A Hodge decomposition spectral sequence for E_n-homology

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Preliminaries

Throughout this talk we'll consider

- a field k
- the category dgmod of nonnegatively graded differential graded k-modules

Hochschild homology

Let A be an augmented k-algebra with augmentation ϵ.

Definition

Hochschild homology $HH_*(A;k)$ with trivial coefficients is the homology of the complex $C_*^{HH}(A;k)$ with

$$C_n^{HH}(A;k) = A^{\otimes n},$$

$$\begin{aligned} d(a_1 \otimes \ldots \otimes a_n) &= \epsilon(a_1)a_2 \otimes \ldots \otimes a_n \\ &+ \sum_i \pm a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &+ (-1)^n a_1 \otimes \ldots \otimes a_{n-1}\epsilon(a_n). \end{aligned}$$

Hochschild homology

Example

Let $A = S(V)$ be the free commutative k-algebra generated by a vector space V. Then

$$HH_n(A; k) = \Omega^n_{A|k} \otimes_A k$$

with

$$\Omega^1_{A|k} = A \otimes A / 1 \otimes ab - a \otimes b - b \otimes a$$

the symmetric A-bimodule of Kähler differentials,

$$\Omega^n_{A|k} = \Lambda^n_A(\Omega^1_{A|k})$$

its exterior powers. Note that $\Omega^1_{A|k} \otimes_A k \cong \mathrm{Indec}(A)$.

André-Quillen homology

Now let A be commutative.
There is a model structure on the category $\mathrm{sAlg}_{\mathrm{com}}$ of simplicial commutative augmented k-algebras by transfer along the adjunction

$$S\colon \mathrm{skmod} \rightleftarrows \mathrm{sAlg}_{\mathrm{com}} \colon V.$$

Let $c(A) \in \mathrm{sAlg}_{\mathrm{com}}$ be the constant simplicial algebra associated to A. A concrete cofibrant replacement of $c(A)$ is given by

$$n \mapsto (S \circ V)^{\circ n+1}(A),$$

face maps: monad structure of $S \circ V$ and multiplication in A,
degeneracies: units.

André-Quillen homology

Definition

Given a cofibrant replacement $X_\bullet \to A$ of A, the André-Quillen homology of order n of A is

$$AQ_*^{[n]}(A|k;k) = \pi_*(\Omega^n_{X_\bullet|k} \otimes_{X_\bullet} k)$$

The classical Hodge decomposition spectral sequence

Theorem (Quillen 70)

There is a spectral sequence

$$E^2_{p,q} = AQ_p^{[q]}(A|k;k) \Rightarrow HH_{p+q}(A;k).$$

The classical Hodge decomposition spectral sequence

Proof.

For $X \to c(A)$ a cofibrant replacement such that each X_i is free commutative, consider the bicomplex

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] & \\
X_0^{\otimes 2} \arrow[d] & X_1^{\otimes 2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
k & k \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

with columns $C_*^{HH}(X_i; k)$ and rows the Moore complex of $X^{\otimes j}$ calculating $\pi_*(X^{\otimes j})$.

The classical Hodge decomposition spectral sequence

Proof (continued).

- $\xrightarrow{\text{horizontal homology}} C_*^{HH}(A;k)$ in column 0
  $\xrightarrow{\text{vertical homology}}$ spectral sequence converges to $HH_*(A;k)$
- $\xrightarrow{\text{vertical homology}} HH_*(X_q;k) \cong \Omega^*_{X_q|k} \otimes_{X_q} k$
  $\xrightarrow{\text{horizontal homology}} AQ^{[*]}(A|k;k)$

The classical Hodge decomposition in characteristic 0

Corollary

If $char(k) = 0$ the spectral sequence collapses at E^2 and

$$HH_n(A|k;k) = \bigoplus_{p+q=n} AQ_p^{[q]}(A|k;k).$$

Higher order Hochschild homology

Comparing the standard simplicial model for $\mathbb{S}^1$ with the Hochschild complex motivates the definition of higher order Hochschild homology:
For A an augmented commutative k-algebra define

$$\mathcal{L}(A;k)\colon \mathrm{Fin} \to k\text{-mod}$$

by

$$\mathcal{L}(A;k)([s]) = A^{\otimes s},$$

$$\mathcal{L}(A;k)(f\colon [s] \to [t])(a_1 \otimes \ldots \otimes a_s) \\ = \prod_{i \in f^{-1}(0)} \epsilon(a_i) (\prod_{i \in f^{-1}(1)} a_i) \otimes \ldots \otimes (\prod_{i \in f^{-1}(t)} a_i).$$

Higher order Hochschild homology

Definition (Pirashvili)

For $1 \leq n < \infty$ the Hochschild homology of order n of A is given by

$$HH_*^{[n]}(A; k) = \pi_*(\mathcal{L}(A; k) \circ \mathbb{S}^n)$$

for a simplicial n-sphere $\mathbb{S}^n$.

- independent of the model chosen for $\mathbb{S}^n$
- $HH^{[1]}(A; k) \cong HH(A; k)$

The Hodge decomposition for higher order Hochschild homology

Theorem (Pirashvili 00)

There is a spectral sequence

$$E^2_{p,q} = T^{(n)}_{p,q}(A;k) \Rightarrow HH^{[n]}_{p+q}(A;k).$$

$$n \text{ odd: } T^{(n)}_{p,q}(A;k) = \begin{cases} AQ^{[j]}_p(A|k;k), & q = nj, \\ 0, & q \neq nj, \end{cases}$$

$$n \text{ even: } T^{(n)}_{p,q} = \begin{cases} \tilde{AQ}^{[j]}_p(A|k;k), & q = nj, \\ 0, & q \neq nj. \end{cases}$$

The Hodge decomposition for higher order Hochschild homology

Theorem

Let char$(k) = 0$.

- *(Pirashvili 00) The spectral sequences collapses at E^2.*
- *(Richter-Z 14) $\tilde{AQ}_p^{[j]}(A|k;k)$ is a variant of higher André-Quillen homology:*

$$\tilde{AQ}_p^{[j]}(A|k;k) \cong \pi_*(S^j_{X_\bullet}(\Omega^1_{X_\bullet|k}) \otimes_{X_\bullet} k)$$

E_∞-algebras

- E_∞-algebras in topology $\leftrightsquigarrow$ infinite loop spaces (Boardman-Vogt, May)
- Algebraic version: An E_∞-algebra is a dg-module with a product associative and commutative up to "all coherent higher homotopies"

Examples include:

- every differential graded commutative algebra
- singular cochains $C^*(X)$ with the cup product

E_n-homology

There is a variant of higher order Hochschild homology for augmented E_∞-algebras A,

$$E_n\text{-homology } H_*^{E_n}(A;k),$$

taking into account the higher homotopies "up to level n".

- If A is strictly commutative, $H_*^{E_n}(A;k) \cong HH_{*+n}^{[n]}(A;k)$.
- Every E_∞-algebra A is an A_∞-algebra, and $H_*^{E_1}(A;k) \cong HH_{*+1}(A;k)$.

E_n-homology

Compute via the bar construction:

Theorem (Fresse 10,11)

There is a functor

$$B^n\colon E_\infty\text{-alg} \to E_\infty\text{-alg}$$

extending the classical n-fold bar construction for commutative algebras.
Furthermore, for any $A \in E_\infty\text{-alg}$

$$H_*^{E_n}(A;k) \cong H_*(\Sigma^{-n} B^n(A))$$

The double complex

For $X \in$ dgmod denote by $E_\infty(X)$ the free E_∞-algebra generated by X.
$\rightsquigarrow$ free simplicial resolution $X_\bullet = E_\infty^{\circ \bullet + 1} \to c(A)$ of E_∞-algebras.
Consider the double complex

$$C_{p,q} = \Sigma^{-n} B^n(X_p)_q$$

with
$C_{p,q} \to C_{p-1,q}$ induced by the simplicial structure,
$C_{p,q} \to C_{p,q-1}$ the differential of $B^n(X_p)$.

The double complex

$$C_{p,q} = \Sigma^{-n} B^n (X_p)_q :$$

- $\xrightarrow{\text{horizontal homology}} \Sigma^{-n} B^n(A)$ in column 0,
  $\xrightarrow{\text{vertical homology}}$ target $H_*^{E_n}(A; k)$.
- $\xrightarrow{\text{vertical homology}} H_*^{E_n}(X_q; k) \cong H_*^{E_n}(E_\infty^{\circ q+1}(A); k)$

What is E_n-homology of a free E_∞-algebra?

E_n-homology of free E_∞-algebras

- E_∞-algebras are algebras over an operad E_∞ with $H_*(E_\infty) = \mathrm{Com}$.
- There is a monad W_∞ such that

$$H_*(E_\infty(X)) \cong W_\infty(H_*X) \quad \text{(May 76)}$$

reflecting the Com-structure but also homology operations existent in $\mathrm{char}(k) > 0$.

E_n-homology of free E_∞-algebras

Proposition

$$H_*^{E_n}(E_\infty(X); k) \cong \Sigma^{-n} W_\infty(\Sigma^n H_*(X)).$$

Proof.

Can form $B(E_\infty)$ just like $B(A)$. There is a spectral sequence

$$\tilde{E}^2_{p,q} = H_*(B(H_*(E_\infty))) \Rightarrow H_*(B(E_\infty)).$$

(Fresse 11) There is a quasiisomorphism $\Lambda^{-1}\mathrm{Com} \to \Sigma^{-1}B(\mathrm{Com})$.
$\rightsquigarrow$ Find quasiisomorphism $\Lambda^{-n}E_\infty \to \Sigma^{-n}B^n(E_\infty)$ and hence a quasiisomorphism

$$\Sigma^{-n}E_\infty(\Sigma^n X) \to \Sigma^{-n}B^n(E_\infty X).$$

$\square$

The Hodge decomposition spectral sequence

Hence

$$\begin{aligned} & H_*^{E_n}(E_\infty^{\circ q+1}(A); k) && \Rightarrow H_{*+q}^{E_n}(A; k) \\ \cong \quad & \Sigma^{-n} W_\infty(\Sigma^n W_\infty^{\circ q}(H_* A))). \end{aligned}$$

Simplicial structure: $d_1, \ldots, d_q$ are easily understood, need to understand d_0.

The Hodge decomposition spectral sequence

If $\mathrm{char}(k) = 0$, then $W_\infty = S$ and d_0 is induced by the projection $S(M) \to M$.
$\Rightarrow$ for the standard resolution $X_\bullet \to H_*A$

$$E^1_{*,q} \cong \Sigma^{-n} S(\Sigma^n(\Omega^1_{X_q} \otimes_{X_q} k))).$$

We recover:

Theorem (Richter-Z 14)

Let $\mathrm{char}(k) = 0$, let A be an augmented E_∞-algebra. There is a spectral sequence

$$E^2_{p,q} = T^{(n)}_{p+n,q} \Rightarrow H^{E_n}_{p+q}(A; k).$$

Work in progress

Work in progress: Hodge spectral sequence in positive characteristic.

There is a morphism of operads

$$\sigma\colon E_\infty \to \Lambda^{-n} E_\infty$$

such that $\sigma(E_n) = I$. $\rightsquigarrow$ conjecture that in any characteristic

$$E^2_{*,*} = \pi_*(\Sigma^{-n} W_\infty(\Sigma^n \mathrm{Indec}_{W_n}(W_\infty^{\bullet+1}(H_* A)))) \Rightarrow H_*^{E_n}(A; k).$$

Work in progress

Work in progress: Hodge spectral sequence for E_n-homology of E_{n+m} -algebras,
by using an E_m-structure on $B^n(E_{n+m})$ to identify

$$H_*^{E_n}(E_{n+m}X) \cong \Sigma^{-n} W_m(\Sigma^n H_*(X)).$$

References

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