ADJOINT SPACES AND FLAG VARIETIES OF p-COMPACT GROUPS

TILMAN BAUER AND NATÀLIA CASTELLANA

ABSTRACT. For a compact Lie group G with maximal torus T, Pittie and Smith showed that the flag variety G/T is always a stably framed boundary. We generalize this to the category of p-compact groups. We replace the geometric argument by a homotopy-theoretic one, showing that the framed bordism class represented by G/T is trivial, even G-equivariantly. As an application and inspired by work by the second author and Kitchloo, we consider an unstable construction of a G-space mimicking the adjoint representation sphere of G. Stably and G-equivariantly, this construction splits off its top cell, which we then shown to be a dualizing spectrum for G.

1. INTRODUCTION

Let G be a compact, connected Lie group of dimension d and rank r with maximal torus T. Left translation by elements of G on the tangent space $\mathfrak{g} = T_eG$ induces a framing of G. The Pontryagin-Thom construction associates to G and this framing an element $[G]$ in the stable homotopy groups of spheres. Many low-dimensional homotopy class are representable by Lie groups in this way: for instance, $[\mathbf{S}^1]$ and $[\mathrm{SU}(2)]$ are the first two Hopf maps. This construction has been extensively studied for example in [Smi74, Woo76, Kna78, Oss82].

The following classical argument shows that the flag variety G/T, while not necessarily framed, is still *stably* framed: since every element in a compact Lie group is conjugate to an element in the maximal torus, the conjugation map $G \times T \to G$, $(g,t) \mapsto gtg^{-1}$, is surjective, and furthermore, it factors through $c\colon G/T \times T \to G$. An element $s \in T$ is called *regular* if the centralizer $C_G(s) \supseteq T$ equals T, or equivalently, if $c|_{G/T\times\{s\}}$ is an embedding. Lie theory says that the set of irregular elements has positive codimension in T. Thus there is a regular element s such that the derivative of c has full rank along $G/T \times \{s\}$. By the tubular neighborhood theorem, it induces an embedding of $G/T \times U$, where U is a contractible neighborhood of s in T. Thus the framing of G can be pulled back to a stable framing of G/T.

Pittie and Smith showed in [Pit75, PS75] that G/T is always the G-equivariant boundary of another framed G-manifold M. In terms of homotopy theory, this implies that the class $[G/T] \in \pi^s_{d-r}$ is trivial.

The first main result of this paper generalizes this fact to $\mathbf{Z}/p$*-local, p-finite groups*, which are up-to-homotopy versions of compact Lie groups. This is a slight generalization of Dwyer and Wilkerson's notion of a *p-compact group* [DW94], which has risen to fame for its power to tackle some long-standing problems in finite loop space theory, among other things.

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A space X is called *p-finite* if $H_*(X;\mathbf{F}_p)$ is finite and $\mathbf{Z}/p$*-local* if whenever $f\colon Y \to Z$ is a mod-p homology equivalence of CW-complexes, then $f^*\colon [Z,X] \to [Y,X]$ is a bijection on homotopy classes. A p-finite, $\mathbf{Z}/p$-local topological group is a p-compact group if $\pi_0(G)$ is a p-group. (In their original definition, Dwyer and Wilkerson allowed G to be a loop space instead of a group; however, any loop space can be rigidified, e.g. by using the geometric realization of Kan's group model of the loops on a simplicial set [Kan56], thus their definition is equivalent.)

Every p-compact group has a maximal torus T unique up to conjugacy [DW94]; that is, there is a monomorphism $T \to G$ with $T \simeq L_p(\mathbf{S}^1)^r$ and r is maximal with this property, where L_p denotes $\mathbf{Z}/p$-localization. By definition, a *monomorphism* of $\mathbf{Z}/p$-local, p-finite loop spaces is a group monomorphism $H \to G$ such that G/H is p-finite (cf. [Bau04] for this slightly nonstandard point of view). Since a maximal torus is always contained in the identity component of a group, this result for p-compact groups immediately generalizes to $\mathbf{Z}/p$-local, p-finite groups.

We shall work throughout in the category of naive G-spectra, i.e. spectra with a G-action, where G is a topological group. This category is equipped with a model structure whose weak equivalence are the so-called hG-equivalences, i. e. G-equivariant maps which are non-equivariantly weak equivalences. A map f is a fibration if its underlying nonequivariant map is, and it is a cofibration if it is a retract of a relative free G-cell complex (cf. [Sch97]. In particular, a G-spectrum is cofibrant if it is a free G-CW-spectrum.

Denote by $\mathbf{S}^0[X]$ the suspension spectrum of a space X with a disjoint base point added.

Definition ([Kle01])**.** Let G be a topological group. Define S_G, the *dualizing spectrum of* G, to be the spectrum of homotopy fixed points of the right action of G on its own suspension spectrum. That is, $S_G = (\mathbf{S}^0[G])^{hG^{\mathrm{op}}}$ as left G-spectra.

In [Bau04], the first author showed that for a connected, d-dimensional p-compact group G, S_G is always homotopy equivalent to a $\mathbf{Z}/p$-local sphere of dimension d. Furthermore, there is a G-equivariant logarithm map $\mathbf{S}^0[G] \to S_G$, where G acts on the left by conjugation. He constructs a Pontryagin-Thom-type map

$$[G/T]\colon S_G \to S_T,$$

which we extend to $\mathbf{Z}/p$-local, p-finite groups in the appendix (A.6). If G is the $\mathbf{Z}/p$-localization of a connected Lie group, then S_G is canonically identified with the suspension spectrum of the one-point compactification of the Lie algebra of G and the map $[G/T]$ with the Pontryagin-Thom construction in framed cobordism.

Theorem 1.1. *Let G be a $\mathbf{Z}/p$-local, p-finite group with maximal torus T such that $\dim(G) > \dim(T)$. Then the Pontryagin-Thom construction $[G/T]\colon S_G \to S_T$ is null-homotopic. If $\tilde{S}_G$ denotes a cofibrant replacement of S_G, the induced map $\tilde{S}_G \to S_T$ is in fact G-equivariantly null-homotopic, with G acting trivially on S_T.*

In the second part of this paper, as an application of Theorem 1.1, we study the relationship between two notions of adjoint objects of p-compact groups. It is an interesting question to ask whether the action of G on S_G actually comes from an unstable action of G on $\mathbf{S}^d$. We will not be able to answer this question here. However, there is an alternative, unstable construction of an adjoint object for a connected p-compact group G inspired by the following theorem:

Theorem 1.2 ([CK02, Mit88]). *Let G be a semisimple, connected Lie group of rank r. There exist subgroups $G_I < G$ for every $I \subsetneq \{1, \dots, r\}$ and a homeomorphism of G-spaces*

$$A_G := \Sigma \operatorname*{hocolim}_{I \subsetneq \{1,\dots,r\}} G/G_I \to \mathfrak{g} \cup \{\infty\}$$

to the one-point compactification of the Lie algebra $\mathfrak{g}$ of G.

We define a functorial G-space A_G for every connected p-compact group G and show:

Theorem 1.3. *There is a G-equivariant map $\alpha \colon \tilde{S}_G \to \mathbf{S}^0[A_G]$ which is an isomorphism in the top homology group. This map induces a G-equivariant splitting $\mathbf{S}^0[A_G] \simeq S_G \vee R$ for some finite G-spectrum R when*

1. *G is the completion of a compact Lie group; or*
2. *p does not divide the order of the Weyl group of G.*

This result links the two notions of adjoint objects together. Unfortunately, A_G is in general not a sphere (cf. Ex. 4.4), and we do not know if the top cell of A_G splits off equivariantly in the cases not covered by Theorem 1.3.

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2. THE STABLE p-COMPLETE SPLITTING OF COMPLEX PROJECTIVE SPACE

2.1. Stable splittings from homotopy idempotents.

Let p be a prime. We denote by L_p the localization functor on spaces with respect to mod-p homology, which coincides with p-completion on nilpotent spaces [BK72]. Let $S = L_p\mathbf{S}^1$ be the p-complete 1-sphere, and set $P = \mathbf{S}^0[BS]$. By a classical result (cf. [Mit85, GR89]),

$$P \simeq \bigvee_{s=0}^{p-2} P_s \tag{2.1}$$

for certain $(2s-1)$-connected spectra P_s. In this section, we will investigate this splitting and its compatibility with certain transfer maps.

Let X be a spectrum, $e \in [X, X]$, and define

$$eX = \operatorname{hocolim}\{X \xrightarrow{e} X \xrightarrow{e} \cdots\}.$$

If e is idempotent, this is a homotopy theoretic analog of the image of e. Any such idempotent e yields a stable splitting $X \simeq eX \vee (1-e)X$. If $\{e_1, \dots, e_n\}$ are a complete set of orthogonal idempotents (this means that each e_i is idempotent, $e_ie_j \simeq *$, and $\mathrm{id}_X \simeq e_1 + \cdots + e_n$), then they induce a splitting $X \simeq e_1X \vee \cdots \vee e_nX$.

We define a partial order on the set of all idempotents in $[X, X]$ by $e \leq f$ iff $ef = e$. Then $\{e_i\}$ is a complete set of minimal (nonzero) orthogonal idempotents if and only if the associated wedge decomposition of X is maximal, i. e. if no factor e_iX can be nontrivially split into a further wedge. The splitting (2.1) is indeed maximal, as can be seen by considering $H^*(P_s; \mathbf{F}_p)$ as a module over the Steenrod algebra.

A complete set of minimal orthogonal idempotents does not need to exists, and if it does, it is not necessarily unique, but any two such sets are conjugate [Mit85, Prop. 1.5]. This leads to the following observation:

Lemma 2.2. *Let X be the p-completion of a suspension spectrum of a CW-complex with noetherian mod-p cohomology and $\{e_1, \dots, e_n\}$ a complete set of minimal orthogonal idempotents in $[X, X]$. Then for any idempotent $f \in [X, X]$, there is a homotopy equivalence*

$$fX \simeq \bigvee_{\alpha=1}^{k} e_{j_\alpha} X$$

over X, for uniquely determined indices j_α.

Proof. By [Hen91, Thm. V], the decomposition $X \simeq \bigvee_{i=1}^{n} e_i X$ is unique up to homotopy equivalence and order. Since $X \simeq fX \vee (1-f)X$, it follows that fX is equivalent to the wedge of a uniquely determined subset $\{e_{j_\alpha}\}$ of $\{e_i X\}$. □

Example 2.3. Let p be a prime and $\zeta \in \mathbf{Z}_p^\times$ a primitive $(p-1)$st root of unity. Denote by $\mu\colon P \to P$ the multiplication by ζ in the $\mathbf{Z}_p$-module $[P, P]$ and by $\psi\colon P \to P$ the map induced by multiplication with ζ on $K(\mathbf{Z}_p, 2)$. Define $e_s\colon P \to P$ by

$$e_s = \frac{1}{p-1} \left(\sum_{i=0}^{p-2} \mu^{-is} \psi^i \right).$$

It is straightforward to check that $\{e_0, \dots, e_{p-2}\}$ is a complete set of orthogonal idempotents in $[P, P]$. They induce the splitting (2.1) by defining $P_s = e_s P$, and this splitting is maximal. This also holds for $p = 2$, although then the splitting is trivial.

Setting $H_*(P) = \mathbf{Z}_p\{x_j\}$ with $|x_j| = 2j$, we have that $(e_i)_*\colon H_*(P) \to H_*(P)$ is given by

$$(e_i)_*(x_j) = \begin{cases} x_j; & j \equiv i \pmod{p-1} \\ 0; & \text{otherwise.} \end{cases} \tag{2.4}$$

2.2. **Transfers as splittings.** Let $1 \to H \xrightarrow{i} G \to W \to 1$ be an extension of compact Lie groups. Then associated to the fibration $W \to BH \to BG$ there are two versions of functorial stable transfer maps:

(1) The Becker-Gottlieb transfer $\overline{\tau}\colon \mathbf{S}^0[BG] \to \mathbf{S}^0[BH]$ [BG75, Section 3]
(2) The stable Umkehr map $\tau\colon BG^{\mathfrak{g}} \to BH^{\mathfrak{h}}$ of Thom spaces of the adjoint representation of the Lie groups [BG75, Section 4].

Both versions can be generalized to a setting where the groups involved are not Lie groups but only $\mathbf{Z}/p$-local and p-finite [Dwy96, Bau04], cf. (A.5). For such a group G, $BG^{\mathfrak{g}}$ is defined to be the homotopy orbit spectrum of G acting on the dualizing spectrum S_G; since $H_*(S_G) = H_*(\mathbf{S}^d; \mathbf{Z}_p)$ by Lemma A.2, we have a (possibly twisted) Thom isomorphism $H_n(BG; \mathcal{H}_d(S_G)) \cong H_{n+d}(BG^{\mathfrak{g}})$. We use the notation $\mathfrak{g}$ for the dualizing spectrum S_G to stress the analogy with Lie algebras and Lie groups.

Note that the Becker-Gottlieb transfer $\overline{\tau}$ factors through the Umkehr map τ in the following way:

$$\begin{aligned} \mathbf{S}^0[BG] \xrightarrow{\tau'} BH^\nu &\xrightarrow{\text{comult.}} BH^\nu \wedge_{BG} \mathbf{S}^0[BH] \\ &\xrightarrow{\text{id} \wedge \Delta} BH^\nu \wedge_{BG} \mathbf{S}^0[BH] \wedge_{BG} \mathbf{S}^0[BH] \xrightarrow{\text{eval} \wedge \text{id}} \mathbf{S}^0[BH] \end{aligned} \tag{2.5}$$

where $\nu = \mathfrak{h} - i^*\mathfrak{g}$ is the virtual normal fibration along the fibers of $BH \to BG$, τ' is τ twisted by $-\mathfrak{g}$, and the right hand side evaluation map is defined by identifying BH^ν with the fiberwise Spanier-Whitehead dual of BH over BG.

Proposition 2.6. *Let $W = C_l$ be a finite cyclic group acting freely on $S = L_p\mathbf{S}^1$, with $l \mid p-1$. Denote by $N = S \rtimes W$ the semidirect product with respect to this action. Then the Becker-Gottlieb transfer map $\overline{\tau}$ factors as*

$$\begin{tikzcd} \mathbf{S}^0[BN] \arrow[r, "\overline{\tau}"] \arrow[d, "c"] & P \\ (e_0 + e_l + \cdots + e_{p-1-l})P, \arrow[ur] & \end{tikzcd}$$

and the induced map c is p-completion.

Proof. Since $p \nmid |W|$, the Serre spectral sequence associated to the group extension $S \xrightarrow{i} N \to W$ is concentrated on the vertical axis and shows that

$$H^*(BN; \mathbf{Z}_p) \cong H^*(BS; \mathbf{Z}_p)^W \cong \mathbf{Z}_p[z^l] \hookrightarrow \mathbf{Z}_p[z] \cong H^*(BS; \mathbf{Z}_p).$$

In this case, the Becker-Gottlieb transfer is nothing but the usual transfer for finite coverings, therefore $i \circ \overline{\tau}$ is multiplication by $|W| = l \in \mathbf{Z}_p^\times$. Setting $I = l^{-1}i \colon P \to \mathbf{S}^0[L_pBN]$, we thus get orthogonal idempotents in $[P, P]$:

$$f = \overline{\tau} \circ I \quad \text{and} \quad e = \mathrm{id}_P - f.$$

Clearly, $e \circ \overline{\tau} \simeq *$, thus the map $\overline{\tau}$ factors through fP and induces an equivalence $\mathbf{S}^0[L_pBN] \to fP$, in particular a mod-$p$ homology isomorphism between $\mathbf{S}^0[BN]$ and fP. The computation of the homology of BN together with (2.4) and Lemma 2.2 implies that $fP \simeq (e_0 + e_l + \cdots + e_{p-1-l})P$ over P. ☐

Proposition 2.7. *Let $S,\ N,\ W$ be as above. Then the stable Umkehr map*

$$\tau \colon BN^{\mathfrak{n}} \to BS^{\mathfrak{s}} \simeq \Sigma P$$

factors as

$$\begin{tikzcd} BN^{\mathfrak{n}} \arrow[r, "\tau"] \arrow[d, "c"] & \Sigma P \\ \Sigma\Big(\sum_{i=0}^{\frac{p-1}{l}-1} e_{(i+1)l-1}\Big)P, \arrow[ur] & \end{tikzcd}$$

The induced map c is p-completion.

Proof. This follows from a similarly simple homological consideration. The S-fibration $\mathfrak{n}$ is not orientable, thus we have a twisted Thom isomorphism

$$\tilde{H}^{n+1}(BN^{\mathfrak{n}}) \cong H^n(BN; \mathcal{H}^1(S; \mathbf{Z}_p))$$

where $\pi_1(BN) = \mathbf{Z}/l$ acts on $H^1(S; \mathbf{Z}_p) \cong \mathbf{Z}_p$ by multiplication by an lth root of unity. Thus is that right?

$$H^i(BN^{\mathfrak{n}}; \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p; & i \equiv -1 \pmod{2l} \\ 0; & \text{otherwise.} \end{cases}$$

The factorization (2.5) of $\overline{\tau}$ through τ

$$\mathbf{S}^0[BN] \xrightarrow{\tau'} BS^\nu \xrightarrow{\text{comult.}} BS^\nu \wedge_{BN} \mathbf{S}^0[BS]$$
$$\xrightarrow{\mathrm{id}\wedge\Delta} BS^\nu \wedge_{BN} \mathbf{S}^0[BS] \wedge_{BN} \mathbf{S}^0[BS] \xrightarrow{\mathrm{eval}\wedge\mathrm{id}} \mathbf{S}^0[BS]$$

simplifies considerably since ν is the trivial 0-dimensional fibration over BS, and the composition of the three right hand side maps is an equivalence.

In Prop. 2.6 it was shown that $I \circ \overline{\tau} = \mathrm{id}_{\mathbf{S}^0[L_pBN]}$, thus the same holds after twisting with $\mathfrak{n}$:

$$\mathrm{id}_{BN^\mathfrak{n}} \colon L_pBN^\mathfrak{n} \xrightarrow{\tau} BS^\mathfrak{s} \to BS^{i^*\mathfrak{n}} \xrightarrow{I^\mathfrak{n}} L_pBN^\mathfrak{n}.$$

If we denote the composition $BS^\mathfrak{s} \to BS^{i^*\mathfrak{n}} \xrightarrow{I^\mathfrak{n}} L_pBN^\mathfrak{n}$ by I, overriding its previous meaning, then $\Sigma^{-1}\tau \circ I$ becomes an idempotent on P. The argument now proceeds as in Prop. 2.6. Using the computation of $H^*(BN^\mathfrak{n};\mathbf{Z}_p)$, we find that $L_p\Sigma^{-1}BN^\mathfrak{n} \simeq (\tau \circ I)P$, and

$$(\tau \circ I)_* = \sum_{i=0}^{\frac{p-1}{l}} (e_{(i+1)l-1})_*$$

□

3. Framing p-compact flag varieties

Before proving Theorem 1.1, we need an alternative description of the Pontryagin-Thom construction (A.6) on G/T.

Lemma 3.1. *The map $[G/T]$ is G-equivariantly homotopic to the map*

$$\tilde{S}_G \xrightarrow{\text{incl}} BG^\mathfrak{g} \xrightarrow{\tau} BT^\mathfrak{t} \simeq \Sigma^r\mathbf{S}^0[BT] \xrightarrow{\Sigma^r\epsilon} \mathbf{S}^r,$$

where $BG^\mathfrak{g}$, $BT^\mathfrak{t}$, and τ are as in Section 2.2, $\epsilon\colon \mathbf{S}^0[BT] \to \mathbf{S}^0$ and all spectra except S_G have a trivial G-action.

Proof. Applying homotopy G-orbits to (A.6), we get a G-equivariant diagram

$$\begin{tikzcd}
\tilde{S}_G \arrow[r, "\tilde{\tau}"] \arrow[d, "\text{incl}"] & G_+ \wedge_T \mathbf{S}^0[T] \arrow[r, "\sim"] & \mathbf{S}^0[G/T] \wedge S_T \arrow[r] \arrow[d] & S_T \arrow[d, equal] \\
BG^\mathfrak{g} \arrow[r, "\tau"] & BT^\mathfrak{t} \arrow[r, "\sim"] & \mathbf{S}^0[BT] \wedge S_T \arrow[r] & S_T
\end{tikzcd}$$

which is commutative by the definition of τ. □

In the proof of Theorem 1.1, certain special subgroups will play an important role. In order to define them we need to recall certain facts about the Weyl group of a p-compact group.

Dwyer and Wilkerson showed in their ground-breaking paper [DW94] that given any connected p-compact group G with maximal torus T, there is an associated *Weyl group* $W(G)$, which is defined as the group of components of the homotopy discrete space of automorphisms of the fibration $BT \to BG$. This generalizes the notion of Weyl groups of compact Lie groups; they are canonically subgroups of $\mathrm{GL}(H_1(T;\mathbf{Z})) = \mathrm{GL}_r(\mathbf{Z}_p)$, and they are so-called finite complex reflection groups.

This means that they are generated by elements (called reflections or, more classically, pseudo-reflections) that fix hyperplanes in $\mathbf{Z}_p^r$. The complete classification of complex reflection groups over $\mathbf{C}$ is classical and due to Shephard and Todd [ST54], the refinement to the p-adic rationals is due to Clark and Ewing [CE74] and the lifting to $\mathbf{Z}_p$ is due to Notbohm [Not96, Not99] and Andersen-Grodal [AGMV08].

Call a reflection $s \in W$ *primitive* if there is no reflection $s' \in W$ of strictly larger order such that $s = (s')^k$ for some k.

Denote by $s \in W$ a primitive reflection of minimal order $l > 1$. Let $T^s < T$ be the fixed point subtorus under s. Since s is primitive,

$$\langle s \rangle = \{w \in W \mid w|_{T^s} = \mathrm{id}_{T^s}\}.$$

Definition. Given a connected p-compact group G and a primitive reflection $s \in W(G)$ of minimal order $l > 1$, define C_s to be the centralizer of T^s in G.

Since G is connected, so is the subgroup C_s [DW95, Lemma 7.8]. Furthermore, C_s has maximal rank because $T < C_s$ by definition, and the inclusion $C_s < G$ induces the inclusion of Weyl groups $\langle s \rangle < W$ [DW95, Thm. 7.6]. Since the Weyl group of C_s is $\mathbf{Z}/l$, the quotient of C_s by its p-compact center, $C_s/Z(C_s)$, can have rank at most 1. By the (almost trivial) classification of rank-1 p-compact groups, we find that its rank is equal to 1 and

$$C_s \cong \left(L_p(\mathbf{S}^1)^{r-1} \times L_p\mathbf{S}^{2l-1}\right)/\Gamma, \tag{3.2}$$

where $L_p\mathbf{S}^{2l-1}$ is simply $L_p\,\mathrm{SU}(2)$ for $l = 2$, and the Sullivan group given by

$$L_p\mathbf{S}^{2l-1} = \Omega L_p\left(L_p(B\mathbf{S}^1)_{h\mathbf{Z}/l}\right)$$

for p odd, and Γ is a finite central subgroup.

Proof of Thm. 1.1. By Lemma 3.1, showing equivariant null-homotopy is equivalent to showing that the map

$$h(G/T)\colon BG^{\mathfrak{g}} \xrightarrow{\tau} BT^{\mathfrak{t}} \xrightarrow{\Sigma^r\epsilon} \mathbf{S}^r$$

is null. Note that for any given subgroup $H < G$ of maximal rank, there is a factorization of τ through $BH^{\mathfrak{h}}$. In particular, we may assume that G is connected. By the dimension hypothesis of the theorem, $W(G)$ is nontrivial. If $H = C_s$ is the subgroup associated to a primitive reflection $s \in W(G)$ of minimal order $l > 1$, then the map $h(C_s/T)$ is the $(r-1)$-fold suspension of $h(L_p\mathbf{S}^{2l-1}/S)$ by (3.2). Therefore, it is enough to prove the theorem for those p-compact groups C_s.

We distinguish two cases.

First suppose that $l = 2$. By the classification of complex reflection groups [ST54], and with the terminology of that paper, this is always the case except when W is a product of any number of groups from the list

$$\{G_4, G_5, G_{16}, G_{18}, G_{20}, G_{25}, G_{32}\}.$$

In this case, the map $h(L_p\,\mathrm{SU}(2)/S)$ is null by Pittie [Pit75, PS75] since the spaces involved are Lie groups, thus $h(G/T) \simeq *$.

Now suppose that $l > 2$. This forces $p > 2$ as well, and since $\langle s \rangle$ acts faithfully on some line in $H^1(T;\mathbf{Z}_p)$ while fixing the complementary hyperplane, we must have that it acts by an lth root of unity, and thus $l \mid p-1$. The proof is finished if we can show that

$$h(L_p\mathbf{S}^{2l-1}/S) = 0$$

where S is the 1-dimensional maximal torus in the Sullivan group $G' := L_p\mathbf{S}^{2l-1}$. To see this, note that the inclusion $S \to G'$ factors through the maximal torus normalizer $N_{G'}(S) \cong S \rtimes \mathbf{Z}/l$, and thus

$$h(G'/S)\colon BG'^{\mathfrak{g}'} \xrightarrow{\tau_1} BN^{\mathfrak{n}} \xrightarrow{\tau_2} \Sigma\mathbf{S}^0[BS] \to \mathbf{S}^1.$$

If $P \simeq \bigvee_{i=0}^{p-2} e_i P$ is any stable splitting of the p-completed complex projective space $P = \mathbf{S}^0[BS]$ induced by idempotents e_i as in the previous section, then the rightmost projection map clearly factors through e_0P, which is the part containing the bottom cell. Since $p > 2$, Proposition 2.7 shows that there is an idempotent $f \in [P, P]$ such that $\tau \simeq f \circ \tau$ and $\tau \circ e_0 = e_0 \circ \tau = 0$, proving the theorem. □

4. The adjoint representation

Let G be a d-dimensional connected p-compact group with maximal torus T of rank r. Choose a set $\{s_1, \dots, s_{r'}\}$ of generating reflections of $W = W(G)$ with r' minimal. The classification of pseudo-reflection groups [ST54, CE74] implies that for G simple, most of the time $r' = r$ (the "well-generated" case), but there are cases where $r' = r + 1$.

Example 4.1 (The group no. 7)**.** Let $p \equiv 1 \pmod{12}$. Let G_7 be the finite group generated by the reflection s of order 2 and the two reflections t, u of order 3, where $s, t, u \in GL_2(\mathbf{Z}_p)$ are given by

$$s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta & -\zeta^7 \\ \zeta & \zeta^7 \end{pmatrix}, \quad u = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta^7 & \zeta^7 \\ -\zeta & \zeta \end{pmatrix}.$$

Here ζ is a primitive 24th root of unity. Note that although possibly $\zeta \notin \mathbf{Z}_p$, $\frac{1}{\sqrt{2}}\zeta \in \mathbf{Z}_p$. In Shephard and Todd's classification, this is the restriction to $\mathbf{Z}_p$ of the complex pseudo-reflection group no. 7. They show that even over the complex numbers, G_7 cannot be generated by two reflections. The associated p-compact group is given by

$$\Omega L_p((BT^2)_{hG_7}).$$

If G is not semisimple (i. e. it contains a nontrivial normal torus subgroup), then r' may be smaller than r. Set $\kappa = r + 1 - r' \geq 0$.

Let $\mathcal{I}_{r'}$ be the poset of proper subsets of $\{1, \dots, r'\}$, and for $I \subseteq \{1, \dots, r'\}$, let T_I be the fixed point subtorus $T^{\langle s_i | i \in I\rangle}$ and $C_I = C_G(T_I)$ be the centralizer in G, which is connected by [DW95, Lemma 7.8].

Definition. Let G be a connected p-compact group. Define the *adjoint space* A_G by the homotopy colimit

$$A_G = \Sigma^\kappa \operatorname*{hocolim}_{I \in \mathcal{I}_r} G/C_I$$

with the induced left G-action, and the trivial G-action on the suspension coordinates.

Theorem 1.2 shows that if G is the p-completion of a connected, semisimple Lie group (in this case $r = r'$ and $\kappa = 1$), then A_G is a d-dimensional sphere G-equivariantly homotopy equivalent to $\mathfrak{g} \cup \{\infty\}$. This holds more generally: if G is a connected, compact Lie group with maximal normal torus T^k then

$$A_G \cong \Sigma^k A_{G/T^k} = (\mathfrak{t} \cup \{\infty\}) \wedge (\mathfrak{g}/\mathfrak{t} \cup \{\infty\}) = \mathfrak{g} \cup \{\infty\}.$$

Lemma 4.2. *Let R be a ring and $\mathcal{C}$ the category of either R-finite CW-complexes or R-finite CW-spectra. For $X \in \mathcal{C}$, denote by $\dim(X) = \dim_R(X)$ the greatest n such that $H_n(X; R) \neq 0$. Let $\mathcal{I}_k$ be the poset of proper subsets of $\{1, \dots, k\}$.*

(1) *If a functor $F\colon \mathcal{I}_k \to \mathcal{C}$ has the property that $\dim F(\emptyset) > \dim F(I)$ for every $I \neq \emptyset$, then*

$$\dim \operatorname{hocolim} F = \dim F(\emptyset) + k - 1.$$

(2) *If $f\colon F \to G$ is a natural transformation of two functors as in (1) such that*

$$f_*(\emptyset)\colon H_{\dim F(\emptyset)}(F(\emptyset)) \xrightarrow{\cong} H_{\dim G(\emptyset)}(G(\emptyset)),$$

then f induces an isomorphism

$$\operatorname{hocolim} f_*\colon H_{\dim \operatorname{hocolim} F}(\operatorname{hocolim} F) \to H_{\dim \operatorname{hocolim} G}(\operatorname{hocolim} G).$$

(3) *Let $F\colon \mathcal{I}_k \to \mathrm{Top}$ be a functor with $F(\emptyset) \simeq \mathbf{S}^n$, $F(I) \simeq *$ for $I \neq \emptyset$. Then $\operatorname{hocolim}_{\mathcal{I}_k} F \simeq \mathbf{S}^{n+k-1}$.*

Proof. Parts (1) and (2) follow from the Mayer-Vietoris spectral sequence [BK72, Chapter XII.5],

$$E^1_{p,q} = \bigoplus_{I \in \mathcal{I}_k,\ |I| = k-1-p} H_q(F(I); R) \Longrightarrow H_{p+q}(\operatorname{hocolim} F; R),$$

along with the observation that under the dimension assumptions of (1), $E^1_{p,q} = 0$ for $q \geq \dim F(\emptyset)$ except for $E^1_{k-1, \dim F(\emptyset)} = H_{\dim F(\emptyset)}(F(\emptyset))$. In particular, this group cannot support a nonzero differential and thus

$$H_i(F(\emptyset); R) \cong H_{i+k-1}(\operatorname{hocolim} F; R) \quad \text{for } i \geq \dim F(\emptyset).$$

Part (3) is a consequence of the Mayer-Vietoris spectral sequence for $R = \mathbf{Z}$. □

Corollary 4.3. *For any connected p-compact group G, $\dim_{\mathbf{Z}/p}(A_G) = \dim_{\mathbf{Z}/p}(G)$.*

Proof. This follows from Lemma 4.2. Indeed, since any C_I ($I \neq \emptyset$) is connected and has the nontrivial Weyl group W_I, its dimension is greater than $\dim T$. So the condition

$$\dim F(\emptyset) = \dim G/T > \dim F(I)$$

is satisfied, and

$$\dim \operatorname{hocolim} F = d - r + r' - 1 = d - \kappa.$$ □

As mentioned at the end of the introduction, for p-compact groups G, A_G is not usually a sphere, as the following example illustrates.

Example 4.4. Let $p \geq 5$ be a prime, and let $G = \mathbf{S}^{2p-3}$ be the Sullivan sphere, whose group structure is given by $BG = L_p\left(BS_{hC_{p-1}}\right)$, where $C_{p-1} \subseteq \mathbf{Z}_p^\times$ acts on $BS = K(\mathbf{Z}_p, 2)$ by multiplication on $\mathbf{Z}_p$. Clearly, G has rank 1, and $\mathcal{I}_1$ consists only of a point, thus $A_G = \Sigma G/T \simeq L_p \Sigma \mathbf{C}P^{p-2}$. Since $p \geq 5$, this is not a sphere.

For the proof of Theorem 1.3 we need a preparatory result. By Lemma A.3, there is a G-equivariant weak equivalence

$$G_+ \wedge_H S_H \to \mathbf{S}^0[G]^{hH^{\mathrm{op}}}.$$

Using the restriction of homotopy fixed points, we thus obtain

$$\tilde{S}_G \to S_G = \mathbf{S}^0[G]^{hG^{\mathrm{op}}} \to \mathbf{S}^0[G]^{hH^{\mathrm{op}}} \xleftarrow{\sim} G_+ \wedge_H S_H \xleftarrow{\sim} G_+ \wedge_H \tilde{S}_H,$$

where $\tilde{S}_G$ and $\tilde{S}_H$ denote cofibrant replacements of S_G and S_H, respectively. Thus we obtain a G-equivariant lift

$$\tilde{\tau}_{G,H} \colon \tilde{S}_G \to G_+ \wedge_H \tilde{S}_H,$$

which we can further compose to $G_+ \wedge_H S_H$; we will call this composition $\tilde{\tau}_{G,H}$ as well.

Proposition 4.5. *Let G be a p-compact group, $P < G$ a maximal rank p-compact subgroup, and $T < P$ a common maximal torus with* $\dim P > \dim T$. *Then the following composition is G-equivariantly null-homotopic:*

$$f_{G,P} \colon \tilde{S}_G \wedge DS_T \to \mathbf{S}^0[G/T] \to \mathbf{S}^0[G/P].$$

The second map is the canonical projection, whereas the first map is adjoint to

$$\tilde{S}_G \xrightarrow{\tilde{\tau}_{G,T}} G_+ \wedge_T \tilde{S}_T \simeq \mathbf{S}^0[G/T] \wedge \tilde{S}_T \to \mathbf{S}^0[G/T] \wedge S_T.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\tilde{S}_G \arrow[r, "\tilde{\tau}_{G,T}"] \arrow[d, "\tilde{\tau}_{G,P}"] & G_+ \wedge_T S_T \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathbf{S}^0[G/T] \wedge S_T \arrow[r, "\mathrm{proj}"] \arrow[d, "\sim"] & \mathbf{S}^0[G/P] \wedge S_T \arrow[d, "\sim"] \\
G_+ \wedge_P \tilde{S}_P \arrow[r, "\mathrm{id} \wedge_P \tilde{\tau}_{P,T}"] & G_+ \wedge_P (P_+ \wedge_T S_T) \arrow[r, "\sim"] & G_+ \wedge_P (\mathbf{S}^0[P/T] \wedge S_T) \arrow[r, "P/T \to *"] & G_+ \wedge_P (\mathbf{S}^0 \wedge S_T).
\end{tikzcd}$$

The upper row is the adjoint of the map $f_{G,P}$ in question. The lower row is the map

$$\tilde{S}_P \to P_+ \wedge_T S_T \to S_T$$

induced up to G. But this map is exactly the homotopy class represented by $[P/T]$ (A.6), thus the assertion follows from Theorem 1.1. □

Proof of Thm. 1.3. Let G be a connected p-compact group whose Weyl group is generated by a minimal set of r' reflections. We want to construct a G-equivariant map $\alpha \colon \tilde{S}_G \to \mathbf{S}^0[A_G]$ which is an isomorphism on the top homology group.

Let $A \colon \mathcal{I}_{r'} \to \mathrm{Top}$ be the functor given by $A(I) = G/C_I$, such that $\Sigma^\kappa \operatorname{hocolim} A = A_G$. Note that, since G is connected, $C_G(T) = T$ [DW94, Proposition 9.1] and $A(\emptyset) = G/T$.

Let $\bar{F} \colon \mathcal{I}_{r'} \to \mathrm{ho}\,\mathrm{Sp}_G$ be the functor to the homotopy category of G-spectra given by $\bar{F}(\emptyset) = S_G \wedge DS_T$ and $\bar{F}(I) = *$ for $I \neq \emptyset$. Then there is a natural transformation $\bar{\Phi}$ of functors into the homotopy category of G-spectra from $\bar{F}$ to $\mathbf{S}^0[A]$ given by

$$\bar{\Phi}(\emptyset) = f_{G,T} \colon F(\emptyset) = \tilde{S}_G \wedge DS_T \to \mathbf{S}^0[G/T]$$

as the map given in Prop. 4.5. Extending this natural transformation by the trivial map for $I \neq \emptyset$ gives a commutative diagram in $\operatorname{ho}\mathrm{Sp}_G$ by virtue of Prop. 4.5.

The strategy of the proof is to lift $\bar{F}$ to a functor $F \colon \mathcal{I}_{r'} \to \mathrm{Sp}_G$ and $\bar{\Phi}$ to a natural transformation Φ into Sp_G. From this we get a G-equivariant map

$$\tilde{S}_G \simeq \mathbf{S}^\kappa \wedge \Sigma^{r'-1}\tilde{S}_G \wedge DS_T \simeq \mathbf{S}^\kappa \wedge \operatorname*{hocolim}_{\mathcal{I}_{r'}} F \to \Sigma^\kappa \operatorname*{hocolim}_{\mathcal{I}_{r'}} \mathbf{S}^0[G/C_I] \simeq \mathbf{S}^0[A_G],$$

which will give us the desired map α.

We will proceed by induction on the number of generating reflections r'. If $r' = 1$ then $A_G = \mathbf{S}^\kappa \wedge G/T$ and $\Phi(\emptyset) = f_{G,T}$.

For $r' > 1$, we can construct the functor F and the natural transformation Φ step by step. Fix a subset I of cardinality k, and assume that F and Φ have been defined for all vertices in the diagram corresponding to I' with $|I'| < k$.

Let $\mathcal{P}(I)$ be the poset of all proper subsets of I. Since F and Φ are defined over $\mathcal{P}(I)$ by induction hypothesis, we have an induced map of homotopy colimits

$$\alpha' \colon \operatorname*{hocolim}_{\mathcal{P}(I)} F \simeq \Sigma^{k-1}\tilde{S}_G \wedge DS_T \to \operatorname*{hocolim}_{\mathcal{P}(I)} A \to \mathbf{S}^0[G/C_I].$$

It is enough to show that α' is G-equivariantly null-homotopic. If this is the case, we define $F(I) = C(\mathrm{hocolim}_{\mathcal{P}(I)} F)$ to be the cone on $\mathrm{hocolim}_{\mathcal{P}(I)} F$ and extend α' to $F(I)$, using a chosen null-homotopy.

Note that $f_{G,T} \colon \tilde{S}_G \wedge DS_T \to \mathbf{S}^0[G/T]$ factors through $G_+ \wedge_{C_I} \tilde{S}_{C_I} \wedge DS_T$. By induction, we know there is a map

$$\Sigma^{k-1}S_{C_I} \wedge DS_T \to \mathbf{S}^0[\operatorname*{hocolim}_{J \in \mathcal{I}_k} C_I/C_J],$$

which splits the top cell. We get a factorization

$$\begin{aligned} \Sigma^{k-1}S_G \wedge DS_T \to \Sigma^{k-1}G_+ \wedge_{C_I} S_{C_I} \wedge DS_T \to \\ G_+ \wedge_{C_I} \mathbf{S}^0[\operatorname*{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \to G_+ \wedge_{C_I} \mathbf{S}^0. \end{aligned} \tag{4.6}$$

It thus suffices to show that in the $\mathcal{I}_k$-diagram

$$\begin{tikzcd}
S_{C_I} \wedge DS_T \arrow[r, Rrightarrow] \arrow[d] & \{*\}_{J \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\mathrm{hocolim}"] \arrow[d] & \Sigma^{k-1}S_{C_I} \wedge DS_T \arrow[d] \\
\mathbf{S}^0[C_I/T] \arrow[r, Rrightarrow] \arrow[d] & \{\mathbf{S}^0[C_I/C_J]\}_{J \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\mathrm{hocolim}"] \arrow[d] & \mathbf{S}^0[\mathrm{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \arrow[d] \\
\mathbf{S}^0 \arrow[r, Rrightarrow] & \{\mathbf{S}^0\}_{I \in \mathcal{I}_k - \{\emptyset\}} \arrow[r, "\mathrm{hocolim}"] & \mathbf{S}^0
\end{tikzcd}$$

the right hand side composition $\Sigma^{k-1}S_{C_I} \wedge DS_T \to \mathbf{S}^0$ is C_I-equivariantly null-homotopic. In the latter diagram, it makes no difference whether the centralizers are taken in C_I or in G. But by Theorem 1.1, the left hand column is already null-homotopic, thus, as a colimit of null-homotopic maps over a contractible diagram, so is the right hand column. $\square$

Conclusion and questions. In this paper, we have compared two imperfect notions of adjoint representations of a p-compact group G. One (S_G) is a sphere, but has a G-action only stably; the other (A_G) is an unstable G-space, but fails to be a sphere. The question remains whether there is an unstable G-sphere whose suspension spectrum is S_G. It might even be true that A_G splits off its top cell after only one suspension, yielding a solution to this problem in the cases where the Weyl group of the rank-r group G is generated by r reflections.

There are also a number of interesting open questions about the flag variety G/T of a p-compact groups:

- By the classification of p-compact groups [AGMV08, AG08], $H^*(G/T;\mathbf{Z}_p)$ is torsion free and generated in degree 2. Can this be seen directly?
- Is there a manifold M such that $L_pM \simeq G/T$, analogous to smoothings of G [BKNP04, BP06]? Is it a boundary of a manifold?
- If such a manifold M exists, can it be given a complex structure?

Appendix A. Stably dualizable groups

The aim of this appendix is to generalize various needed results from [Bau04] to the class of $\mathbf{Z}/p$-local, p-finite groups. In that paper, the first author restricted attention to p-compact groups, which have the additional property that its group of components is a p-group. This assumption is never really needed, but we want to give short proofs of the relevant results for the sake of completeness nevertheless.

Lemma A.1. *Let $H < G$ be an inclusion of $\mathbf{Z}/p$-local, p-finite groups, and let X be a non-equivariant spectrum. Then the H^{op}-action on the mapping spectrum $\mathrm{map}(G_+, X)$ gives a weak equivalence, natural in H and G:*

$$\mathrm{map}(G_+, X)^{hH^{\mathrm{op}}} \simeq \mathrm{map}(G/H_+, X).$$

Proof.

$$\begin{aligned}\mathrm{map}(G, X)^{hH^{\mathrm{op}}} &\simeq \mathrm{map}^{H^{\mathrm{op}}}(EH_+, \mathrm{map}(G, X)) \simeq \mathrm{map}^{H^{\mathrm{op}}}((EH \times G)_+, X)\\ &\simeq \mathrm{map}((EH \times G)_+/H, X) \simeq \mathrm{map}(G/H_+, X).\end{aligned}$$

$\square$

Lemma A.2. *Let G be a $\mathbf{Z}/p$-local, p-finite group of dimension d. Then the dualizing spectrum S_G is equivalent to a $\mathbf{Z}/p$-local sphere of the same dimension d, and the inclusion of the identity component $G_0 \hookrightarrow G$ induces a G_0-equivariant equivalence $S_{G_0} \to S_G$.*

Proof. Let $\pi = \pi_0 G$ be the finite group of components. Then G_0-equivariantly, $G \simeq \mathrm{map}(\pi, G_0)$ and since the suspension functor $\mathbf{S}^0[-]$ sends coproducts to wedges,

$$\mathbf{S}^0[G]^{hG_0} = \mathrm{map}_{G_0}((EG_0)_+, \mathbf{S}^0[G]) \simeq \mathrm{map}(\pi, \mathbf{S}^0[G_0]^{hG_0}) = \mathrm{map}(\pi, S_{G_0}),$$

which is a finite wedge of $\mathbf{Z}/p$-local spheres by [Bau04, Cor. 23]. Then

$$\mathbf{S}^0[G]^{hG} = \left(\mathbf{S}^0[G]^{hG_0}\right)^{h\pi} \simeq \mathrm{map}(\pi, S_{G_0})^{h\pi} \simeq S_{G_0}$$

by Lemma A.1. Moreover, the inclusion of the unit component $G_0 \to G$ induces the inclusion of the identity wedge factor $S_{G_0} \hookrightarrow \mathbf{S}^0[G]^{hG_0}$ and hence an equivalence with S_G. □

Lemma A.3. *Let $H < G$ be an inclusion of $\mathbf{Z}/p$-local, p-finite groups. Then there is an hG-equivalence, natural in G:*

$$G_+ \wedge_H S_H \to \mathbf{S}^0[G]^{hH^{\mathrm{op}}}.$$

Proof. In the case of connected H and G, this is [Bau04, Lemma 19]. In general, the natural map

$$G_+ \wedge_H \mathrm{map}(EH_+, \mathbf{S}^0[H]) \to \mathrm{map}(EH_+, G_+ \wedge_H \mathbf{S}^0[H]) \simeq \mathrm{map}(EH_+, \mathbf{S}^0[G])$$

induces a G-equivariant map $\phi\colon G_+ \wedge_H S_H \to \mathbf{S}^0[G]^{hH^{\mathrm{op}}}$ by passing to H^{op}-homotopy fixed points. Non-equivariantly, $G_+ \wedge_H S_H$ splits as $\mathrm{map}(\pi_0 G/\pi_0 H, G_0 \wedge_{H_0} S_{H_0})$ and

$$\begin{aligned}\mathbf{S}^0[G]^{hH^{\mathrm{op}}} &\simeq \left(\mathrm{map}(\pi_0 G, \mathbf{S}^0[G_0])^{hH_0^{\mathrm{op}}}\right)^{\pi_0 H^{\mathrm{op}}} \\ &\simeq \mathrm{map}(\pi_0 G, \mathbf{S}^0[G_0]^{hH_0^{\mathrm{op}}})^{\pi_0 H^{\mathrm{op}}} \simeq \mathrm{map}(\pi_0 G/\pi_0 H, \mathbf{S}^0[G_0]^{hH_0^{\mathrm{op}}})\end{aligned}$$

by Lemma A.1, and ϕ respects this splitting. By [Bau04, Lemma 19], ϕ is a weak equivalence on every wedge summand, hence a weak equivalence. □

Lemma A.4. *Let $H < G$ be a monomorphism of $\mathbf{Z}/p$-local, p-finite groups. Then there is zigzag of hG-equivalences*

$$G_+ \wedge_H S_H \simeq D\mathbf{S}^0[G/H] \wedge S_G.$$

Moreover, for inclusions $K < H < G$ of $\mathbf{Z}/p$-local, p-finite groups, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{S}^0[G]^{hH^{\mathrm{op}}} \arrow[d, "\mathrm{res}"] & G_+ \wedge_H S_H \arrow[l, "\sim"'] \arrow[r, dotted, leftrightarrow, "\sim"] & D\mathbf{S}^0[G/H] \wedge S_G \arrow[d, "D(\mathrm{proj})\wedge\mathrm{id}"] \\
\mathbf{S}^0[G]^{hK^{\mathrm{op}}} & G_+ \wedge_K S_K \arrow[l, "\sim"'] \arrow[r, dotted, leftrightarrow, "\sim"] & D\mathbf{S}^0[G/K] \wedge S_G
\end{tikzcd}$$

Proof. In [Bau04, Prop. 22], the first author constructed a weak equivalence

$$S_G \wedge D\mathbf{S}^0[G] \to \mathbf{S}^0[G]$$

for connected p-compact groups G, which is equivariant with respect to two different G-actions. The first is multiplication on $D\mathbf{S}^0[G]$ and $\mathbf{S}^0[G]$ and the standard (conjugation) action on S_G, and the second one is right multiplication on $D\mathbf{S}^0[G]$ and $\mathbf{S}^0[G]$ and the trivial action on S_G. Rognes [Rog08, Thm. 3.1.4] extended this proof to stably dualizable groups, in particular to $\mathbf{F}_p$-local, p-finite groups. Taking H-homotopy fixed points with respect to that second action, we obtain hG-equivalences

$$D\mathbf{S}^0[G/H] \wedge S_G \xleftarrow{\sim} D\mathbf{S}^0[G]^{hH} \wedge S_G \xrightarrow{\sim} (D\mathbf{S}^0[G] \wedge S_G)^{hH^{\mathrm{op}}} \xrightarrow{\sim} \mathbf{S}^0[G]^{hH^{\mathrm{op}}},$$

where the left hand map is the equivalence from Lemma A.1. Composing with the natural equivalence of Lemma A.3 gives the result.

For the naturality statement, consider the following diagram:

$$\begin{tikzcd}
D\mathbf{S}^0[G/H] \wedge S_G \arrow[d] & D\mathbf{S}^0[G]^{hH^{\mathrm{op}}} \wedge S_G \arrow[l] \arrow[r] \arrow[d] & (D\mathbf{S}^0[G] \wedge S_G)^{hH^{\mathrm{op}}} \arrow[r] \arrow[d] & \mathbf{S}^0[G]^{hH^{\mathrm{op}}} \arrow[d] \\
D\mathbf{S}^0[G/K] \wedge S_G & D\mathbf{S}^0[G]^{hK^{\mathrm{op}}} \wedge S_G \arrow[l] \arrow[r] & (D\mathbf{S}^0[G] \wedge S_G)^{hK^{\mathrm{op}}} \arrow[r] & \mathbf{S}^0[G]^{hK^{\mathrm{op}}}
\end{tikzcd}$$

The left hand square commutes by Lemma A.1, the other two for trivial reasons. □

Lemma A.4 does not provide a G-equivariant map in either direction, but if X is a cofibrant G-spectrum (i. e. a free G-CW spectrum) then any map $X \to D\mathbf{S}^0[G/H] \wedge S_G$ lifts uniquely up to homotopy to a G-map $X \to G_+ \wedge_H S_H$. In particular, we get a G-map

$$\tilde{\tau}\colon \tilde{S}_G \xrightarrow{\mathrm{id} \wedge D\epsilon} \tilde{S}_G \wedge D\mathbf{S}^0[G/H] \underset{\text{Lemma A.4}}{\simeq} G_+ \wedge_H S_H,$$

where $\epsilon\colon \mathbf{S}^0[X] \to \mathbf{S}^0$ is given by applying the functor $\mathbf{S}^0[-]$ to $X \to *$. By passage to G-homotopy orbits, we obtain a transfer map

$$\tau\colon BG^{\mathfrak{g}} = (S_G)_{hG} \to (G_+ \wedge_H S_H)_{hG} \simeq (S_H)_{hH} = BH^{\mathfrak{h}}, \tag{A.5}$$

which coincides with the stable Umkehr map for fiber bundles when H, G are Lie groups [Bau04, Thm. 4].

If $T < G$ is a sub-torus in a $\mathbf{Z}/p$-local, p-finite group then we can use $\tilde{\tau}$ to define a G-equivariant map

$$[G/T]\colon \tilde{S}_G \xrightarrow{\tilde{\tau}} G_+ \wedge_T S_T \simeq \mathbf{S}^0[G/T] \wedge S_T \xrightarrow{\epsilon} S_T \tag{A.6}$$

where the homotopy equivalence holds because S_T has a homotopy trivial T-action as T is homotopy abelian. This map generalizes the Pontryagin-Thom construction [Bau04, Section 5].

References

[AG08] Kasper K. S. Andersen and Jesper Grodal. The classification of 2-compact groups. *J. Amer. Math. Soc.*, 2008.

[AGMV08] K. K. S. Andersen, J. Grodal, J. M. Møller, and A. Viruel. The classification of p-compact groups for p odd. *Ann. of Math. (2)*, 167(1):95–210, 2008.

[Bau04] Tilman Bauer. p-compact groups as framed manifolds. *Topology*, 43(3):569–597, 2004.

[BG75] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[BKNP04] Tilman Bauer, Nitu Kitchloo, Dietrich Notbohm, and Erik Kjær Pedersen. Finite loop spaces are manifolds. *Acta Math.*, 192:5–31, 2004.

[BP06] Tilman Bauer and Erik Kjær Pedersen. The realizability of local loop spaces as manifolds. *K-Theory*, 37(3):329–339, 2006.

[CE74] Allan Clark and John Ewing. The realization of polynomial algebras as cohomology rings. *Pacific J. Math.*, 50:425–434, 1974.

[CK02] Natàlia Castellana and Nitu Kitchloo. A homotopy construction of the adjoint representation for Lie groups. *Math. Proc. Cambridge Philos. Soc.*, 133(3):399–409, 2002.

[DW94] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.

[DW95] W. G. Dwyer and C. W. Wilkerson. The center of a p-compact group. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 119–157. Amer. Math. Soc., Providence, RI, 1995.

[Dwy96] W. G. Dwyer. Transfer maps for fibrations. *Math. Proc. Cambridge Philos. Soc.*, 120(2):221–235, 1996.

[GR89] Brayton Gray and Nigel Ray. Splitting $\mathbf{C}P^\infty$ and $B\mathbf{Z}/p^n$ into Thom spectra. *Math. Proc. Cambridge Philos. Soc.*, 106(2):263–271, 1989.

[Hen91] Hans-Werner Henn. Finiteness properties of injective resolutions of certain unstable modules over the Steenrod algebra and applications. *Math. Ann.*, 291(2):191–203, 1991.

[Kan56] Daniel M. Kan. Abstract homotopy. IV. *Proc. Nat. Acad. Sci. U.S.A.*, 42:542–544, 1956.

[Kle01] John R. Klein. The dualizing spectrum of a topological group. *Math. Ann.*, 319(3):421–456, 2001.

[Kna78] K. Knapp. Rank and Adams filtration of a Lie group. *Topology*, 17(1):41–52, 1978.

[Mit85] Stephen A. Mitchell. Splitting $B(\mathbf{Z}/p)^n$ and BT^n via modular representation theory. *Math. Z.*, 189(1):1–9, 1985.

[Mit88] Stephen A. Mitchell. Quillen's theorem on buildings and the loops on a symmetric space. *Enseign. Math. (2)*, 34(1-2):123–166, 1988.

[Not96] D. Notbohm. p-adic lattices of pseudo reflection groups. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 337–352. Birkhäuser, Basel, 1996.

[Not99] D. Notbohm. For which pseudo-reflection groups are the p-adic polynomial invariants again a polynomial algebra? *J. Algebra*, 214(2):553–570, 1999.

[Oss82] Erich Ossa. Lie groups as framed manifolds. *Topology*, 21(3):315–323, 1982.

[Pit75] Harsh V. Pittie. Generalized flag manifolds bound equivariantly. *Proc. Amer. Math. Soc.*, 47:263–264, 1975.

[PS75] Harsh V. Pittie and Larry Smith. Generalized flag manifolds as framed boundaries. *Math. Z.*, 142:191–193, 1975.

[Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008.

[Sch97] Stefan Schwede. Spectra in model categories and applications to the algebraic cotangent complex. *J. Pure Appl. Algebra*, 120(1):77–104, 1997.

[Smi74] Larry Smith. Framings of sphere bundles over spheres, the plumbing pairing, and the framed bordism classes of rank two simple Lie groups. *Topology*, 13:401–415, 1974.

[ST54] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[Woo76] R. M. W. Wood. Framing the exceptional Lie group G_2. *Topology*, 15(4):303–320, 1976.

INSTITUTIONEN FÖR MATEMATIK
KUNGLIGA TEKNISKA HÖGSKOLAN
LINDSTEDTSVÄGEN 25
10044 STOCKHOLM, SWEDEN
E-mail address: `tilmanb@kth.se`

DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
08193 BELLATERRA, SPAIN
E-mail address: `natalia@mat.uab.es`