, v_0) to (p_0', v_0') has similar actions of $\Gamma^1(F)$ and $\Gamma^1(F')$. If the genus of F is at least 2, or F has a boundary, then there is a short exact sequence

$$1 \longrightarrow \pi_1(\mathbf{S}F') \stackrel{\Psi}{\longrightarrow} \Gamma^1(F, F') \longrightarrow \Gamma(F, F') \longrightarrow 1$$

in the sense that the group $\pi_1(\mathbf{S}F')$ acts freely on the middle term with quotient $\Gamma(F, F')$.

DEFINITION Let $(F, \mathfrak{s})$, $(F', \mathfrak{s}')$ be spin surfaces as above. Define

$$\tilde{G}_{\mathfrak{s},\mathfrak{s}'}(F, F') = \{ f \in \Gamma^1 \mid f^* \sigma' = \sigma \} / (\ker \sigma'),$$

where $\ker \sigma'$ acts by Ψ. It is clear that this defines a groupoid $\tilde{G}$ of extended spin mapping classes.

To prove Theorem 3, let F_1, F_2 be surfaces with a common set of boundaries S, $F = F_1 \cup_S F_2$, and let $\mathfrak{s}$ be a spin structure on F which restricts to $\mathfrak{s}_i$ on F_i. Let σ, σ_1 and σ_2 be the representing homomorphisms. Suppose $\mathbf{S}F$ has a base point which actually lies in the interior of $\mathbf{S}F_1$. The inclusion $F_1 \hookrightarrow F$ induces a map of short exact sequences.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbf{S}F_1) & \longrightarrow & \Gamma^1(F_1) & \longrightarrow & \Gamma(F_1) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(\mathbf{S}F) & \longrightarrow & \Gamma^1(F) & \longrightarrow & \Gamma(F) & \longrightarrow & 1 \end{array}$$

Since the spin structure on F restricts to the one on F_1,

$$\begin{array}{ccccc} K_{\sigma_1} & \longrightarrow & G^1_{\sigma_1}(F_1) & \longrightarrow & G_{\mathfrak{s}_1}(F_1) \\ \downarrow & & \downarrow & & \downarrow \\ K_\sigma & \longrightarrow & G^1_\sigma(F) & \longrightarrow & G_{\mathfrak{s}}(F) \end{array}$$

commutes, and we have a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \tilde{G}_{\mathfrak{s}_1}(F_1) & \longrightarrow & G_{\mathfrak{s}_1}(F_1) & \longrightarrow & 1 \\ & & \| & & \downarrow \beta & & \downarrow \gamma & & \\ 1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \tilde{G}_{\mathfrak{s}}(F) & \longrightarrow & G_{\mathfrak{s}}(F) & \longrightarrow & 1. \end{array} \tag{13}$$

Proof of Theorem 3. Let g be the genus of F_1. Naturality of the Hochschild–Serre spectral sequence for (13) gives us a morphism γ_* of spectral sequences

$$E^2_{*,*} = H_*(G_{\mathfrak{s}_1}(F_1); H_*(\mathbb{Z}/2)) \Longrightarrow H_*(\tilde{G}_{\mathfrak{s}_1}(F_1))$$

and

$$E'^2_{*,*} = H_*(G_{\mathfrak{s}}(F); H_*(\mathbb{Z}/2)) \Longrightarrow H_*(\tilde{G}_{\mathfrak{s}}(F)).$$

Let $l = 2$ if F_2 has a boundary component which stays boundary in F, and $l = 7$ otherwise. Then γ is an isomorphism for $g \geqslant 4s + l$ by Theorems 10 and 2. We will now show by induction that

- $\gamma^r_{s,t}$ is an isomorphism for $g \geqslant 4s + 2r(r-1) + l - 4$.

The choice of l guarantees the validity for $r = 2$. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d^r_{s,t}) & \longrightarrow & E^r_{s,t} & \xrightarrow{d} & \operatorname{im}(d) \subseteq E^r_{s+r,t-r+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma^r_{s,t}} & & \downarrow{\scriptstyle \gamma^r_{s,t}} & & \downarrow{\scriptstyle \gamma^r_{s+r,t-r+1}} & & \\ 0 & \longrightarrow & \ker(d'^r_{s,t}) & \longrightarrow & E'^r_{s,t} & \xrightarrow{d'} & \operatorname{im}(d') \subseteq E'^r_{s+r,t-r+1} & \longrightarrow & 0 \end{array}$$

The vertical arrows are isomorphisms if

$$\gamma^r_{s,t}\colon E^r_{s,t} \longrightarrow E'^r_{s,t} \quad \text{and} \quad \gamma^r_{s+r,t-r+1}\colon E^r_{s+r,t-r+1} \longrightarrow E'^r_{s+r,t-r+1}$$

are isomorphisms. Now γ^{r+1} fits into the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d^r_{s,t}) & \longrightarrow & \mathrm{im}(d^r_{s+r,t-r+1}) & \longrightarrow & E^{r+1}_{s,t} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma^r} & & \downarrow{\scriptstyle \gamma^r} & & \downarrow{\scriptstyle \gamma^{r+1}} & & \\ 0 & \longrightarrow & \ker({d'}^r_{s,t}) & \longrightarrow & \mathrm{im}({d'}^r_{s+r,t-r+1}) & \longrightarrow & {E'}^{r+1}_{s,t} & \longrightarrow & 0. \end{array}$$

So, by induction, it is isomorphic if

1. $g \geqslant 4s + 4r(r-1) + l - 4$, and
2. $g \geqslant 4(s+r) + 2r(r-1) + l - 4 = 4s + 2(r+1)r + l - 4$.

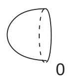
These inequalities are satisfied by hypothesis.

Since these spectral sequences are concentrated in the first quadrant, we have $E^\infty_{s,t} = E^{s+t+2}_{s,t}$ and $\gamma^\infty_{s,t} = \gamma^{s+t+2}_{s,t}$. So, $\gamma^2_{s,t}, \dots, \gamma^{s+t+1}_{s,t}$ are all isomorphic if $g \geqslant 4s + 2(s+t+1)(s+t) + l - 4$, and the latter is smaller than $2(s+t)^2 + 6(s+t) + l - 4$.

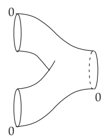
5. A category of 2-spin bordisms

In this section, we will construct a strictly symmetric monoidal 2-category $\mathcal{S}$ whose objects are representatives of closed, compact 1-manifolds, in other words $\mathbb{N}_0$, whose morphisms are compact spin surfaces that bound the source and target 1-manifolds, and whose 2-morphisms are spin mapping classes of these surfaces. This category is closely related to the one Tillmann considers in [**18**].

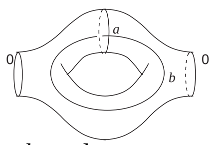
Rather informally, construct a category $\mathcal{S}^{(1)}$ as follows. For details, consult [**18**] or [**16**] or [**3**]. Take the following surfaces:

(1) a disc $D =$ 

with its unique spin structure as a morphism $0 \to 1$,

(2) a pair of pants $P =$ 

with the spin structure whose associated quadratic form Q evaluates to 0 on all boundaries as a morphism $2 \to 1$,

(3) a torus $T =$ 

as a morphism $1 \to 1$, such that Q evaluates to 0 everywhere except on $a + b$, and

(4) the same torus T, but such that $Q(a) = Q(b) = 1$.

Then, adjoin to this every spin surface that can be obtained by either disjoint union or gluing (= composition). Moreover, add labels to the boundary components of the surfaces and formally add to the morphisms an operation 'relabelling' of the symmetric group. As in [**18**], this gives a symmetric monoidal category $\mathcal{S}^{(1)}$ (with the operation $\sqcup$).

REMARK Because of the particular choice of spin structures on the tori (5), (5), every possible spin surface has a representative in the category $\mathcal{S}^{(1)}$. The reason for this is that the isomorphism type of genus g spin surfaces (F, Q) with n boundary components on each of which Q evaluates to 0 is given by the Arf invariant

$$\operatorname{Arf}(Q) = \sum_i Q(a_i)Q(b_i) \in \mathbb{Z}/2,$$

where (a_i, b_i) is a symplectic basis of $H_1(F, \partial F)$. Torus (5) has Arf invariant zero, torus (5) one. The Arf invariant of any spin surface with boundary can be flipped by attaching an Arf one torus.

We now extend the above bordism category to a strictly symmetric monoidal 2-category by adding 2-morphisms between two isomorphic spin surfaces F, F'. We construct the ordinary spin bordism category $\mathcal{S}$ by setting $\operatorname{Hom}_{\mathcal{S}}((F, \mathfrak{s}), (F', \mathfrak{s}')) = G_{\mathfrak{s},\mathfrak{s}'}(F, F')$, the extended spin bordism category $\tilde{\mathcal{S}}$ by $\operatorname{Hom}_{\tilde{\mathcal{S}}}(F, F') = \tilde{G}_{\mathfrak{s},\mathfrak{s}'}(F, F')$. Note that $\mathcal{S}^{(1)}$ does not contain closed surfaces as 1-morphisms. It is therefore not a problem that Masbaum's $\tilde{G}$ groups are undefined for closed surfaces of genus at most 1.

Now let $\mathcal{S}_1$ be the 1-nerve of $\mathcal{S}$, that is, it is the category enriched over simplicial sets that has the same objects as $\mathcal{S}$, but the morphisms between m and $n \in \operatorname{ob}(\mathcal{S}_1) = \mathbb{N}_0$ are the nerve of the category $\mathcal{S}(m, n)$. Now $\mathcal{S}$ contains many morphisms $m \to n$, according to different genera, spin structures, number of components and also different constructions of isomorphic surfaces. Since there are relatively few 2-morphisms, the morphism space $\mathcal{S}_1(m, n)$ splits into many components. However, one does not change the homotopy type of $\mathcal{S}_1(m, n)$ if one replaces it by the classifying space of a skeleton of $\mathcal{S}(m, n)$. The space $\mathcal{S}_1(m, 1)$ is rather simple because a bordism from m circles to 1 circle is automatically connected:

$$\mathcal{S}_1(m, 1) \simeq \coprod_{g \in \mathbb{N}_0, \epsilon \in \mathbb{Z}/2} BG_{\mathfrak{s}(\epsilon)}(F_{g,m+1}),$$

where $\mathfrak{s}(\epsilon)$ is any chosen spin structure of Arf invariant ϵ on $F_{g,m+1}$.

$\mathcal{S}_1$ inherits the symmetric monoidal structure of $\mathcal{S}$. So, if we apply the nerve functor once again, we get a bisimplicial set $\mathcal{N}\mathcal{S}$ which is an E_∞ space. Since there is a morphism from 0 to any n (take n disjoint discs), $\mathcal{N}\mathcal{S}$ is connected. Of course, all of this applies to $\tilde{\mathcal{S}}$ as well, and the results from [**13**, **15**] imply the following.

COROLLARY 14 *$\mathcal{N}\mathcal{S}$ and $\mathcal{N}\tilde{\mathcal{S}}$ are infinite loop spaces.*

We need to retrieve information on the original groups $G_{\mathfrak{s}}$ and $\tilde{G}_{\mathfrak{s}}$ out of these spaces. Let $F_0 \subset F_1 \subset F_2 \subset \cdots$ be a sequence of spin surfaces with one boundary component such that F_i is obtained from F_{i-1} by attaching a torus of type (5). Define the stable spin mapping class groups $G_\infty = \varinjlim_g G_{\mathfrak{s}}(F_g)$ and $\tilde{G}_\infty = \varinjlim_g \tilde{G}_{\mathfrak{s}}(F_g)$. Although the choice of torus (5) seems non-canonical, it becomes immaterial in the homology of the direct limit. Indeed, let $F'_0 \subset F'_1 \cdots$ be another system of spin surfaces, where F'_i is obtained from F'_{i-1} by attachment of a torus with arbitrary spin structure. For every F_g, choose an attachment of a torus $i_g\colon F_g \longrightarrow \overline{F}_{g+1}$ such that there is an isomorphism $\Phi\colon \overline{F}_{g+1} \longrightarrow F'_{g+1}$; similarly, for every F'_g, choose an attachment of a torus $i'_g\colon F'_g \longrightarrow \overline{F}'_{g+1}$ such that there is an isomorphism $\Psi\colon \overline{F}'_{g+1} \longrightarrow F_{g+1}$. These assemble to a diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_g & \longrightarrow & F_{g+1} & \longrightarrow & \cdots \\ & & {\scriptstyle \Phi\circ i_g} & \times & {\scriptstyle \Psi\circ i'_g} & & \\ \cdots & \longrightarrow & F'_g & \longrightarrow & F'_{g+1} & \longrightarrow & \cdots \end{array}$$

which does not necessarily commute; however, $\Psi \circ i'_{g+1} \circ \Phi \circ i_g$ is the inclusion $F_g \to F_{g+2}$, followed by a spin automorphism of F_{g+2}. Thus after applying the functor G to the diagram, it becomes commutative up to inner automorphisms, and thus the diagonal maps induce a homology isomorphism in the direct limit.

The proof of the main theorem is now the same as Tillmann's [**17**, section 3] for the case of no spin structures, with only one exception: whereas the ordinary mapping class groups Γ are perfect, the spin mapping class groups are not. Thus Quillen's plus construction (which is defined for a perfect subgroup) must be replaced by the more general Bousfield localization with respect to integral homology. It is also worth noting that the group of components of $\Omega\mathcal{NS}$ is now $\mathbb{Z} \times \mathbb{Z}/2$, corresponding to genus and Arf invariant.

Thus the proof of the main theorem is finished, as is the following version for Masbaum's groups.

COROLLARY 15 *There is a homology equivalence*

$$\mathbb{Z} \times \mathbb{Z}/2 \times B\tilde{G}_\infty \to \Omega(\mathcal{N}\tilde{\mathcal{S}}).$$

Therefore, the homology localization of $B\tilde{G}_\infty$ *is an infinite loop space.*

6. An α_∞ map

Let L denote the universal complex line bundle over $\mathbb{C}P^\infty$, and for any virtual vector bundle V, let $(\mathbb{C}P^\infty)^V$ denote its Thom spectrum. In this section, a map $\alpha_\infty \colon \mathbb{Z} \times \mathbb{Z}/2 \times BG_\infty \longrightarrow \Omega^\infty(\mathbb{C}P^\infty)^{-L\otimes L}$ is constructed. We do not know whether this map is a homology equivalence (similar to the map of the same name in [**11**]), but we give some positive indication. The constructed map makes the following diagram of infinite loop spaces commute.

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z}/2 \times L_H BG_\infty & \longrightarrow & \Omega^\infty(\mathbb{C}P^\infty)^{-L^2} \\ \downarrow & & \downarrow \\ \mathbb{Z} \times L_H B\Gamma_\infty & \longrightarrow & \Omega^\infty(\mathbb{C}P^\infty)^{-L} \end{array}$$

Here L_H denotes homology localization. The bottom map is defined in [**10**, **11**] as follows. Since for all smooth oriented surfaces F of genus at least 2, $B\,\mathrm{Diff}(F) \simeq B\Gamma(F)$, there is a universal fibration with fibre F

$$F \to E \to B\Gamma(F),$$

where, explicitly, $E \simeq E\,\mathrm{Diff}^+(F) \times_{\mathrm{Diff}^+(F)} F$.

The tangent bundle along the fibres is classified by a map $t : E \to \mathbb{C}P^\infty$. The stable Umkehr map $\Sigma^\infty B\Gamma(F)_+ \to E^\nu$ to the Thom spectrum of the stable normal bundle along the fibres composes with the Thomification of t to give a map of spectra $\alpha_F : \Sigma^\infty B\Gamma(F)_+ \to (\mathbb{C}P^\infty)^{-L}$. These assemble to give a map of E_∞-spaces

$$\coprod_F B\Gamma(F) \to \Omega^\infty(\mathbb{C}P^\infty)^{-L},$$

where F runs through representatives of the isomorphism classes of certain surfaces. Since the target is an infinite loop space, this map lifts to the group completion of the source,

$$\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \to \Omega^\infty(\mathbb{C}P^\infty)^{-L},$$

and is a homotopy equivalence by [**11**].

If $(F, \mathfrak{s})$ is a spin surface and we replace $\Gamma(F)$ by $G_\mathfrak{s}(F)$, the spin structure specifies a lift of the map t to the source of the map $\mathbb{C}P^\infty \xrightarrow{2} \mathbb{C}P^\infty$ classifying the square of the universal line bundle. Thus α_F lifts to the Thom space of the inverse of that bundle, $(\mathbb{C}P^\infty)^{-L^2}$, and so does α_∞.

By counting spin structures of Arf invariant 0 and boundary value 0 on a surface $F = F_{g,n}$, one can easily see that $G_\mathfrak{s}(F) < \Gamma(F)$ is an index $2^{g-1}(2^g + 1)$ subgroup. Since for every prime p there are infinitely many g such that $p \nmid 2^g + 1$, the transfer implies that $i_*\colon H_*(G_\infty; \mathbb{Z}[\frac{1}{2}]) \longrightarrow H_*(\Gamma_\infty; \mathbb{Z}[\frac{1}{2}])$ is a split injection. It is not clear whether or not i_* is an equivalence away from 2; however, the map $\Omega^\infty(\mathbb{C}P^\infty)^{-L^2} \to \Omega^\infty(\mathbb{C}P^\infty)^{-L}$ is.

As further evidence supporting that α_∞ might be an equivalence, an Adams spectral sequence shows that $\pi_0(\Omega^\infty(\mathbb{C}P^\infty)^{-L^2}) = \mathbb{Z} \oplus \mathbb{Z}/2$ and $\pi_1(\Omega^\infty(\mathbb{C}P^\infty)^{-L^2}) = \mathbb{Z}/4$; and of course, $\pi_2(\Omega^\infty(\mathbb{C}P^\infty)^{-L^2}; \mathbb{Q}) = \mathbb{Q}$. This agrees with the homotopy of $\mathbb{Z} \times \mathbb{Z}/2 \times L_H BG_\infty$ [**8**]; in fact, this is all the concrete knowledge about the homology of BG_∞ (or homotopy of its homology localization) at the time of this writing.

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