Additional Structures on E_n-Cohomology

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E_n-algebras

Reminder: E_n-algebras:

- The little n-cubes operad $\mathcal{C}_n$ is an operad in Top_* and plays an important role in understanding n-fold loop spaces.
- E_n-operads in dgmod are operads weakly equivalent to the operad formed by chains on little n-cubes.
- E_n-algebras are equipped with a multiplication associative up to all higher homotopies, commutative up to homotopies of a certain level.

We fix an E_n-operad from now on.

Cohomology for algebras over an operad

For every Σ_*-cofibrant operad $\mathcal{O}$ in dgmod :

- There is a semi-model structure on $\mathcal{O}$-Alg $\rightsquigarrow$ can define the $\mathcal{O}$-cohomology of an $\mathcal{O}$-algebra A with coefficients in a representation M of A as

$$H^*_{\mathcal{O}}(A; M) = H^*(\mathrm{Der}_{\mathcal{O}}(Q_A, M))$$

  for a cofibrant replacement $Q_A \to A$.

E_n-homology and -cohomology

Certain cases yield well known notions of (co)homology:

Example

- E_1-(co)homology is Hochschild (co)homology.
- E_∞-(co)homology is Γ-(co)homology.
- E_n-homology of a commutative algebra is higher order Hochschild homology.

The classical bar complex

Recall that for an augmented dga algebra A the reduced bar construction $BA = (T^c(\Sigma\overline{A}), \partial)$ is given by

- the tensor coalgebra on the suspension of the augmentation ideal $\overline{A}$
- with differential twisted by utilising the multiplication in A.
- If A is graded commutative, $B(A)$ is a graded commutative dga algebra and we can iterate this construction.

The bar complex for E_n-algebras

Theorem (Fresse 2011)

(i) *There is a functor*

$$B^n \colon E_n\mathrm{-alg} \to \mathrm{dgmod}$$

which coincides with the classical n-fold bar construction if restricted to graded commutative dga algebras.

(ii) *For any sufficiently good E_n-algebra A we have*

$$H_*^{E_n}(A, k) \cong H_*(\Sigma^{-n} B^n A)$$

for $1 \leq n \leq \infty$.

E_n-(co)homology with coefficients via the bar complex

This can be extended to include coefficients:

Theorem (Fresse)

For any nice augmented commutative algebra A and A-module M there is a twisting differential

$$\partial\colon B^n(A) \otimes A \to B^n(A) \otimes A$$

such that

$$H_*^{E_n}(A, M) \cong H_*(\Sigma^{-n} B^n(A) \otimes M, \partial')$$

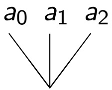
and

$$H^*_{E_n}(A, M) \cong H_*(\mathrm{Hom}_k(\Sigma^{-n} B^n(A), M), \partial'').$$

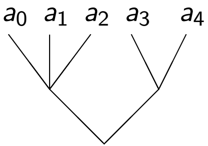
The iterated bar complex and trees

Typical elements in $B^n(A)$ can be visualised as planar fully grown trees with n levels and leaves labeled by elements in $\overline{A}$:

- Typical elements in $B(A)$ are elements in $(\Sigma\overline{A})^{\otimes k}$:



- Typical elements in $B^2(A)$ are elements in $(\Sigma\overline{B(A)})^{\otimes j}$:



- ... and so on.

E_n-cohomology of functors

Definition

Let Epi_n^+ be the category with:

- Objects: planar fully grown trees with n levels,
- Morphisms generated by mimicking the differential in $B^n(A)$ and the twist ∂.

E_n-cohomology of functors

Definition

Let

$$F\colon \mathrm{Epi}_n^{+\,op} \to k{-}\mathrm{mod}$$

be a functor.

The *E_n-cohomology of F* is defined as

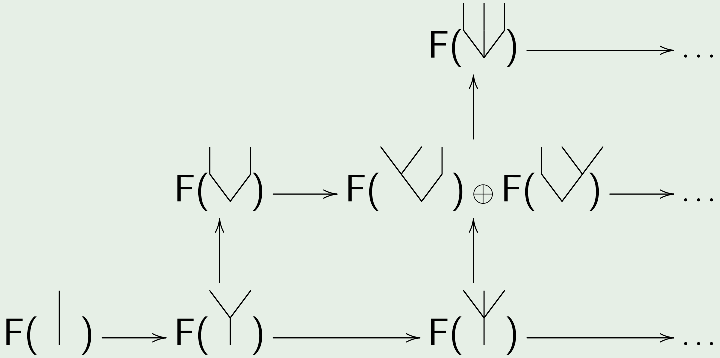
$$H^*_{E_n}(F) = H_*(\mathrm{Tot}\, C^{*,\dots,*}_{E_n}(F))$$

with $C_{E_n}(F)$ an n-fold complex indexed over trees with differentials like in $(\mathrm{Hom}_k(B^n(A), M), \partial')$.

E_n-cohomology of functors

Example

$C_{E_2}(F)$:



E_n-cohomology of functors

Remark

Let A be a commutative algebra and M an A-module. Let

$$\mathcal{L}(A, M)\colon \mathrm{Epi}_n^{+\,op} \to k{-}\mathrm{mod}$$

be the following functor:

- $t \mapsto \mathrm{Hom}_k(A^{\otimes r}, M)$ *for t a tree with r leaves,*
- *morphisms in Epi_n^+ induce multiplication, permutations and the action of A on M according to how they operate on the top leaves.*

Then

$$H^*_{E_n}(\mathcal{L}(A; M)) = H^*_{E_n}(A; M)$$

E_n-cohomology as functor cohomology

Theorem (Livernet-Richter 2011, Z)

Let k be a field. There exists a functor $b\colon \mathrm{Epi}_n^{+\,op} \to k{-}\mathrm{mod}$ such that for all $F\colon \mathrm{Epi}_n^{+\,op} \to k{-}\mathrm{mod}$ we have

$$H^*_{E_n}(F) \cong \mathrm{Ext}_*^{\mathrm{Epi}_n^+}(b, F).$$

Cohomology operations

- Recall: cohomology operations in singular cohomology can be defined by using that singular cohomology is representable.
- In the derived setting one has the Yoneda product: For projective/injective resolutions P_b, I_F

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathrm{Epi}_n^+}(b,b) \otimes \mathrm{Ext}^j_{\mathrm{Epi}_n^+}(b,F) \arrow[d, equal] \\
H_{i+j}(\mathrm{Mor}_{\mathrm{Epi}_n^+}(P_b,b) \otimes \mathrm{Mor}_{\mathrm{Epi}_n^+}(b,I_F)) \arrow[d] \\
H_{i+j}(\mathrm{Mor}_{\mathrm{Epi}_n^+}(P_b,I_F)) = \mathrm{Ext}^{i+j}_{\mathrm{Epi}_n^+}(b,F)
\end{tikzcd}$$

A negative result

Unfortunately no operations arise this way:

Theorem (Z)

We have

$$H^*_{E_n}(b) = \begin{cases} k, & * = 0, \\ 0, & * > 0 \end{cases}$$

for all $1 \leq n < \infty$.

Gerstenhaber structures

The Hochschild cohomology $HH^*(A, A)$ of an associative algebra A has the structure of a Gerstenhaber algebra, consisting of

- an associative graded commutative product
$$\cup \colon HH^*(A, A) \otimes HH^*(A, A) \to HH^*(A, A).$$
- a graded Lie bracket
$$HH^*(A, A) \otimes HH^*(A, A) \to HH^{*-1}(A, A)$$
constructed from the homotopy $\cup_1$ for the homotopy commutativity of $\cup$
- satisfying a Poisson relation.

Gerstenhaber structures

- The operad $H_*(E_2)$ encodes Gerstenhaber algebras.
- Deligne's conjecture: The Gerstenhaber structure on $HH^*(A, A)$ stems from an action of E_2 on $C^*(A, A)$.
- Higher Deligne conjecture: There is an E_{n+1}-action on suitable cochains calculating $H^*_{E_n}(A; A)$ (Hu-Kriz-Voronov, Lurie, Francis, Ginot-Tradler-Zeinalian,...).

Higher cup products

Goal

- *Use the definition of* $H^*_{E_n}(A, A)$ *by a multicomplex indexed over trees to define* $\cup, \cup_1, \ldots, \cup_n$*.*
- *Define a graded Lie bracket on* $H^*_{E_n}(A, A)$ *as the graded commutator of* $\cup_n$*.*

Partial result

- *The product* $\cup$ *"obviously" exists (coalgebra structure on* $B^n(A)$*).*
- *There exists an* $\cup_1$ *on* $C^{*,\ldots,*}_{E_n}(A; A)$*.*

Thanks

Thank you for your attention!

References

Main references:

- Benoit Fresse, *Iterated bar complexes of E-infinity algebras and homology theories*, Alg. Geom. Topol. **11** (2011), 747–838.
- Muriel Livernet, Birgit Richter, *An interpretation of E_n-homology as functor homology,*, Math. Z. **269 (1)** (2011), 193–219.