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Bachelor thesis project proposal

The computation of stationary states of rotating Bose-Einstein condensates

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1. Background and general goal

This project is devoted to solving nonlinear Schrödinger equations with an angular momentum rotation. Such equations occur in the context of rotating superfluids. One example are rotating Bose-Einstein condensates. Such condensates are formed when a dilute gas of Bosons is trapped in a magnetic potential and cooled down to ultra-low temperatures close to absolute zero. It is characterized by the property that the particles can no longer be separated from each other. They lose their identity and behave like one single super-atom, which in particular has a superfluid character. Superfluidity is expressed through a lattice of density singularities (vortices). In this project we aim at simulating realistic scenarios and to study the number of vortices and the patterns that they form. This requires to solve nonlinear eigenvalue problems of the following structure: We seek $u : \mathcal{D} \subset \mathbb{R}^3 \to \mathbb{C}$ that describes the quantum state of the condensate with $\int_{\mathcal{D}} |u(\mathbf{x})|^2 \, d\mathbf{x} = 1$ and a chemical potential (eigenvalue) $\lambda > 0$ such that

$$\begin{aligned} -\frac{1}{2}\triangle u + V\, u + \mathrm{i}\boldsymbol{\Omega} \cdot (\mathbf{x} \times \nabla)\, u + \beta |u|^2 u &= \lambda u \qquad \text{in } \mathcal{D}, \\ u &= 0 \qquad \text{on } \partial\mathcal{D}, \end{aligned}$$

where we denote $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Here, V characterizes the magnetic trapping potential that confines the system (by adjusting V to some trap frequencies) and the nonlinear term $\beta|u|^2 u$ describes the species of the bosons and how they interact with each other. In particular, β depends on the number of bosons, their individual mass and their scattering length. The term $\mathrm{i}\boldsymbol{\Omega} \cdot (\mathbf{x} \times \nabla)\, u$ characterizes the angular rotation of the condensate, where $\boldsymbol{\Omega} \in \mathbb{R}^3$ defines the angular velocity. The operator $\mathbf{L} = (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z) := -\mathrm{i}\, (\mathbf{x} \times \nabla) = \mathbf{x} \times \mathbf{P}$ describes the angular momentum, with $\mathbf{P} = -\mathrm{i}\nabla$ denoting the momentum operator.

2. Tasks

The following list is preliminary and can change upon the students personal interests and preferences.

- Description of the model and its physical background.

- Working out a setup of physical relevance (i.e. scaling and possible choices / parameter constellations for V, β and $\mathbf{\Omega}$).
- Stating possible numerical schemes for solving the eigenvalue problem.
- Implementing one (or more) of the numerical methods.
- Studying (i.e. simulating) the vortex formation and the arising patterns depending on the chosen parameter configurations in the equation.

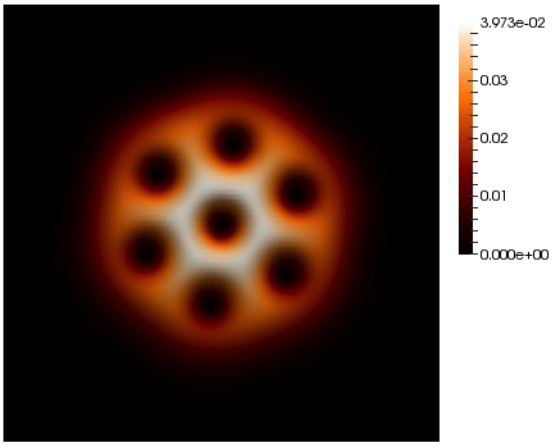

Figure 1: Simulated ground-state state density $|u|^2$ of a rotating Bose-Einstein condensate. The black holes are density singularities.

References

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