ON THE STRUCTURE OF ABELIAN HOPF ALGEBRAS

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ABSTRACT. We study the structure of the category of graded, connected, commutative and cocommutative Hopf algebras of finite type over a perfect field k of characteristic p. Every p-torsion object in this category is uniquely a direct sum of explicitly given indecomposables. This gives rise to a similar classification of not necessarily p-torsion objects that are either free as commutative algebras or cofree as cocommutative coalgebras. We also completely classify those objects that are indecomposable modulo p.

1. Introduction

Building on work of Hopf [Hop41] and Borel [Bor54], Milnor and Moore's classical paper [MM65] described the the structure of graded, connected, commutative Hopf algebras over a field k *as algebras* and a classification *as Hopf algebras* in the case where they are primitively generated. In the simplest case, where the Hopf algebras in question are abelian (i.e., commutative and cocommutative), this is the end of the story if the characteristic of k is 0. In that case, the primitives and indecomposables coincide and thus H is primitively generated. However, a classification in positive characteristic seemed less approachable. Some years later, Schoeller [Sch70] observed that the category of abelian Hopf algebras over a perfect field k of characteristic p is abelian and thus, by the Freyd-Mitchell embedding theorem, isomorphic to the category of graded modules over a ring, which she described. This is Dieudonné theory in a graded setting.

Although Dieudonné theory has been very successfully applied in various topological and algebraic contexts, it seems that the original question of classification has fallen into a bit of neglect. This paper aims to fix this, to the extent possible.

Let k be a perfect field of characteristic p. All Hopf algebras H considered here are over k, nonnegatively graded, connected (i.e. $H_0 = k$), abelian, and of finite type (i.e. H_i is a finite-dimensional k-vector space for all $i \geq 0$), and we will just refer to them as "abelian Hopf algebras".

If H is a *graded-commutative* Hopf algebra and $\mathrm{char}(k) > 2$ then there is a natural splitting $H = H^{\mathrm{even}} \otimes H^{\mathrm{odd}}$ [Bou96, Prop. A.4] where H^{even} is concentrated in even degrees and H^{odd} is the exterior algebra on the odd-dimensional primitive elements of H. Thus odd Hopf algebras are simply classified the the underlying graded k-vector space of primitive elements. We will therefore concentrate on commutative Hopf algebras (i.e. not graded-commutative) without losing generality.

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The following classification result concerns abelian Hopf algebras H that are p-torsion in the sense that the multiplication-by-p map $[p]$ in the abelian group $\mathrm{End}(H)$ is trivial. In this form, it was proven by Touzé [Tou21, Section 9] in the context of "exponential functors", using results from [CB18] on representations of string algebras:

Theorem 1.1. *Every abelian p-torsion Hopf algebra decomposes uniquely into a tensor product of indecomposable Hopf algebras. An abelian p-torsion Hopf algebra is indecomposable if and only if it is isomorphic to a Hopf algebra $H(r,m,I)$ described below, indexed by a tuple (r,m,I) with $r \in \mathbf{N}_0$, $m \in \mathbf{N}_0 \cup \{\infty\}$, $I \subset \{1,\dots,m\}$ (resp. $I \subset \mathbf{N}$ if $m = \infty$).*

Similarly to the classification of finitely generated modules over PIDs, the uniqueness statement is not (falsely) claiming that the decomposition is natural. It is instead to be understood as saying that the unordered sequence of indices (r,m,I) occurring in the decomposition into indecomposables is the same for every such decomposition.

Explicitly, the Hopf algebra $H(r,m,I)$ is given as an algebra by

$$H(r,m,I) = k[x_0,x_1,\dots,x_m]/\left(x_{i-1}^p - \begin{cases} x_i; & i \in I \\ 0; & i \notin I \end{cases}\right), \quad |x_i| = rp^i,$$

and it has a unique coalgebra structure that makes it indecomposable as a Hopf algebra. Note that $H(r,m,I)$ has dimension 1 in degrees $r, 2r, \dots, (p^{m+1}-1)r$ and is trivial in all other dimensions.

A similarly satisfying classification does not exist for general (non-p-torsion) abelian Hopf algebras. There are indecomposable Hopf algebras of arbitrarily large dimension in any given degree. The situation greatly improves if H is free as an algebra or cofree as a coalgebra:

Theorem 1.2. *Given any tuple (r,m,I), there exists a unique Hopf algebra $H_f(r,m,I)$ which is free as an algebra, and a unique Hopf algebra $H_c(r,m,I)$ which is cofree as a coalgebra, such that $H_f(r,m,I)/[p] \cong H(r,m,I) \cong H_c(r,m,I)/[p]$. Any abelian Hopf algebra which is free as an algebra or cofree as a coalgebra decomposes uniquely into a tensor product of Hopf algebras isomorphic to $H_f(r,m,I)$ (resp. $H_c(r,m,I)$).*

Note that the condition of being free as an algebra is strictly weaker than being free over a coalgebra, or being a projective object in Hopf algebras.

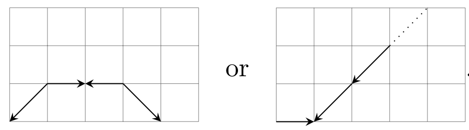
In the general case, we can at least classify all Hopf algebras H with a given indecomposable reduction modulo $[p]$. We define:

Definition. A *basic Hopf graph* is a (finite or infinite) connected chain of arrows with end points in $\mathbf{N}_0 \times \mathbf{N}_0$ with the following properties:

(1) Every arrow goes either from (i,j) to $(i+1,j)$, from $(i+1,j)$ to (i,j), from $(i,j+1)$ to $(i+1,j)$, or from $(i+1,j+1)$ to (i,j).
(2) There is an arrow with end point $(0,0)$.
(3) Every arrow shares each of its end point with exactly one other arrow, unless the end point lies on the x-axis (i.e. is of the form $(i,0)$), in which case it may share its end point with no other arrow.

Denote by $\mathcal{H}$ the set of basic Hopf graphs.

So, a basic Hopf graph looks something like this:



To any basic Hopf graph Γ, we can associate a pair (m_Γ, I_Γ) as in Thm. 1.1, where m_Γ is the horizontal length of Γ and $i \in I_\Gamma$ iff there is a right or right-down pointing arrow from $i-1$ to i in Γ.

In the following theorem, certain Hopf algebras are classified by the behaviour of the Frobenius (pth power) map $F\colon H_i \to H_{pi}$ and its dual, the Verschiebung $V\colon H_{pi} \to H_i$:

Theorem 1.3. *For every basic Hopf graph Γ, there exists a unique Hopf algebra $H = H(\Gamma)$ such that $H/[p] \cong H(0, m_\Gamma, I_\Gamma)$ and with the properties:*

1. *$V\colon H_{p^i} \to H_{p^{i-1}}$ is injective iff Γ contains an arrow from (i,j) to $(i-1,j)$ or an arrow from $(i-1,j)$ to $(i,j-1)$;*
2. *$F\colon H_{p^{i-1}} \to H_{p^i}$ is injective iff Γ contains an arrow from $(i-1,j)$ to (i,j) or an arrow from (i,j) to $(i-1,j-1)$.*

Any Hopf algebra whose mod-$[p]$ reduction is indecomposable is isomorphic to $H(r,\Gamma)$ for exactly one pair (r,Γ). Here $H(r,\Gamma)$ denotes the Hopf algebra $H(\Gamma)$ with all degrees multiplied by r: $H(r,\Gamma)_i = H(\Gamma)_{\frac{i}{r}}$.

Once translated into statements about Dieudonné modules, none of the methods and proofs in this paper use sophisticated techniques.

An early attempt att classifying abelian Hopf algebras was made in [Wra67], but that paper contains some key errors that invalidate its main results, such as that the category of abelian Hopf algebras does not have projectives or injectives (it does), that the homological dimension is 1 (it is 2 [Sch70]), and a classification of indecomposable objects. One such error is the assumption that two Hopf algebras which are isomorphic both as algebras and as coalgebras, must be isomorphic as Hopf algebras [Wra67, 4.9]. A counterexample to this is Kuhn's "cautionary example" [Kuh20, Example 1.13].

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2. p-typical splittings and the Dieudonné equivalence

Continue to let k be a perfect field of characteristic p.

Recall the classical Dieudonné equivalence in the graded context [Sch70]. Denote by $W(k)$ the ring of p-typical Witt vectors of k and by $\mathrm{frob}\colon W(k) \to W(k)$ its Frobenius, i.e. the unique lift of the pth power map on k to a linear map on $W(k)$. Denote by DMod the category of Dieudonné modules, i.e. positively graded $W(k)$-modules together with additive operators $F\colon M_n \to M_{pn}$ and $V\colon M_{pn} \to M_n$, for each n, satisfying

- $FV = VF = p$
- $F(\lambda x) = \mathrm{frob}(\lambda) F(x)$ for $\lambda \in W(k)$, $x \in M$;
- $V(\mathrm{frob}(\lambda) x) = \lambda V(x)$ for $\lambda \in W(k)$, $x \in M$.

By convention, $V(x) = 0$ unless $p \mid |x|$.

Theorem 2.1 ([Sch70]). *There is an equivalence of abelian categories*

$$D\colon \mathrm{Hopf} \to \mathrm{DMod}$$

Let us call a module $M \in \mathrm{DMod}$ *p-typical of type j*, where j is an integer coprime to p, if $M_i = 0$ unless $i = jp^a$ for some $a \geq 0$. Denote the full subcategory of p-typical Dieudonné modules of type j by $\mathrm{DMod}^{(j)}$. Similarly, a graded, connected, abelian Hopf algebra H is called *p-typical* of type j if $H_i = 0$ unless $i = jp^k$ for some k. Denote the full subcategory of p-typical Hopf algebras of type j by $\mathrm{Hopf}^{(j)}$.

Lemma 2.2 ([Sch70, §2]). *A Hopf algebra H is p-typical of type j iff $D(H)$ is a p-typical Dieudonné module of type j. There are splittings of categories*

$$\mathrm{DMod} \simeq \prod_{(j,p)=1} \mathrm{DMod}^{(j)} \quad \text{and} \quad \mathrm{Hopf} \simeq \prod_{(j,p)=1} \mathrm{Hopf}^{(j)}$$

compatible with the Dieudonné functor D. Finally, for any j coprime to p, there is an equivalence of categories

$$\Lambda^{(j)}\,\mathrm{DMod}^{(j)} \simeq \mathrm{Mod}_{\mathcal{R}},$$

where $\mathcal{R} = W(k)[s,t]/(st-p)$ is a $\mathbf{Z}$-graded commutative ring with $|s| = 1$, $|t| = -1$, and $\mathrm{Mod}_{\mathcal{R}}$ denotes the category of nonnegatively graded modules over it.

Proof. The preservation of p-typicality under the Dieudonné functor follows from its construction. The p-typical splitting on the module level is obvious because F and V preserve the type j. The corresponding splitting of Hopf algebras follows.

Finally, construct a functor $\Lambda^{(j)}\colon \mathrm{DMod}^{(j)} \to \mathrm{Mod}_{\mathcal{R}}$ by

$$\Lambda^{(j)}(M)_n = M_{jp^n} \quad \text{with } W(k)\text{-module structure given by } \lambda.x = \lambda^{p^n} x$$

and define $s\colon \Lambda^{(j)}(M)_n \to \Lambda^{(j)}(M)_{n+1}$ by $s(x) = F(x)$ and similarly $t(x) = V(x)$. Since k is perfect, frob is bijective on $W(k)$ and $\Lambda^{(j)}$ is an equivalence of categories. □

Essentially the same results were also obtained by Ravenel [Rav75]. Using them, the classification of abelian Hopf algebras is thus reduced to the classification of modules in $\mathrm{Mod}_{\mathcal{R}}$.

Let

$$D^{(j)}\colon \mathrm{Hopf}^{(j)} \xrightarrow{D} \mathrm{DMod}^{(j)} \xrightarrow{\Lambda^{(j)}} to\, \mathrm{Mod}_{\mathcal{R}} \tag{2.3}$$

denote the composite equivalence between p-typical Hopf algebras of type j and modules over $\mathcal{R}$.

3. The classification of abelian p-torsion Hopf algebras

In this section, we give a self-contained and elementary proof of Thm. 1.1. The result itself is due to Touzé [Tou21].

As an object of an abelian category, any abelian Hopf algebra H has a multiplication-by-p map, classically denoted by $[p]\colon H \to H$. This map is given by the p-fold comultiplication followed by the p-fold multiplication:

$$[p]\colon H \xrightarrow{\Delta^{p-1}} H^{\otimes p} \xrightarrow{\mu^{p-1}} H.$$

In this section, we classify p-torsion Hopf algebras, i. e. abelian Hopf algebras for which $[p] = 0$. From an algebro-geometric point of view, these represent $\mathbf{F}_p$-module schemes rather than abelian group schemes. Since the Dieudonné equivalence is an equivalence of abelian categories, the full subcategory p-torsion Hopf algebras is equivalent to the category of p-torsion Dieudonné modules. By Lemma 2.2, the category of p-typical p-torsion Hopf algebras (of any type) is thus equivalent to the category of nonnegatively graded modules over $\mathcal{R}/(p) \cong k[s,t]/(st)$.

Definition. Let $0 \leq m \leq \infty$ and $I \subseteq \{1,\dots,m\}$ ($I \subseteq \mathbf{N}$ if $m = \infty$). Define an $\mathcal{R}/(p)$-module

$$M(m,I) = k\langle x_0, x_1, \dots, x_m\rangle \quad (k\langle x_0, x_1, \dots\rangle \text{ if } m = \infty)$$

with

$$tx_i = \begin{cases} x_{i-1}; & i \notin I \\ 0; & i \in I \end{cases} \quad \text{and} \quad sx_{i-1} = \begin{cases} x_i; & i \in I \\ 0; & i \notin I. \end{cases}$$

One can graphically depict $M(m,I)$ by a chain of arrows pointing either left or right, depending on whether or not $i \in I$:

$$M(5,\{1,3,4\})\colon x_0 \xrightarrow{s} x_1 \xleftarrow{t} x_2 \xrightarrow{s} x_3 \xrightarrow{s} x_4 \xleftarrow{t} x_5.$$

Obviously, neither the arrow labels nor the "x_i" carry any additional information, so we omit them.

We define a total order on the set of all (m,I) as follows: Let $\alpha(m,I) \in \prod_{i=1}^{\infty}\{-1,0,1\}$ be the sequence given by

$$\alpha(m,I) = \begin{cases} 1; & i \leq m, i \in I \\ -1; & i \leq m, i \notin I \\ 0; & i > m. \end{cases}$$

We then say that $(m,I) \leq (m',I')$ iff $\alpha(m,I) \leq \alpha(m',I')$ in the lexicographic order.

Example 3.1. $\rightarrow\ \geq\ \cdot\ \geq\ \leftarrow\rightarrow\ \geq\ \leftarrow\ \geq\ \leftarrow\leftarrow$

Note that this order relation is not a well-ordering: $\{(m,\emptyset) \mid m \geq 0\}$ does not have a minimal element.

The following result allows us to always split certain submodules of the form $M(m,I)$ off of a $\mathcal{R}/(p)$-module M:

Theorem 3.2. *Let M be a nonnegatively graded module over $\mathcal{R}/(p)$. For some (m,I), suppose we have an injection $i\colon M(m,I) \to M$, and suppose that (m,I) is minimal with this property. Then i has a retraction.*

Proof. We will freely consider $M(m,I)$ as a submodule of M and identify $i(x_j)$ with x_j.

The structure of the argument is as follows. In a first step, we construct an "orthogonal" submodule $V_M(m,I)$, so that $M(m,I) \oplus V_M(m,I) \leq M$. As a second step, we construct a complement M' of $M(m,I) \oplus V_M(m,I)$ as k-vector spaces in a particular way that allows us to show, in the third step, that although M' is not an $\mathcal{R}/(p)$-module complement to $M(m,I) \oplus V_M(m,I)$, $V_M(m,I) \oplus M'$ is an $\mathcal{R}/(p)$-complement to $M(m,I)$.

Step 1. Define the submodule $V_M(m,I) < M$ as follows. For $0 \leq a < b \leq m$, define the truncation $M(m,I)_a^b$ to be the subquotient of $M(m,I)$ generated by

$x_a, \dots, x_{b-1}$ with $tx_a = 0$, $sx_{b-1} = 0$. Set

$$V_M(m,I) = \bigcup_{\substack{a \in I \cup \{0\} \\ a<b\in I \\ f\colon M(m,I)_a^b \to M}} \operatorname{im}(f),$$

or $V_M(m,I) = 0$ if $I = \emptyset$. For every $i \geq 0$, let $a \in I \cup \{0\}$ be the largest index smaller or equal to i. Then we have that

$$V_M(m,I)_i = \bigcup_{\substack{a<b\in I \\ f\colon M(m,I)_a^b \to M}} \operatorname{im}(f)_i$$

is an increasing union of k-vector spaces and hence itself a k-vector space. Hence $V_M(m,I)$ is a sub-$\mathcal{R}/(p)$-modules of M. I claim that $M(m,I) \cap V_M(m,I) = 0$. It suffices to show that $x_i \notin V_M(m,I)_i$ for any $0 \leq i \leq m$, where x_i is the defining generator of $M(m,I)$ in degree i. Suppose to the contrary that there exists a map $f\colon M(m,I)_a^b \to M$ such that $f(x_i') = x_i$, where we denote the defining generators of $M(m,I)_a^b$ by x_i' to avoid confusion. We may choose $a = \max\{j \in I \cup \{0\} \mid j \leq i\}$ and b minimal. Then we must have that $f(x_{b-1}') \neq 0$ and $sx_{b-1}' = 0$ by definition. Thus $(x_0, \cdots, x_{i-1}, x_i', \dots, x_{b-1}')$ defines an injective map $M(b-1, I \cap \{1, \dots, b-1\}) \to M$, contradicting the minimality of (m,I).

Thus we conclude that $N := M(m,I) \oplus V_M(m,I) \subseteq M$.

Step 2. We proceed by induction to construct a k-linear complement M' of N in M in such a way that $M' \oplus V_M(m,I)$ is a $\mathcal{R}/(p)$-module.

Let M_0' be any k-vector space complement of N_0 in M. Suppose M_{i-1}' is constructed. For the construction of M_i', we distinguish between two cases:

(1) $i \notin I$ (including the case $i > m$).

I claim that $t^{-1}M_{i-1}' + N_i = M_i$. Indeed, let $y \in M_i$ be any element. By induction, $ty = x + z$ for $x \in N_{i-1}$ and $z \in M_{i-1}'$. We have that $t\colon N_i \to N_{i-1}$ is surjective by construction of $M(m,I)$ and $V_M(m,I)$. Thus there exists $\tilde{x} \in N_i$ such that $t\tilde{x} = x$. Then we have that $t(y - \tilde{x}) = z \in M_{i-1}'$, hence $y = \tilde{x} + (y - \tilde{x}) \in N_i + t^{-1}M_{i-1}'$, proving the claim. Define M_i' to be a complement of N_i inside $t^{-1}M_{i-1}'$.

(2) $i \in I$.

We define M_i' to be an arbitrary complement of N_i in M_i containing sM_{i-1}'. For this to work, we need to see that $sM_{i-1}' \cap N_i = 0$. So suppose that there exists a $y \in M_{i-1}'$ with $sy = \alpha x_i + z$ with $z \in V_M(m,I)$. Let $f\colon M(m,I)_a^b \to M$ be a map such that $z = f(x_i)$; we may without loss of generality choose $a = i$. Let a^- be the largest element in $I \cup \{0\}$ smaller than a. Then the existence of the map

$$\tilde{f}\colon M(m,I)_{a^-}^b \to M; \quad \tilde{f}(x_j) = \begin{cases} f(j); & j \geq a \\ t^{a-j-1}(y - \alpha x_{i-1}); & a^- \leq j < a. \end{cases}$$

shows that $y - \alpha x_{i-1} \in V_M(m,I)$, and hence $y \in M(m,I) \oplus V_M(m,I)$, a contradiction.

Step 3. It remains to show that $V_M(m,I) \oplus M'$ is closed under multiplication with s and t. Since $V_M(m,I)$ already is, we only have to show this for elements of M'. Let $y \in M_i'$ and $ty = \alpha x_{i-1} + z$, where $x_{i-1} \in M(m,I)_{i-1}$ is the defining generator and $z \in V_M(m,I) \oplus M'$. To show $\alpha = 0$, we again distinguish cases:

(1) $i > m+1$. There is nothing to show since $M(m,I)_{i-1} = 0$.
(2) $i \le m$, $i \in I$. Then $0 = sty = \alpha s x_{i-1} + sz = \alpha x_i + sz$. Since tz and x_i are linearly independent, $\alpha = 0$.
(3) $i \le m+1$, $i \notin I$. Then $\alpha = 0$ because $ty \in M'_{i-1}$ by construction.

Similarly, for $sy = \alpha x_{i+1} + z$, we have the cases

(1) $i \ge m$. There is nothing to show since $M(m,I)_i = 0$.
(2) $i+1 \notin I$. Then $0 = tsy = \alpha t x_{i+1} + tz = \alpha x_i + tz$. Since tz and x_i are linearly independent, $\alpha = 0$.
(3) $i+1 \in I$. By construction of M'_{i+1}, $F(y) \in M'$ and hence $\alpha = 0$.

□

Before continuing to study the $\mathcal{R}$-modules $M(m,I)$, we need a technical lemma about inverse limits of zigzag diagrams.

Definition. For $0 \le n \le \infty$, let I_n be the poset category with objects a_i, b_i for $0 \le i < n$ and $a_i < b_i$ for all $i \le m$, $a_{i+1} < b_i$ for all $i < m$.

We say that a functor $F\colon I_n \to \mathrm{Mod}_k$ into vector spaces *has a zeroless limit* if there exists an element $(x_i, y_i) \in \lim F \subseteq \prod_{i=0}^{n-1} F(a_i) \times \prod_{i=0}^{n-1} F(b_i)$ such that for all i, $x_i \neq 0$ and $y_i \neq 0$. (The second inequality implies the first by functoriality.)

Lemma 3.3. *Let $F\colon I_\infty \to \mathrm{mod}_k$ be a functor into finite-dimensional k-vector spaces and denote its restriction to I_n by F_n. Then F has a zeroless limit if and only if for every $n \ge 0$, F_n has a zeroless limit.*

Proof. If $(x_i, y_i) \in \lim F$ with $x_i, y_i \neq 0$ exists, it is clear that $(x_i, y_i)_{0 \le i < n} \in \lim F_n$, so this direction is clear.

For the other direction, write $M_n = F(a_n)$, $N_n = F(b_n)$, $i_n\colon M_n \to N_n$ for $F(a_n \to b_n)$, and $p_n\colon M_{n+1} \to N_n$ for $F(a_{n+1} \to b_n)$. Conversely, given M_n, N_n with maps as above, write $F_n(M,N)$ for the functor on I associated with these vector spaces. Note that $G_n(M,N) = \lim F_n(M,N)$ form an inverse system by restriction of functors and $\lim F(M,N) \cong \lim_n G_n(M,N)$ naturally.

We first consider the case where all i_n are injective and all p_n are surjective. Since $M_0 \neq 0$ because $\lim F_0 \neq 0$, we can choose $x_0 \in M_0$ and then, inductively, $y_n = i_n(x_n)$ and $x_{n+1} \in p_n^{-1}(y_n)$ arbitrarily. Since i_n are injective, this results in nonzero x_n and y_n.

Next, consider the case where all p_n are surjective, but i_n are arbitrary. For any n, let K_n be the colimit

$$K_n = \operatorname{colim}(i_n^{-1}(0) \hookrightarrow i_n^{-1} p_n i_{n+1}^{-1}(0) \hookrightarrow i_n^{-1} p_n i_{n+1}^{-1} p_{n+1} i_{n+2}^{-1}(0) \hookrightarrow \cdots) \le M_n$$

and

$$L_n = p_n^{-1} K_n \le N_n.$$

Since the M_n are finite-dimensional, the ascending chains in the colimits have to stabilize at a finite stage. For this reason, the inverse system $G_n(K,L)$ is Mittag–Leffler.

By the assumption on F_n for finite n, K_n is a proper subspace of M_n. Consider the short exact sequence of functors

$$0 \to F(K,L) \to F(M,N) \to F(M/K, L/N) \to 0,$$

resulting in a short exact sequence

$$0 \to G_n(K,L) \to G_n(M,N) \to \lim G_n(M/K, N/L) \to 0$$

and hence an exact sequence

$$\lim_n G_n(M,N) \to \lim_n G_n(M/K,N/L) \to \lim^1_n G_n(K,L).$$

Since the $\lim^1$-term is zero by the Mittag-Leffler property, the first map is surjective. By the previous case, $\lim_n G(M/K,N/L) = \lim F(M/K,N/L)$ contains an element (x_n,y_n) with all x_n, y_n nonzero, which can be lifted to $\lim F(M,N)$.

Finally, consider the general case, where p_n and i_n are arbitrary. Then for every n, there is an intersection of a decreasing sequence

$$B_n = \lim(i_n(M_n) \supseteq i_n p^{n-1} i_{n-1} M_{n-1} \supseteq \cdots) \leq N_n$$

and

$$A_{n+1} = p^n(B_n) \leq M_{n+1}; A_0 = M_0.$$

These sequences become stationary by the finite dimensionality of the M_n and N_n, or simply because of the nonnegative grading, and A_n and B_n are nontrivial by the assumption on F_n. Moreover, $p_n\colon A_{n+1} \to B_n$ is surjective. By the previous case, $\lim F(A,B)$ contains an element with nontrivial components, whose inclusion into $\lim F(M,N)$ still satisfies the same property. □

Lemma 3.4. *Let $M \in \mathrm{Mod}_{\mathcal{R}/(p)}$ be of finite type with $M_0 \neq 0$. Then the set*

$$S = \{(m,I) \mid \text{ there exists an injection } M(m,I) \hookrightarrow M\}$$

has a minimal element.

Proof. I claim that every descending chain in S has a lower bound in S. Indeed, suppose that $(m_0,I_0) > (m_1,I_1) > \cdots$ is such a chain. Since there are only finitely many pairs (m,I) for fixed m, we are done if the set of m_i is bounded. Thus suppose that $m_i \to \infty$ and for each $m \geq 0$, all but finitely many $I_i \cap \{1,\dots,m\}$ are equal. Thus by passing to a subchain, we may assume that $m_i \leq m_{i+1}$ and $I_{i+1} \cap \{1,\dots,m_i\} = I_i$. Let $I = I_0 \cup I_1 \cup \cdots$. I claim that $(\infty, I) \in S$.

Inductively construct a sequence m_j of nonnegative integers as follows. Let $m_0 = 0$. Given m_j, let $m_{j+1} = \begin{cases} j+1; & j+1 \in I; \\ j; & j+1 \notin I \end{cases}$. There are maps $i_j\colon M_j \to M_{m_j}$ given by

$$i_j(x) = \begin{cases} sx; & j+1 \in I; \\ x; & j+1 \notin I \end{cases}$$

and $p_j\colon M_{j+1} \to M_{m_j}$ given by

$$p_j(x) = \begin{cases} tx; & j+1 \notin I; \\ x; & j+1 \in I. \end{cases}$$

Note that an injective map $M(\infty,I) \to M$ correspond exactly to a sequence of elements $(x_j \in M_j - \{0\}, y_j \in M_{m_j} - \{0\})$ such that $i_j(x_j) = y_j$ and $p_j(x_{j+1}) = y_j$, i.e. to an element of $\lim F(M,N)$ with nonvanishing components, in the notation of Lemma 3.3. By assumption, there exist elements with nonvanishing components in $\lim F_n(M,N)$ for each n, hence the result follows from the lemma. □

Remark 3.5. For finite k, this proof can be greatly simplified by using that the sets $\lim F_n(M,N) - \{0\}$ are finite and nonempty and thus have a nonempty limit.

Corollary 3.6. *An $\mathcal{R}/(p)$-module of finite type is indecomposable if and only if it is isomorphic to a suspension of some $M(m,I)$.*

Proof. By construction, $M(m,I)$ is indecomposable. Given any nontrivial $\mathcal{R}/(p)$-module M of connectivity $s \geq -1$, the desuspension $\Sigma^{-s-1}M$ has a direct summand isomorphic to $M(m,I)$ by Lemma 3.4, hence $\Sigma^{s+1}M(m,I)$ is a direct summand of M. In particular, if M is indecomposable, $M \cong \Sigma^{s+1}M(m,I)$. □

By induction, we obtain:

Corollary 3.7. *Let M be a nonnegatively graded module over $\mathcal{R}/(p)$ of finite type. Then M is isomorphic to a direct sum of suspensions of indecomposable modules of the form $M(m,I)$ defined above.* □

We next address the question of uniqueness of the decomposition of Cor. 3.7

Lemma 3.8. *There exists a map $f\colon M(m,I) \to M(m',I')$ with $f(x_0) \neq 0$ if and only if $(m,I) \geq (m',I')$.*

Proof. First note that $\operatorname{End}(M(m,I)) \cong k$, given by multiples of the identity map.

If $(m,I) \neq (m',I')$ then there is an index $d \leq \min(m,m')$ of "last equality", i.e. maximal with the property that $d \leq \min(m,m')$ and $I \cap \{1,\dots,d\} = I' \cap \{1,\dots,d\}$.

One of the following cases applies when $(m,I) > (m',I')$:

(1) $d = m < m'$ and $m+1 \notin I'$. Then $(m,I) > (m',I')$ and the standard inclusion $x_i \mapsto x_i'$ is a map of $\mathcal{R}/(p)$-modules.
(2) $m+1 \in I$ and ($d = m' < m$ or $m+1 \notin I'$). In this case, the identity map up to degree m can be extended by 0 in higher degrees.

Similarly, when $(m,I) < (m',I')$, one of the following cases applies:

(1) $d = m' < m$ and $m+1 \notin I$. Then no nontrivial self-map $M(m',I')$ can be extended to $M(m,I)$.
(2) ($d = m < m'$ or $m+1 \notin I$) and $m+1 \in I'$. Then $(m,I) < (m',I')$ and no nontrivial map $M(m,I) \to M(m,I)$ can be extended to $M(m',I')$.

□

Corollary 3.9. *Let $M \cong \bigoplus_{i=1}^n \Sigma^{r_i} M(m_i,I_i) \xrightarrow[\cong]{\phi} \bigoplus_{j=1}^m \Sigma^{r_j'} M(m_j',I_j')$. Then $m = n$ and the unordered sequences (r_i,m_i,I_i) and (r_j',m_j',I_j') are equal.*

Proof. Without loss of generality, assume that $M_0 \neq 0$ and that (m_1,I_1) is the minimal element such that $M(m_i,I_i)$ injects into M. Consider the restriction of the isomorphism ϕ to $M(m_1,I_1)$,

$$M(m_1,I_1) \to \bigoplus_{j=1}^m \Sigma^{r_j'} M(m_j',I_j').$$

Since every $(m_j',I_j') \geq (m_1,I_1)$ and by Lemma 3.8, every map $M(m_1,I_1) \to M(m_j',I_j')$ is zero in degree 0 if $(m_j',I_j') > (m_1,I_1)$, there must be an index j such that $(m_1,I_1) = (m_j',I_j')$. The proof is finished by induction. □

With this, all ingredients are in place to prove the first main theorem.

Proof of Thm. 1.1. Let $r = jp^{r'}$ with $p \nmid j$.

Define $H(r,m,I) = D^{(j)}\Sigma^{r'}M(m,I)$, using the p-typical Dieudonné equivalence (2.3). By Lemma 2.2, every abelian Hopf algebra splits naturally and uniquely into a tensor product of p-typical parts of various types, so it suffices to show the theorem for p-typical Hopf algebras of type j. The existence of the splitting follows from Cor. 3.7, and the uniqueness from Cor. 3.9.

It suffices to show that the characterization of $H(r,m,I)$ in the introduction is correct. Let H be any abelian Hopf algebra isomorphic as algebras to

$$k[x_0,x_1,\dots,x_m]/\left(x_{i-1}^p - \begin{cases} x_i; & i\in I \\ 0; & i\notin I\end{cases}\right), \quad |x_i| = rp^i.$$

Then H is j-typical, and $M = D^{(j)}(H) \cong \langle x_0,x_1,\dots,x_m\rangle$ with $s(x_{i-1} = x_i)$ iff $i \in I$. In order for H, and hence M, to be indecomposable, it is necessary that $tx_i = x_{i-1}$ iff $i \notin I$, because otherwise, M would split as $\langle x_0,\dots,x_{i-1}\rangle \oplus \langle x_i,\dots,x_m\rangle$. Thus $M \cong \Sigma^{r'} M(m,I)$. □

Example 3.10. Primitively generated Hopf algebras are Hopf algebras for which the canonical map $PH \to \tilde{H} \to QH$ from primitives to indecomposables is surjective. For the associated $\mathcal{R}$-modules M, this translates to the map ${}_tM \to M/sM$ being surjective, where ${}_tM = \{m \in M \mid tm = 0\}$. Thus for each $m \in M_i$, there exists an $m' \in M_{i-1}$ such that $t(m + sm') = 0$. By induction, $tm' = 0$ and hence $tm = 0$. This shows that primitively generated Hopf algebras have trivial Verschiebung, and in particular are p-torsion. Thus they split into copies of $H(r,m,I)$. The Hopf algebra $H(r,m,I)$ is primitively generated exactly if $I = \{1,\dots,m\}$ (resp. $I = \mathbf{N}$ when $m = \infty$), recovering the abelian case of the classification in [MM65, Thm. 7.16]. Finite type is actually not a necessary assumption in this case because *any* connected $\mathbf{F}_p[s]$-module splits into a sum of cyclic modules [Web85].

4. Hopf algebras that are free as algebras or cofree as coalgebras

A classification of general abelian Hopf algebras seems unfeasible; we will give some hopefully illuminating examples of the complexity of the problem in the next section. However, we can classify the Hopf algebras that reduce to the Hopf algebras $H(m,I)$ of Section 3 and use this to classify all Hopf algebras that are either free as algebras or cofree as coalgebras.

Note that a Hopf algebra that is free on a coalgebra is also free as an algebra, but not vice versa. Similarly, a Hopf algebra that is cofree on an algebra is also cofree as a coalgebra, but not vice versa.

Definition. An abelian Hopf algebra H is called *basic* if its mod-p reduction $H/[p]$ is indecomposable and $H_1 \neq 0$.

By Thm. 1.1, H is thus basic iff $H/[p] \cong H(0,m,I)$ for some $m \geq 0$ and $I \subseteq \{1,\dots,m\}$.

Recall the graphical model of "basic Hopf graphs" from the introduction, which will be used for indexing basic Hopf algebras.

Projection to the x-axis associates to each basic Hopf graph Γ a pair (m_Γ, I_Γ), where m is the maximal x-coordinate of an arrow end point (possibly ∞) and $i \in I$ iff there is an arrow $(i-1,j)$ to (i,j') for some j, j'. Moreover, each Hopf graph Γ comes with a vector v_Γ of y-coordinates of length $m_\Gamma + 1$.

Conversely, a pair (m,I) and a vector v of length $m+1$ gives rise to a unique basic Hopf graph iff

(1) $v_0 = 0$
(2) if $i \in I$ then $v_i \in \{v_{i-1} - 1, v_{i-1}\}$;
(3) if $i \notin I$ then $v_i \in \{v_{i-1}, v_{i-1} + 1\}$;
(4) if $m < \infty$ then $v_m = 0$.

Fig. 4.1 and 4.2 show the two examples of basic Hopf graphs from the introduction together with data m, I, v explained above.

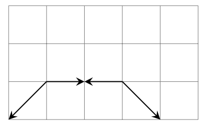


Figure 4.1. $m_\Gamma = 4$, $I_\Gamma = \{2, 4\}$, $v_\Gamma = (0, 1, 1, 1, 0)$

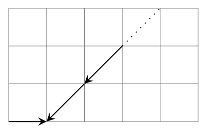


Figure 4.2. $m_\Gamma = \infty$, $I_\Gamma = \{1\}$, $v_\Gamma = (0, 0, 1, 2, 3, \dots)$

Definition. Let Γ be a basic Hopf graph. Define an $\mathcal{R}$-module $M(\Gamma)$ as follows:

$$M(\Gamma)_i = \begin{cases} W(k)/p^{(v_\Gamma)_i+1}\langle x_i\rangle; & i \leq m_\Gamma \\ 0; & i > m_\Gamma \end{cases}$$

together with s- and t-multiplications given, for $1 \leq i \leq m$, by

$$sx_{i-1} = \begin{cases} x_i; & i \in I_\Gamma \\ px_i; & i \notin I_\Gamma \end{cases}; \quad tx_i = \begin{cases} px_{i-1}; & i \in I_\Gamma \\ x_{i-1}; & i \notin I_\Gamma. \end{cases} \tag{4.3}$$

We call a $\mathcal{R}$-module M *basic* if its associated Hopf algebra $(D^{(j)})^{-1}(M)$ is basic, or equivalently, by Thm. 1.1, if $M/(p) \cong M(m, I)$ for some (m, I).

Theorem 4.4. *An $\mathcal{R}$-module M of finite type is a basic $\mathcal{R}$-module with $M/(p) \cong M(m, I)$ if and only if $M \cong M(\Gamma)$ for some basic Hopf graph Γ with $(m_\Gamma, I_\Gamma) = (m, I)$. Moreover, M is uniquely determined by the properties*

(1) *$t\colon M_i \to M_{i-1}$ is injective iff Γ contains an arrow from (i, j) to $(i-1, j)$ or an arrow from $(i-1, j)$ to $(i, j-1)$, and*
(2) *$s\colon M_{i-1} \to M_i$ is injective iff Γ contains an arrow from $(i-1, j)$ to (i, j) or an arrow from (i, j) to $(i-1, j-1)$.*

Proof. It is evident from the definition that $M(\Gamma)/(p) \cong M(m_\Gamma, I_\Gamma)$, so the "only if" direction is clear.

Let M be a basic $\mathcal{R}$-module with $M/(p) \cong M(m, I)$. Because M is of finite type, $M_i = W_{v_i+1}(k)$ for some v_i for $i \leq m$ and $M_i = 0$ for $i > m$. Note that the condition $p = st = ts$ implies that $v_i \in \{v_{i-1} - 1, v_i, v_{i+1}\}$ for all $i \geq 0$.

We inductively construct lifts $\tilde{x}_i \in M_i$ of the defining generators $x_i \in M(m, I)_i$ and show that the v_i satisfy the conditions for being v_Γ of a basic Hopf graph Γ with $(m, I) = (m_\Gamma, I_\Gamma$. Since $tM_0 = 0$ for dimensional reasons, $pM_0 = 0$ as well, so $M_0 = k$ and $v_0 = 0$. Let $\tilde{x}_0$ be the unique lift of x_0.

Suppose now that $\tilde{x}_0, \dots, x_{i-1}$ are constructed satisfying (4.3). If $i \in I$, define $\tilde{x}_i = s(x_{i-1})$. Then $\tilde{x}_i$ is a lift of x_i and hence a generator, and we have that

$t\tilde{x}_i = tsx_{i-1} = px_{i-1}$, so (4.3) is satisfied for the index i. Since $s\colon M_{i-1} \to M_i$ is an surjection (it maps a generator to a generator), we cannot have $v_i = v_{i-1}+1$.

Conversely, if $i \notin I$, but $i \leq m$, $t\colon M_i/(p) \to M_{i-1}/(p)$ is bijective, hence $t\colon M_i \to M_{i-1}$ is surjective. Let $\tilde{x}_i$ be any preimage of $\tilde{x}_{i-1}$ under t. Then $\tilde{x}_i$ is a lift of x_i and hence a generator. Again, $v_i = v_{i-1}-1$ is not possible because of the surjectivity of $t\colon M_i \to M_{i-1}$. □

Proof of Thm. 1.3. Given Γ, define $H(\Gamma) = (D^{(1)})^{-1}(M(\Gamma)$, using the p-typical Dieudonné functor of type 1 from (2.3). By construction, $H(\Gamma)/[p] \cong H(0, m_\Gamma, I_\Gamma)$. For any p-typical Hopf algebra H of type 1 with $M = D^{(1)}(H)$, we have that

$$D^{(1)}(F\colon H_{p^i} \to H_{p^{i+1}}) = s\colon M_i \to M_{i+1}$$

and

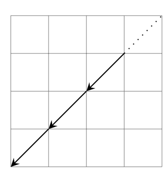
$$D^{(1)}(V\colon H_{p^{i+1}} \to H_{p^i}) = t\colon M_{i+1} \to M_i$$

The claimed classification of basic Hopf algebras thus follows directly from Thm. 4.4. Moreover, let H be any Hopf algebra whose mod-p reduction is indecomposable. Let r be the smallest positive integer such that $H_r \neq 0$. Then by Thm. 1.1, H is concentrated in degrees rp^j. Denote by H' the regraded Hopf algebra with $H'_j = H_{rp^j}$ and $M' = D^{(1)}(H')$ satisfies $M'_0 \neq 0$, $M = \Sigma^{r'} M'$. Thus $M' = M(\Gamma)$ is basic, and $H \cong H(r, \Gamma)$. □

It seems that most abelian Hopf algebras one encounters "in nature" are tensor products of basic ones. We will not attempt to support this claim with many examples, but one important one is the following:

Example 4.5. Consider the Hopf algebra $H = H^*(BU; k) \cong k[c_1, c_2, \dots]$, where BU is the classifying space of the infinite unitary group and c_i are the universal Chern classes in degree $2i$. By dimensional considerations, H splits into a tensor product of one p-typical part of type j for each $p \nmid j$. The type-1 part Λ_p is the Hopf algebra representing the p-typical Witt vector functor. It is basic: $\Lambda_p \cong H(\Gamma)$ for

Γ:



Lemma 4.6. *An abelian, connected, graded Hopf algebra H of finite type is free as an algebra iff the Frobenius F on DH is injective. It is cofree as a coalgebra iff the Verschiebung V on DH is surjective.*

Proof. The Frobenius map of H is a Hopf algebra morphism $F\colon H \to H(1)$ inducing the maps $F\colon DH \to DH(1)$ of Dieudonné modules. If H is free as an algebra, F will therefore be injective. Conversely, by the classification of algebra stuctures on connected, graded Hopf algebras [MM65], a Hopf algebra that is not free as an algebra has a noninjective Frobenius. The argument for the Verschiebung is analogous. □

This gives a simple characterization of when an $\mathcal{R}$-module corresponds to a free resp. cofree Hopf algebra:

Corollary 4.7. *The Hopf algebra $H(\Gamma)$*

- *free as an algebra if Γ consists of horizontal right arrows and diagonal left arrows only;*
- *cofree as a coalgebra if Γ consists of horizontal left arrows and diagonal left arrows only.*

Thus the only basic Hopf algebra that is both free as an algebra and cofree as a coalgebra is Λ_p of Example 4.5.

Theorem 4.8. *Let H be an abelian Hopf algebra. If H is free as an algebra or free as a coalgebra then H is a tensor product of basic Hopf algebras.*

Proof. Without loss of generality, suppose that H is p-typical and M its associated $\mathcal{R}$-module. Choose a decomposition

$$\bigoplus_{i=1}^{n} \bar{\gamma}_i \colon \bigoplus_{i=1}^{n} \Sigma^{r_i} M(m_i, I_i) \xrightarrow{\cong} M/(p)$$

as in Thm. 1.1. By Thm. 1.3, there are basic Hopf graphs Γ_i and maps

$$\gamma_i \colon \Sigma^{r_i} M(\Gamma_i) \to M$$

such that $\bigoplus_{i=1}^{n} \gamma_i$ is a surjective map and $\bar{\gamma}_i = \gamma_i/(p)$.

We denote the defining generator of $\Sigma^{r_i} M(\Gamma_i)$ in degree j by $x_{i,j}$ and identify it with its image in M under γ_i. To see that $\bigoplus_{i=1}^{n} \gamma_i$ is an injection, we need to show that

$$\sum_{i=1}^{n} \alpha_i x_{i,j} = 0 \Longrightarrow \alpha_i x_{i,j} = 0 \text{ for all } \alpha_i \in k. \tag{4.9}$$

We show this by induction in the degree j. For $j = 0$, $M(\Gamma_i)_0 \cong M(m_i, I_i)_0$ is p-torsion, and by definition, the nonzero $x_{i,0}$ are linearly independent. Assume that (4.9) holds for $j-1$. Since modulo p, the $x_{i,j}$ are linearly independent, we have that $p \mid \alpha_i$ for all i. But $p = st$, and since s is injective by assumption, we have that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p} t x_{i,j} = 0.$$

We have that $t x_{i,j} \in \{x_{i,j-1}, p x_{i,j-1}\}$, so by induction, $\frac{\alpha_i}{p} t x_{i,j} = 0$ and hence $s(\frac{\alpha_i}{p} t x_{i,j}) = \alpha_i x_{i,j} = 0$ for all i.

This completes the proof that $\bigoplus_{i=1}^{n} \gamma_i$ is an isomorphism.

The result for Hopf algebras which are cofree as coalgebras follows from dualization. □

Proof of Thm. 1.2. Given Thm. 4.8, the only thing that remains is to classify those basic Hopf algebra of mod-p type (m, I) which are free as an algebras (the other case being dual). This corresponds to s being injective. The characterization of basic $\mathcal{R}$-modules with injective s from Thm. 4.4 shows that there is exactly one basic Hopf graph Γ with this property for any given (m, I). □

We will now compare the property of H being free as an algebra to two stronger conditions. The first condition is being a projective object. The second condition is, morally, to be free over a graded, connected coalgebra. However, this notion is unnecessarily restrictive and is incompatible with the property of being p-typical –

a p-typical Hopf algebra can never be free in this sense. Instead, we consider the property of being free over a *p-polar* graded coalgebra [Bau22a, Bau22b].

Definition. A *graded p-polar k-coalgebra* is a graded vector space C together with a k-linear map

$$\Delta\colon C_{pn} \to \Big(C_n \otimes \cdots \otimes C_n\Big)^{\Sigma_p}$$

which, with this structure, is a retract of a graded k-coalgebra.

The dual definition of this notion was given in [Bau22b], along with the dual version of the following theorem:

Theorem 4.10. *The functor with associates to a graded k-coalgebra the free graded Hopf algebra over it factors naturally through the category of p-polar coalgebras.*

In other words, there exists a free Hopf algebra functor on p-polar coalgebras, and the usual free Hopf algebra functor on coalgebras really only depends on the underlying p-polar coalgebra structure. This free functor sends p-typical p-polar coalgebras (in the obvious meaning) to p-typical Hopf algebras of the same type.

Proposition 4.11. *Let $H = H(r, \Gamma)$ be an irreducible abelian Hopf algebra. Then*

(1) *H is cofree on a graded, connected p-polar coalgebra iff Γ has $m = \infty$, $I_H = \{i \mid i \geq n\}$, $(v_H)_i = \min(i, n)$ for $0 \leq n \leq \infty$ (Figs. 4.12 and 4.13)*

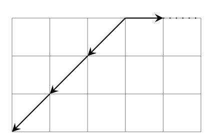


FIGURE 4.12. $m = \infty$, $I = \{4, 5, \ldots\}$, $v_H = (0, 1, 2, 3, 3, 3, \ldots)$

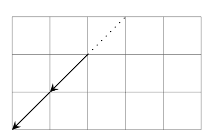


FIGURE 4.13. $m = \infty$, $I = \emptyset$, $v_H = \{0, 1, 2, 3 \ldots\}$

(2) *H is projective if, in addition, $p \nmid r$.*

Proof. (1) In [Kuh20], Kuhn showed that for every finitely generated $\mathcal{R}/(s)$-module M, there exists a Hopf algebra $H(M)$ with indecomposables M, and in [Bau22b], the author showed that $H(M)$ is free on a p-polar coalgebra and (noncanonically) unique with this property. Moreover, the construction H sends direct sums to tensor products. Indecomposable $\mathcal{R}/(s)$-modules are suspensions of modules of the form $k[t]/(t^{n+1})$, with corresponding Hopf algebra isomorphic to $H(\Gamma_n)$ with Γ_n as displayed in Fig. 4.12 for $n = 4$, or of the form $k[t, t^{-1}]/k[t]$, with corresponding Hopf algebra $H(\Gamma_\infty)$ as displayed in Fig. 4.13.

(2) This is well-known, e.g. by [Sch70, Théorème 3.2].

□

5. Bestiary

In this section, we collect a few examples that show that our results are sharp in many respects.

Example 5.1. This example is to show that the grading is crucial. Suppose H is an ungraded abelian Hopf algebra over a perfect field k. By [Fon77], H naturally splits into $H^u \otimes H^m$, where H^u denotes the unipotent part (also known as conilpotent) and H^m is the part of multiplicative type. The latter is classified by an abelian group A with a continuous action of the absolute Galois group Γ of k; if the Γ-action is trivial then the associated Hopf algebra is just the group ring $k[A]$. So a classicifation of H^m, under some finiteness assumptions, is feasible. Note that connected, graded Hopf algebras are automatically unipotent for degree reasons.

Dieudonné theory works as expected for ungraded Hopf algebras, and unipotent Hopf algebras correspond to Dieudonné modules on which V acts nilpotently. Let W be an n-dimensional $\mathbf{F}_p$-vector space, $\phi\colon W \to W$ an endomorphism, and construct a Dieudonné module $M_\phi = V \oplus V$ with $V(x,y) = (0,x)$ and $F(x,y) = (0,\phi(x))$. Then it is easy to see that $M_\phi \cong M_\psi$ if and only if ϕ and ψ are conjugate, and M_ϕ is decomposable iff W has a ϕ-invariant direct sum decomposition. So already this finite-dimensional example shows that the moduli of indecomposable modules is rather large and, in an imprecise meaning, positive-dimensional (it grows with the size of k).

Example 5.2. p-torsion connected abelian Hopf algebras that are not of finite type do not need to decompose into indecomposables. Let

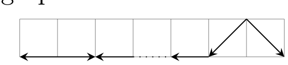
$$M_{p^n} = \prod_{i \geq n} k; \quad t(x_i, x_{i+1}, \dots) = (0, x_i, x_{i+1}, \dots).$$

This $\mathcal{R}/(p)$- module is not a sum of cyclic modules. It is also not a sum of finitely generated modules, or even a sum of indecomposable modules.

It is possible that countable dimension suffices, cf. [Web85, Theorem 2].

Example 5.3. In this example, we will construct arbitrarily large $\mathcal{R}$-modules (and hence Hopf algebras), showing in particular that non-p-torsion Hopf algebras do not decompose into tensor products of basic Hopf algebras.

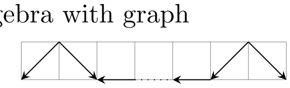
Let Γ_i be the basic Hopf graph



with $m = i + 4$.

One checks readily that there are no nontrivial maps $M(\Gamma_i) \to M(\Gamma_j)$ unless $i = j$.

There is a unique nontrivial extension $0 \to \Sigma k \to M_i \to M(\Gamma_i) \to 0$ for each i, resulting in a basic Hopf algebra with graph



Thus $\operatorname{Ext}^1(M(\Gamma_i), \Sigma k) \cong k\langle \alpha_i \rangle$. For any $N > 0$, the class

$$(\alpha_1, \dots, \alpha_N) \in \operatorname{Ext}^1\Big(\bigoplus_{i=1}^N M(\Gamma_i), \Sigma k\Big) \cong \bigoplus_{i=1}^N \operatorname{Ext}^1(M(\Gamma_i), \Sigma k)$$

thus defines an $\mathcal{R}$-module M. I claim that this module is indecomposable. We use the following result, which is a special case of [HW06, Thm. 2.3]:

Lemma 5.4. *Let Q be a finitely generated R-module of positive depth, T an indecomposable finitely generated R-module of finite length, and $\alpha \in \operatorname{Ext}^1(Q,T)$. Suppose that for each $f \in \operatorname{End}(Q) - \{0\}$, $f^*\alpha \neq 0$. Then α represents an indecomposable module.*

In our case, $Q = \bigoplus_{i=1}^N M(\Gamma_i)$, $T = \Sigma k$, and $\alpha = (\alpha_1, \ldots, \alpha_k)$. We have that $\operatorname{End}(Q) \cong k^N$ because there are no nontrivial maps $M(\Gamma_i) \to M(\Gamma_j)$ for $i \neq j$. For $f = (f_1, \ldots, f_N) \in \operatorname{End}(Q)$, we have that $f^*\alpha = (f_1\alpha_1, \ldots, f_N\alpha_N)$ and since all α_i are nontrivial, $f^*\alpha \neq 0$.

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