APPLICATION OF UNIFORM DISTRIBUTION TO HOMOGENIZATION OF A THIN OBSTACLE PROBLEM WITH $p-$LAPLACIAN

ARAM L. KARAKHANYAN AND MARTIN H. STRÖMQVIST

Abstract. In this paper we study the homogenization of $p-$Laplacian with thin obstacle in a perforated domain. The obstacle is defined on the intersection between a hyperplane and a periodic perforation.

We construct the family of correctors for this problem and show that the solutions for the $\varepsilon-$problem converge to a solution of a minimization problem of similar form but with an extra term involving the mean capacity of the obstacle. The novelty of our approach is based on the employment of quasi-uniform convergence. As an application we obtain Poincaré's inequality for perforated domains.

1. Introduction

Let $\Gamma = \{x \in \mathbb{R}^d : x\nu = 0\}$ where $|\nu| = 1$ is a fixed unit vector. We define the $\varepsilon-$cube $Q_\varepsilon = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^d$ and $Q_\varepsilon(x) = x + Q_\varepsilon$. The k:th hole is defined by

$$T_\varepsilon^k = a_\varepsilon T + k\varepsilon, \quad k \in \mathbb{Z}^d \tag{1}$$

where

$$a_\varepsilon = \varepsilon^{d/(d-p+1)} \tag{2}$$

and T is a fixed compact subset of Q_1. Thus $T_\varepsilon^k \subset\subset Q_\varepsilon^k$.

The union of all holes is denoted by

$$T_\varepsilon = \bigcup_{k\in\mathbb{Z}^d} T_\varepsilon^k, \tag{3}$$

and the trace of Γ on T_ε by

$$\Gamma_\varepsilon = \Gamma \cap T_\varepsilon. \tag{4}$$

The main problem we are concerned with is formulated as follows:

Problem: For $\psi \in L^\infty \cap W^{1,p}(\Omega), p > 1$, such that $\psi \le 0$ on $\partial\Omega$, we let $\psi_\varepsilon = \psi\chi_{\Gamma_\varepsilon}$ where χ_D is the characteristic function of a set $D \subset \mathbb{R}^d$. For any $\sigma \in L^q(\Omega)$, $1/p + 1/q = 1$, find the stable solution of

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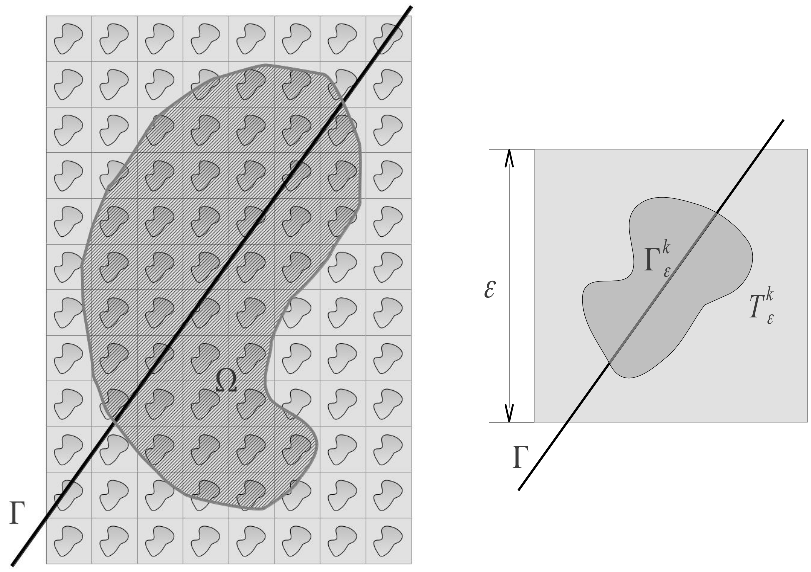

(A) The perforated domain Ω.　　(B) The structure of the cell.

Figure 1. Microstructure

$$\mathcal{J}_p = \int_\Omega [|\nabla v(x)|^p - \sigma(x)v(x)]\,dx \longrightarrow \min, \quad v \in \mathcal{O}_\varepsilon, \tag{5}$$

as $\varepsilon \longrightarrow 0$ where the class of admissible functions is defined as

$$\mathcal{O}_\varepsilon = \{v \in W_0^{1,p}(\Omega), v \geq \psi_\varepsilon\}. \tag{6}$$

The answer will be given in Theorems A and B below.

In order to study the behavior of u_ε, as $\varepsilon \longrightarrow 0$, we shall construct an auxiliary function w_ε, called *corrector*, and with its help derive the form of the stable (or homogenized) problem that the limit $u_0 = \lim_{\varepsilon\to 0} u_\varepsilon$ solves.

The boundary value problem in perforated domains have been considered by several authors. There is a vast literature devoted to this type of problems, see [3], [10]. The free boundary problems, particularly the *classical* obstacle problem are discussed in [3], [12]. Homogenization in randomly perforated domains has also attracted much attention during the last decade, see [4] and the references therein.

The *thin* obstacle problem has been studied in [9] for Laplace's operator, where **Theorem A** below is proved for $p = 2$. In [9] the convergence of correctors (see **H1**-**H3** in Section 3) is proved through covering arguments and the construction of barriers to the correctors.

In the process of extending the result to other values of p, we develop some new techniques in this paper that we believe to be more robust. Our main contribution is the employment of quasi-uniform convergence for the family of solutions u_ε. The use of quasi-uniform convergence simplifies some of the more technical parts of the proof of convergence even in the case of the standard Laplacian treated in [9]. In particular, it greatly facilitates the proof of **H2** and **H3** below in section 3. This method is applicable to any homogenization problem in a perforated domain where the capacity associated with the governing operator is comparable to p-capacity.

It is worthwhile to point out that with the aid of Choquet's capacity defined by

$$\text{F-cap}(K, Q) = \inf\left\{\int_Q F(x, \nabla\phi), \phi \in C_0^\infty(Q), \phi \geq 1 \text{ on } K\right\}$$

(see [5] page 2) one can extend our results to a more general class of problems of similar sort. Notably, the homogenization of the quasilinear operator $\mathcal{L}_F v = \operatorname{div}(\nabla_\xi F(x, \nabla v))$ with $\nabla F_\xi(\cdot, \xi) = (\frac{\partial F(\cdot,\xi)}{\partial \xi_1}, \dots, \frac{\partial F(\cdot,\xi)}{\partial \xi_d})$, associated with the functional

$$\mathcal{J} = \int_\Omega [F(x, \nabla v(x)) - \sigma(x)v(x)]\, dx, \quad v \in \widetilde{\mathcal{O}_\varepsilon} = \{v \in W_0^{1,F}(\Omega), v \geq \psi_\varepsilon\}$$

as its Euler-Lagrange equation, can be treated by similar method provided that F fulfills some structure conditions under which Choquet's capacity is comparable with $p-$capacity for some $p > 1$. The handling of more general functionals can be tightened up, as it will become clear from our exposition, but at the cost of lengthier presentation. Thus for the sake of brevity we decided to elucidate our method for the $p-$Laplacian.

A major difference between the p-Laplacian and the standard Laplacian treated in [9] is the proof of lower semicontinuity and convergence of the energy of the solution to (5). When $p \neq 2$ the functional to be minimized is no longer quadratic, which makes it signifacantly more difficult to analyse the effect of perturbing the solution.

The paper is organized as follows: In section 2 we state our main results, Theorem A and Theorem B. Theorem B is essentially a consequence of Theorem A and is proved in section 2. We also present an idea of the proof of Theorem A in section 2. In section 3 we construct correctors. Here we rely a lot on capacity techniques. In section 4 our main theorems are proved by exploiting properties of the correctors. Section 5 is an appendix where we have gathered some results that are used frequently in the paper, mostly capacity techniques and results from the theory of uniform distribution.

2. Main results

Throughout this paper we make the following assumptions:

(A_1) $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and $\sigma \in L^q(\Omega)$, where $1/p + 1/q = 1$.

(A_2) The compact set T from which the holes are constructed must be sufficiently regular in order for the mapping

$$t \mapsto \text{p-cap}(\{\Gamma + t\nu\} \cap T)$$

to be continuous. This is satisfied if, for example, T has a Lipschitz boundary. We shall need this when passing to the limit in Riemann sums, see Lemma 4.

(A_3) The size of the holes is

$$a_\varepsilon = \varepsilon^{d/(d-p+1)}.$$

This is the critical size that gives rise to an interesting effective equation for (5).

(A_4) The exponent p in (5) is in the range

$$1 < p < \frac{d+2}{2}.$$

This is to ensure that the holes are large enough that we are able to effectively estimate the intersections between the hyper plane Γ and the holes T_ε, of size a_ε. See the proof of Lemma 4 for details.

We would also like to point out that a translation of Γ does not affect the structure of the results, i.e.

$$\Gamma = \{x : (x - x_0)\nu = 0\}$$

works equally well. We have chosen $x_0 = 0$ only to simplify notation.

2.1. **Main Results.** The main results of this paper are formulated below.

Theorem A. *Let u_ε be the minimizer of*

$$\mathcal{J}_p = \int_\Omega [|\nabla v(x)|^p - \sigma(x)v(x)]\, dx, \quad v \in \mathcal{O}_\varepsilon,$$

and let ν be the normal of the hyper plane Γ. Then the following holds for a.e. $\nu \in S^{d-1}$:

Under the assumptions (A_1)-(A_4), u_ε converges weakly to a $u_0 \in W_0^{1,p}(\Omega)$ which is the minimizer of

$$(7) \qquad \begin{cases} \mathcal{J}_p^0 = \int_\Omega [|\nabla v(x)|^p - \sigma(x)v(x)]\, dx + c_\nu \int_\Gamma ((\psi - u)^+)^p d\mathcal{H}^{d-1}, \\ v \in W_0^{1,p}(\Omega). \end{cases}$$

The constant c_ν in (7) is the *mean capacity* of T with respect to the hyper plane Γ with normal ν:

$$(8) \qquad c_\nu = \int_{-\infty}^{\infty} \text{p-cap}(\{\Gamma + t\nu\} \cap T) dt.$$

The p-capacity of a set $E \in \mathbb{R}^d$, for $1 < p < d$, is defined by

$$\text{p-cap}(E) = \lim_{R\to\infty} \inf \left\{ \int_{B_R} |\nabla v|^p dx : v \in W_0^{1,p}(B_R) \text{ and } v = 1 \text{ on } E \right\}.$$

As a consequence of **Theorem A** we obtain the following Poincaré type inequality for the perforated domain,

Theorem B. *There is a tame constant $C > 0$ (independent of ε) such that*

$$\int_\Omega |u|^p \leq C \int_\Omega |\nabla u|^p$$

for any $u \in \Gamma_\varepsilon W^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : v = 0$ on $\Gamma_\varepsilon \cap \Omega\}$.

Proof. The proof is based on the fact that the constant C in the Poincaré inequality behaves like $1/\,\text{p-cap}(\{u = 0\})$. We recall Theorem 2.9 in [5]:

For any $p \in [1, \infty)$, there exists a constant $K = K(p, d)$ such that

$$\int_\Omega |u|^p dx \leq \frac{K}{\text{p-cap}(\{u = 0\})} \int_\Omega |\nabla u|^p dx, \tag{9}$$

for all $u \in W^{1,p}(\Omega)$ such that $\|\nabla u\|_{L^p(\Omega)} \neq 0$.

Since $u \in \Gamma_\varepsilon W^{1,p}(\Omega)$ we know that $\Gamma_\varepsilon \cap \Omega \subset \{u = 0\}$. Hence it is enough to show that $\text{p-cap}(\Gamma_\varepsilon \cap \Omega)$ is bounded from below uniformly in ε. The capacity of $\Gamma_\varepsilon \cap \Omega$ (with respect to $\mathbb{R}^d$) is given by

$$\text{p-cap}(\Gamma_\varepsilon \cap \Omega) = \lim_{R\to\infty} \inf_{\mathcal{K}_R} \int_{B_R} |\nabla v|^p dx, \tag{10}$$

where

$$\mathcal{K}_R = \left\{ v \in W_0^{1,p}(B_R) : v = 1 \text{ on } \Gamma_\varepsilon \cap \Omega \right\}.$$

It is a routine matter to show that the inf is assumed in (10) for any R. Call the minimizer v_ε^R and choose R so large that

$$\text{p-cap}(\Gamma_\varepsilon \cap \Omega) \geq \frac{1}{2} \int_{B_R} |\nabla v_\varepsilon^R|^p dx. \tag{11}$$

Now, the proof of Theorem A can be applied to determine the limit of $\int_{B_R} |\nabla v_\varepsilon^R|^p dx$, as $\varepsilon \to 0$. Simply, take $\Omega = B_R$, $\sigma = 0$ and let ψ be any smooth function with compact support in B_R such that $\psi = 1$ on $\Gamma \cap \Omega$. Then, using Corollary 10 and Lemma 11 we find that

$$\lim_{\varepsilon\to 0} \int_{B_R} |\nabla v_\varepsilon^R|^p dx = \inf_{W_0^{1,p}(B_R)} \int_{B_R} |\nabla v|^p + c_\nu \int_{\Gamma\cap\Omega} |(1 - v)^+|^p d\mathcal{H}^{d-1}. \tag{12}$$

The right hand side of (12) cannot be zero. Indeed, then we would have $\nabla v = 0$ which implies $v = 0$, but then the second term would be

$$c_\nu \mathcal{H}^{d-1}(\Gamma \cap \Omega) > 0.$$

The theorem now follows from (11), (12) and (9). $\square$

2.2. **Idea of proof of Theorem A.** First of all we need some quantitative information about Γ_ε, see (4). In particular, if $A \subset \Gamma$ and the surface measure $\mathcal{H}^{d-1}(A) = c$ we would like to determine the number of $k \in \mathbb{Z}^d$ such that $A \cap T_\varepsilon^k \neq \emptyset$. Call this number A_ε. An easy volume argument shows that the number of $k \in \mathbb{Z}^d$ for which $A \cap Q_\varepsilon^k \neq \emptyset$ is

$$N_\varepsilon = \varepsilon^{1-d}\mathcal{H}^{d-1}(A) + O(\varepsilon^{2-d}).$$

An intersection between Γ and Q_ε^k corresponds to a real number t_k such that

$$\Gamma \cap Q_\varepsilon^k = \{\Gamma + t_k\nu\} \cap Q_\varepsilon,$$

where ν is the normal of Γ. If the intersections $\Gamma \cap Q_\varepsilon^k$ for different k are uniformly distributed over Q_ε, i.e. if the sequence $\{t_k\}$ has a uniform distribution, we should expect that $A_\varepsilon/N_\varepsilon$ is comparable to the relative size of T_ε^k in Q_ε^k, i.e. to $a_\varepsilon/\varepsilon$. This is indeed the case. For a.e. ν on the unit sphere S^{d-1}, where ν is the normal of Γ, we have

$$\left|\frac{A_\varepsilon}{N_\varepsilon} - \frac{a_\varepsilon}{\varepsilon}\right| = O(\varepsilon^{1-\delta}), \quad \text{for any } \delta > 0. \tag{13}$$

See section 3 for a proof and more precise statement.

Having determined the frequency at which Γ and T_ε intersect, we choose a_ε in such a way that Γ_ε has finite positive capacity as $\varepsilon \to 0$: If $1 < p < d$, we expect that each intersection contributes $O(a_\varepsilon^{d-p})$ to the capacity of Γ_ε, hence we expect a total capacity of

$$\text{p-cap}(\Gamma_\varepsilon) = O(A_\varepsilon a_\varepsilon^{d-p}) = O\left(\varepsilon^{1-d}\frac{a_\varepsilon}{\varepsilon}a_\varepsilon^{d-p}\right) = O\left(\frac{a_\varepsilon^{d-p+1}}{\varepsilon^d}\right).$$

This leads to the choice of a_ε in (2). However, in order to determine the number of intersections A_ε satisfactorily, a_ε cannot be too small. Recalling (13), we need

$$\varepsilon^{1-\delta} = o(a_\varepsilon/\varepsilon), \quad \text{for some } \delta > 0. \tag{14}$$

This is why we require (A_4). If $p = d$, each intersection contributes even more to the capacity, which leads to a choice of a_ε which is not compatible with (14). Finally, for $p > d$, a single point has positive capacity and functions in $W^{1,p}(\Omega)$ are Hölder continuous and the type of weak convergence illustrated in **Theorem A** cannot occur.

The remaining part of the proof consists of constructing correctors w_ε satisfying the properties **H1**, **H2** and **H3**, and exploiting these properties to prove **Theorem A**. This is done in Section 3. Much of the techniques used here are standard but we also introduce new techniques relying on capacity methods, in particular when proving the convergence of $\Delta_p w_\varepsilon$ in a certain weak sense (**H3**).

3. Correctors

Throughout this section we assume that the conditions $(A_1)-(A_4)$ on the data are fulfilled. We are going to construct a sequence of functions w_ε satisfying the following three conditions *for a.e.* $\nu \in S^{d-1}$:

H1 $0 \le w_\varepsilon \le 1$ in $\mathbb{R}^d$, $w_\varepsilon = 1$ on Γ_ε and $w_\varepsilon \rightharpoonup 0$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^d)$,

H2 $\int\limits_\Omega |\nabla w_\varepsilon|^p f dx \to c_\nu \int\limits_\Gamma f d\mathcal{H}^{d-1}$, for any $f \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$,

H3 (weak continuity) for any $\phi_\varepsilon \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} \sup\limits_{\varepsilon>0} \|\phi_\varepsilon\|_{L^\infty(\Omega)} < \infty, \\ \phi_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \text{ and } \phi_\varepsilon \rightharpoonup \phi \in W^{1,p}_0(\Omega), \end{cases}$$

we have

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle \to \langle \mu, \phi \rangle$$

with

$$\mu = c_\nu \mathcal{H}^{d-1} \llcorner \Gamma, \tag{15}$$

where c_ν is given by (8) and $\mathcal{H}^s$ is the $s-$dimensional Hausdorff measure.

Remark 1. The actual restriction on ν comes from the fact that the intersections between Γ and the lattice set $\bigcup_k T^k_\varepsilon$ need to have sufficiently even distribution in order for averaging to occur. Specifically, this imposes some restrictions on the components in the vector

$$\alpha = (\alpha_1, \ldots, \alpha_{d-1}) = \left(\frac{\nu_1}{\nu_d}, \ldots, \frac{\nu_{d-1}}{\nu_d} \right).$$

A *necessary* condition on the class of normals for which our theorems hold is that at least one α_i is irrational. This excludes all rational normals, i.e. normals of the type

$$\nu \in \mathbb{R}\mathbb{Z}^d = \{tk : t \in \mathbb{R},\ k \in \mathbb{Z}^d\}.$$

However, a *sufficient* condition is that the sequence $\{n\alpha_i\}_{n=1}^N$ has low enough discrepancy D_N for some i, see Definition 16 for a definition. In fact, for a given p in the range $(1, (d+2)/2)$, the discrepancy of $\{n\alpha_i\}_{n=1}^N$ has to satisfy $D_N = O(N^{\delta-1})$, where $\delta = \delta(p) \in (0,1)$. In turn, a necessary condition for this is that α_i is *of type* η for some η close enough to 1 depending on p, where the type of the irrational number α_i is the sup of all γ such that

$$\liminf_{q\to\infty} q^\gamma \min_{z\in\mathbb{Z}} |q\alpha_i - z| = 0, \qquad q \in \mathbb{Z}.$$

This does not in general hold for α_i coming from an irrational (non-rational) vector, but holds for a.e. ν.

3.1. **Definition and basic properties of the corrector.** The corrector is defined by its restriction to any Q_ε^k. Let $\Gamma_\varepsilon^k = \Gamma \cap T_\varepsilon^k$ and let

$$B_{\varepsilon/2}^k = \{x \in \mathbb{R}^d : |x - \varepsilon k| < \varepsilon/2\}.$$

Then the k:th corrector w_ε^k is defined as

$$\begin{cases} w_\varepsilon^k = 1 & \text{on } \Gamma_\varepsilon^k, \\ \Delta_p w_\varepsilon^k = 0 & \text{in } B_{\varepsilon/2}^k \setminus \Gamma_\varepsilon^k, \\ w_\varepsilon^k = 0 & \text{on } Q_\varepsilon^k \setminus B_{\varepsilon/2}^k. \end{cases} \tag{16}$$

That is, w_ε^k is the capacitary potential of Γ_ε^k in $B_{\varepsilon/2}^k$. Note that w_ε^k and w_ε^l have disjoint supports for $k \neq l$. The full corrector is

$$w_\varepsilon = \sum_{k \in \mathbb{Z}} w_\varepsilon^k.$$

The p-Laplacian of w_ε consists of two measures:

$$-\Delta_p w_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^-, \tag{17}$$

$$\text{supp}\mu_\varepsilon^+ \subset \bigcup_k \partial B_{\varepsilon/2}^k, \quad \text{supp}\mu_\varepsilon^- \subset \bigcup_k \Gamma_\varepsilon^k = \Gamma_\varepsilon. \tag{18}$$

Lemma 2. *Let $k \in \mathbb{Z}^d$ and w_ε^k be the k:th corrector. Then we have:*

(i) *The k:th intersection Γ_ε^k and the k:th corrector w_ε^k satisfy*

$$\int_{Q_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = c(\varepsilon, k)\, \text{p-cap}(\Gamma_\varepsilon^k), \tag{19}$$

where $c(\varepsilon, k) \to 1$ uniformly w.r.t. k as $\varepsilon \to 0$.

(ii) *For each $k \in \mathbb{Z}^d$ such that $\Gamma_\varepsilon^k \neq \emptyset$, there is a unique $t = t(k) \in \mathbb{R}$ such that*

$$\Gamma_\varepsilon^k = a_\varepsilon(\{\Gamma + t\nu\} \cap T) + k\varepsilon, \tag{20}$$

where ν is the unit normal of the hyperplane Γ.

(iii) *Furthermore,*

$$\text{p-cap}(\Gamma_\varepsilon^k) = a_\varepsilon^{d-p}\, \text{p-cap}((\{\Gamma + t\nu\} \cap T)). \tag{21}$$

Proof. The claim (ii) follows from geometric considerations. Recall that

$$\Gamma_\varepsilon^k = \Gamma \cap \{a_\varepsilon T + k\varepsilon\}.$$

Next (iii) follows from the fact that, if $p < d$, then

$$\text{p-cap}(\rho E + x_0) = \rho^{d-p}\, \text{p-cap}(E),$$

for any $E \subset \mathbb{R}^d$ and any $x_0 \in \mathbb{R}^d$.

To prove (i) we employ a variational formulation of (16):

$$\begin{cases} w_\varepsilon^k \in \mathcal{K}_\varepsilon^k = \{w \in W_0^{1,p}(B_{\varepsilon/2}^k) \text{ and } w = 1 \text{ on } \Gamma_\varepsilon^k\}, \\ \displaystyle\int_{B_{\varepsilon/2}^k} |\nabla w_\varepsilon^k|^p dx = \inf_{\mathcal{K}_\varepsilon^k} \int_{B_{\varepsilon/2}^k} |\nabla w|^p dx. \end{cases}$$

This problem is translation invariant, i.e. we may omit the translation by $k\varepsilon$ and replace Γ_ε^k by $a_\varepsilon(\{\Gamma + t\nu\} \cap T)$. Performing the change of variables $x \mapsto a_\varepsilon x$, we see that

$$\inf \left\{ \int_{B_{\varepsilon/2}} |\nabla w|^p dx : \ w \in W_0^{1,p}(B_{\varepsilon/2}) \text{ and } w = 1 \text{ on } a_\varepsilon(\{\Gamma + t\nu\} \cap T) \right\} =$$
$$= a_\varepsilon^{d-p} \inf_{\mathcal{K}_\varepsilon} \int_{B_{\varepsilon/2a_\varepsilon}} |\nabla w|^p dx,$$

where

$$\mathcal{K}_\varepsilon = \left\{ w \in W_0^{1,p}(B_{\varepsilon/2a_\varepsilon}) \text{ and } w = 1 \text{ on } \{\Gamma + t\nu\} \cap T \right\}.$$

Now, for $p < d$,

$$\text{p-cap}(E) = \lim_{R\to\infty} \inf \left\{ \int_{B_R} |\nabla w|^p dx : \ w \in W_0^{1,p}(B_R) \text{ and } w = 1 \text{ on } E \right\},$$

for any $E \subset \mathbb{R}^d$. Since $\varepsilon/2a_\varepsilon \to \infty$ as $\varepsilon \to 0$, we obtain that

$$\lim_{\varepsilon\to 0} \inf_{\mathcal{K}_\varepsilon} \int_{B_{\varepsilon/2a_\varepsilon}} |\nabla w|^p dx = \text{p-cap}(\{\Gamma + t\nu\} \cap T), \tag{22}$$

as $\varepsilon \to 0$. Since (22) is zero when t does not belong to the compact set $\{t \in \mathbb{R} : \{\Gamma + t\nu\} \cap T \neq \emptyset\}$, the convergence is uniform w.r.t. t, hence w.r.t. k, by (ii). $\square$

Our next goal is to give a simple geometric characterization of the intersections $\Gamma_\varepsilon^k = \Gamma \cap T_\varepsilon^k$, see Figure 2 and (23)-(24) below. We may assume that $\nu_d \neq 0$ (by choosing appropriate coordinate system) and can therefore write

$$\Gamma = \{x \in \mathbb{R}^d : x \cdot \nu = 0\} = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d = \alpha x'\},$$

where $\alpha \in \mathbb{R}^{d-1}$, $\alpha_i = \nu_i/\nu_d$. Let $k = (k', k_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$. In order to have $\Gamma_\varepsilon^k \neq \emptyset$, that is, in order for Γ to intersect with $a_\varepsilon T + \varepsilon k$, the x_d coordinate of Γ at $x' = \varepsilon k'$ needs to be close enough to k_d. At $x' = \varepsilon k'$, the d :th coordinate of Γ is $x_d = \alpha\varepsilon k'$. There is an interval I_ε such that $\Gamma_\varepsilon^k \neq \emptyset$ if and only if

$$\alpha\varepsilon k' - \varepsilon k_d \in I_\varepsilon, \tag{23}$$

see Figure 2.

Thus, in order for an intersection to occur the point $(\varepsilon k', \varepsilon k'\alpha) \in \Gamma$ must satisfy

$$\varepsilon k'\alpha \in I_\varepsilon \pmod{\varepsilon},$$

or equivalently,

$$k'\alpha \in \varepsilon^{-1} I_\varepsilon \pmod{1}. \tag{24}$$

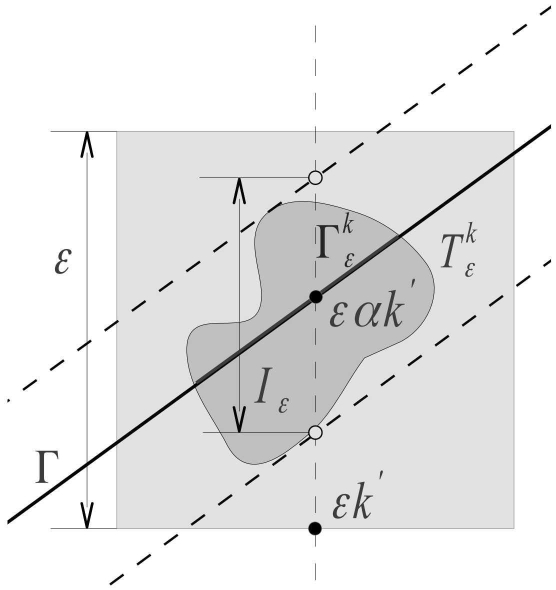

FIGURE 2. Γ intersects T^k_ε above $\varepsilon k'$ if and only if $\varepsilon k'\alpha \in I_\varepsilon$.

This observation will be used later in proving the properties of correctors.

3.2. **Verifying H1 and H2.** In our first lemma we estimate the number of $k \in \mathbb{Z}^d$ that satisfies (20) for a certain range of t.

Lemma 3. *Let $R' = (a_1, b_1) \times \cdots \times (a_{d-1}, b_{d-1}) \subset \mathbb{R}^{d-1}$ and let $A_{\varepsilon,\eta}(\tau)$ be the number of $k = (k', k_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$ such that $k' \in \varepsilon^{-1}R'$ and (20) holds with $t \in (\tau - \eta/2, \tau + \eta/2)$, where $\eta > 0$ is a small fixed number. Let further*

$$N_\varepsilon = \#\{k' \in \varepsilon^{-1}R' \cap \mathbb{Z}^{d-1}\}.$$

Then there is an interval $I_{\varepsilon,\eta}(\tau)$ of length $\eta\frac{a_\varepsilon}{\nu_d}$ such that

$$A_{\varepsilon,\eta}(\tau) = \#\{k' \in \varepsilon^{-1}R' : k'\alpha \in \varepsilon^{-1}I_{\varepsilon,\eta}(\tau) \pmod 1\}, \tag{25}$$

and for a.e. $\nu \in S^{d-1}$,

$$\left| \frac{A_{\varepsilon,\eta}(\tau)}{N_\varepsilon} - \frac{a_\varepsilon \eta}{\varepsilon \nu_d} \right| = O(\varepsilon^{1-\delta}), \tag{26}$$

for any $\delta > 0$.

Proof. According to (20), there is a unique $t_k \in \mathbb{R}$ such that

$$\Gamma^k_\varepsilon = a_\varepsilon(\{\Gamma + t_k\nu\} \cap T) + k\varepsilon.$$

If k is under the hypothesis of the lemma, then

$$t_k \in \left(\tau - \frac{\eta}{2}, \tau + \frac{\eta}{2}\right). \tag{27}$$

Thus the sets

$$a_\varepsilon(\{\Gamma + t_k\nu\} \cap T)$$

in question lie between two copies of Γ a distance $a_\varepsilon\eta$ apart in the normal direction of Γ. The distance between these planes in the x_d-direction is therefore $a_\varepsilon\eta/\nu_d$, with a corresponding interval $I_{\varepsilon,\eta}$, cf. Figure 2. Thus t_k satisfies (27) if and only if $k = (k', k_d)$ satisfies

$$\alpha\varepsilon k' - \varepsilon k_d \in I_{\varepsilon,\eta}, \tag{28}$$

where $\alpha\varepsilon k'$ is the x_d-coordinate of $\Gamma = \{(x', x_d) : x_d = \alpha x'\}$ at $x' = \varepsilon k'$. We note that (28) holds for some k_d if and only if

$$\alpha\varepsilon k' \in I_{\varepsilon,\eta} \pmod{\varepsilon},$$

or equivalently, if and only if

$$\alpha k' \in \varepsilon^{-1} I_{\varepsilon,\eta} \pmod{1}.$$

This proves (25).

To prove (26), we only need to note that the left hand side of (26) is bounded by the discrepancy of the sequence

$$\{k'\alpha : k' \in \varepsilon^{-1}R'\},$$

and apply Proposition 19, with $N_i = [(b_i - a_i)\varepsilon^{-1}]$ (see Appendix). □

Lemma 4. *Let $R = (a_1, b_1) \times \cdots \times (a_d, b_d) \subset \mathbb{R}^d$. Then for a.e. $\nu \in S^{d-1}$,*

$$\lim_{\varepsilon \to 0} \int_R |\nabla w_\varepsilon|^p dx = c_\nu \int_{\Gamma \cap R} d\mathcal{H}^{d-1}, \tag{29}$$

where c_ν is given by (8)

Proof. We observe that if ν is not paralell to any of the coordinate axes (which we may assume since these ν form a set of measure zero), then the left- and right-hand-sides of (29) do not change if we replace R by its closure $\overline{R}$. If $R \cap \Gamma = \emptyset$, then $w_\varepsilon|_R = 0$ for ε small enough, which gives (29). If $R \cap \Gamma \neq \emptyset$, let $\tilde{R} = ((R \cap \Gamma)' \times (a_d, b_d))$, where $(R \cap \Gamma)'$ is the projection of $R \cap \Gamma$ on $\{x_d = 0\}$. Then $\overline{R} \cap \Gamma = \overline{\tilde{R}} \cap \Gamma$ and $R = \tilde{R} \cup H$, where

$$\lim_{\varepsilon \to 0} \int_H |\nabla w_\varepsilon|^p dx = c_\nu \int_{\Gamma \cap H} d\mathcal{H}^{d-1} = 0.$$

Thus we may as well assume that

$$R \cap \Gamma = \{(x', \alpha x') : x' \in R'\},$$

where $R' = (a_1, b_1) \times \cdots \times (a_{d-1}, b_{d-1})$. Let $K_{\varepsilon,\eta}(\tau)$ be the set of $k = (k', k_d) \in \mathbb{Z}^d$ such that

(*a*) $k' \in \varepsilon^{-1}R' \cap \mathbb{Z}^{d-1}$
(*b*) Γ_ε^k satisfies (20) for some $t \in (\tau - \frac{\eta}{2}, \tau + \frac{\eta}{2})$, $\quad \eta > 0$.

By Lemma 3, there exists an interval $I_{\varepsilon,\eta}(\tau)$ of length $\eta\frac{a_\varepsilon}{\nu_d}$ such that

$$K_{\varepsilon,\eta}(\tau) = \{k' \in \varepsilon^{-1}R' \cap \mathbb{Z}^{d-1},\ k'\alpha \in \varepsilon^{-1}I_{\varepsilon,\eta}(\tau) \pmod{1}\}.$$

Let $A_{\varepsilon,\eta}(\tau) = \#K_{\varepsilon,\eta}(\tau)$ and let $N_\varepsilon = \#\{\varepsilon^{-1}R' \cap \mathbb{Z}^{d-1}\}$. Then Lemma 3 tells us that for a.e. $\nu \in S^{d-1}$,

$$\left| \frac{A_{\varepsilon,\eta}(\tau)}{N_\varepsilon} - \frac{\eta a_\varepsilon}{\varepsilon \nu_d} \right| = O(\varepsilon^{1-\delta}),$$

for any $\delta > 0$.

For any $k \in K_{\varepsilon,\eta}(\tau)$,

$$\int_{Q_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = c_k(\varepsilon,\eta) a_\varepsilon^{d-p} \,\text{p-cap}(\{\Gamma + \tau\nu\} \cap T), \tag{30}$$

where

$$c_k(\varepsilon,\eta) \to 1 \text{ uniformly as } \varepsilon \to 0 \text{ and } \eta \to 0. \tag{31}$$

This follows from Lemma 2 and the fact that $\tau \mapsto \text{p-cap}(\{\Gamma + \tau\nu\} \cap T)$ is continuous thanks to assumption (A_2). Using the fact that

$$N_\varepsilon = \varepsilon^{1-d}\mathcal{H}^{d-1}(R') + O(\varepsilon^{2-d}) = \varepsilon^{1-d}\nu_d \mathcal{H}^{d-1}(R \cap \Gamma) + O(\varepsilon^{2-d}),$$

we get

$$A_{\varepsilon,\eta}(\tau) = \frac{\eta a_\varepsilon}{\varepsilon^d}\mathcal{H}^{d-1}(\Gamma \cap R) + O(\varepsilon^{2-d-\delta} + \eta a_\varepsilon \varepsilon^{1-d}). \tag{32}$$

The error term in (32) is $o(a_\varepsilon/\varepsilon^d)$ if and only if $\varepsilon^{1-\delta} = o(a_\varepsilon/\varepsilon)$ (for some $\delta > 0$). Since $a_\varepsilon = \varepsilon^{d/(d-p+1)}$ this is equivalent to $p < \frac{d+2}{2}$ (see assumption (A_4)).

From (30) we find

$$\begin{aligned} &\sum_{k \in K_{\varepsilon,\eta}(\tau)} \int_R |\nabla w_\varepsilon^k|^p dx = \\ &= \sum_{k \in K_{\varepsilon,\eta}(\tau)} c_k(\varepsilon,\eta) a_\varepsilon^{d-p} \,\text{p-cap}(\{\Gamma + \tau\nu\} \cap T). \end{aligned} \tag{33}$$

Since the sum in (33) consists of $A_{\varepsilon,\eta}(\tau)$ terms $c_k(\varepsilon,\eta)$, we have

$$\sum_{k \in K_{\varepsilon,\eta}(\tau)} c_k(\varepsilon,\eta) = A_{\varepsilon,\eta}(\tau) \sum_{k \in K_{\varepsilon,\eta}(\tau)} \frac{c_k(\varepsilon,\eta)}{A_{\varepsilon,\eta}(\tau)} = A_{\varepsilon,\eta}(\tau) c_\tau(\varepsilon,\eta),$$

where $c_\tau(\varepsilon,\eta) \to 1$ uniformly w.r.t. τ as $\varepsilon, \eta \to 0$. This follows from (31) and the uniform continuity of $\tau \mapsto \text{p-cap}(\{\Gamma + \tau\nu\} \cap T)$.

Thus

$$\begin{aligned} &\sum_{k \in K_{\varepsilon,\eta}(\tau)} \int_R |\nabla w_\varepsilon^k|^p dx = \\ &= A_{\varepsilon,\eta}(\tau) c_\tau(\varepsilon,\eta) a_\varepsilon^{d-p} \,\text{p-cap}(\{\Gamma + \tau\nu\} \cap T) \\ &= \mathcal{H}^{d-1}(\Gamma \cap R)\left[\frac{\eta a_\varepsilon}{\varepsilon^d} c_\tau(\varepsilon,\eta) a_\varepsilon^{d-p}\right] \text{p-cap}(\{\Gamma + \tau\nu\} \cap T) + E_\varepsilon \\ &= \eta \mathcal{H}^{d-1}(\Gamma \cap R) c_\tau(\varepsilon,\eta) \,\text{p-cap}(\{\Gamma + \tau\nu\} \cap T) + E_\varepsilon \end{aligned} \tag{34}$$

where E_ε is an error term such that $\lim_{\varepsilon \to 0} E_\varepsilon = 0$.

Let t_1, t_2, $t_1 < t_2$ satisfy

$$T \subset \{\Gamma + t\nu : t_1 \le t \le t_2\}.$$

Note that we may take $|t_i| \le \sqrt{d}$, $i = 1, 2$. Now make a uniform partition of $[t_1, t_2]$ into intervals $[\tau_i - \frac{\eta}{2}, \tau_i + \frac{\eta}{2}]$ of length η. Using (34) we obtain the following estimate:

$$\begin{aligned} \int_R |\nabla w_\varepsilon|^p dx &= \sum_{\tau_i} \sum_{k \in K'_{\varepsilon,\eta}} \int_R |\nabla w_\varepsilon^k|^p dx \ge \\ &\ge \sum_{\tau_i} \eta \mathcal{H}^{d-1}(\Gamma \cap R) c_{\tau_i}(\varepsilon, \eta) \inf_{\tau_i - \frac{\eta}{2} \le \tau \le \tau_i + \frac{\eta}{2}} \text{p-cap}(\{\Gamma + \tau\nu\} \cap T) + E_\varepsilon. \end{aligned} \tag{35}$$

Taking the inferior limit we find

$$\begin{aligned} &\liminf_{\varepsilon \to 0} \int_R |\nabla w_\varepsilon|^p dx \ge \\ &\ge \sum_{\tau_i} \eta \mathcal{H}^{d-1}(\Gamma \cap R) c_{\tau_i}(\eta) \inf_{\tau_i - \frac{\eta}{2} \le \tau \le \tau_i + \frac{\eta}{2}} \text{p-cap}(\{\Gamma + \tau\nu\} \cap T), \end{aligned} \tag{36}$$

where $c_{\tau_i}(\eta) = \lim_{\varepsilon \to 0} c_{\tau_i}(\varepsilon, \eta)$. Of course, taking supremum instead of infimum in (35) and the superior limit in (36) will result in a reverse inequality for the superior limit. Passing then to the limit $\eta \to 0$, the continuity of $\tau \mapsto \text{p-cap}(\{\Gamma + \tau\nu\} \cap T)$ allows us to conclude that

$$\lim_{\varepsilon \to 0} \int_R |\nabla w_\varepsilon|^p dx = \mathcal{H}^{d-1}(\Gamma \cap R) \int_{t_1}^{t_2} \text{p-cap}(\{\Gamma + \tau\nu\} \cap T) d\tau,$$

which is (29)

□

3.2.1. **Proof of H1**. This follows easily from Lemma 4: Since $\lim_{\varepsilon \to 0} w_\varepsilon = 0$ outside of Γ, it is enough to show that

$$\int_R |\nabla w_\varepsilon|^p dx \tag{37}$$

is bounded for any rectangle $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$. This is indeed a consequence of Lemma 4.

3.2.2. **Proof of H2**. It is convenient to divide the proof into several steps.

Step 1. *Compactness.*
Let f be continuous with compact support and let $R \supset \text{supp} f$ be a rectangle. Since

$$\left| \int_\Omega |\nabla w_\varepsilon|^p f dx \right| \le \|f\|_{L^\infty} \int_R |\nabla w_\varepsilon|^p dx,$$

we can identify $|\nabla w_\varepsilon|^p dx$ with a Borel measure $d\mu_\varepsilon$, by the Riesz representation theorem, see [11] p.40. By the Banach-Alaoglu theorem, there is a Borel measure μ such that $\mu_\varepsilon \rightharpoonup^* \mu$. It is clear that $\text{supp}\mu \subset \Gamma$.

Step 2. *The measure μ is absolutely continuous with respect to $d-1$ dimensional Lebesgue measure (or $\mathcal{H}^{d-1}$) on Γ.*
Let $K \subset \Gamma$ be compact and let $\delta > 0$. Then there is a family of rectangles $\{R_j\}$, which we may assume to be finite and disjoint, such that

$$K \subset \bigcup_j R_j \cap \Gamma$$

and

$$\mathcal{H}^{d-1}(K) \geq \mathcal{H}^{d-1}(\bigcup_j R_j \cap \Gamma) - \delta = \sum_j \mathcal{H}^{d-1}(R_j \cap \Gamma) - \delta.$$

By (29), we have

$$\begin{aligned}\mu(K) &\leq \sum_j \mu(R_j \cap \Gamma) = \lim_{\varepsilon \to 0} \sum_j \int_{R_j} |\nabla w_\varepsilon|^p dx = c_\nu \sum_j \mathcal{H}^{d-1}(R_j \cap \Gamma) \\ &\leq c_\nu(\mathcal{H}^{d-1}(K) + \delta),\end{aligned}$$

which proves the claim because we are allowed to send $\delta \to 0$.

Step 3. $\mu = c_\nu \mathcal{H}^{d-1}$.
By the Radon-Nikodym theorem, there exists $g \in L^1_{\text{loc}}(\Gamma, d\mathcal{H}^{d-1})$ such that

$$\mu(E) = \int_E g d\mathcal{H}^{d-1},$$

for any measurable set $E \subset \Gamma$. To prove that $g = c_\nu$, we only need to use the fact that g can be approximated by step functions (characteristic functions on rectangles) in L^1 norm and apply (29):

Let $\delta > 0$ and let

$$g_n = \sum_{j=1}^n c_j \chi_{R_j}$$

be a step function such that $\|g - g_n\|_{L^1(\Gamma, \mathcal{H}^{d-1})} < \delta$. We may assume that the rectangles R_j are disjoint. Let $R \subset \Gamma$ be any rectangle and set $\widetilde{R}_j = R \cap R_j$. Then

$$\begin{aligned}\int_R |g - c_\nu| d\mathcal{H}^{d-1} &\leq \int_R |g - g_n| d\mathcal{H}^{d-1} + \int_R |g_n - c_\nu| d\mathcal{H}^{d-1} \\ &= \int_R |g - g_n| d\mathcal{H}^{d-1} + \sum_{j=1}^n |c_j - c_\nu| \mathcal{H}^{d-1}(\widetilde{R}_j).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{j=1}^{n}|c_j-c_\nu|\mathcal{H}^{d-1}(\widetilde{R}_j) &= \sum_{j=1}^{n}\left|\int_{\widetilde{R}_j}(g_n-g)d\mathcal{H}^{d-1}\right| \\ &\le \sum_{j=1}^{n}\int_{\widetilde{R}_j}|g_n-g|d\mathcal{H}^{d-1} \\ &= \int_R |g_n-g|d\mathcal{H}^{d-1} < \delta.\end{aligned}$$

This shows that

$$\int_R |g-c_\nu|d\mathcal{H}^{d-1} < 2\delta$$

for any rectangle R and any $\delta>0$, hence $g=c_\nu$.

Step 4. *Going from $f\in C_c^\infty(\Omega)$ to $f\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ using capacity techniques.*

Now assume $f\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and let $\{f_n\}_{n=1}^\infty$ be a sequence of smooth, compactly supported functions such that $f_n\to f$ in $W_0^{1,p}(\Omega)$. According to Theorem 2.1. in [5], $f_n\to f$ quasi-uniformly on Ω (see also Definition 13 and Theorem 14 in Appendix). That is, for any $\delta>0$ there is a set $E_\delta\subset\Omega$ such that $f_n\to f$ uniformly on $\Omega\setminus E_\delta$ and p-cap$(E_\delta)<\delta$. Write

$$\int_\Omega |f_n-f||\nabla w_\varepsilon|^p dx = \int_{\Omega\setminus E_\delta}|f_n-f||\nabla w_\varepsilon|^p dx + \int_{E_\delta}|f_n-f||\nabla w_\varepsilon|^p dx = I_1+I_2.$$

Since $\int_\Omega |\nabla w_\varepsilon|^p dx$ is bounded and $f_n\to f$ uniformly on $\Omega\setminus E_\delta$, $I_1\to 0$ as $n\to\infty$ and $\varepsilon\to 0$. Next we estimate I_2. Let R_δ be a union of rectangles such that $R_\delta\supset E_\delta$ and

$$\mathcal{H}^{d-1}(R_\delta\cap\Gamma)\le\mathcal{H}^{d-1}(E_\delta\cap\Gamma)+\delta. \tag{38}$$

Then

$$\int_{E_\delta}|f_n-f||\nabla w_\varepsilon|^p dx \le C\int_{R_\delta}|\nabla w_\varepsilon|^p dx \to Cc_\nu\mathcal{H}^{d-1}(R_\delta\cap\Gamma).$$

Now, using the fact that

$$\begin{aligned}\mathcal{H}^{d-1}(E_\delta\cap\Gamma) &\le C\,\text{p-cap}(E_\delta\cap\Gamma)^{(d-1)/(d-p)} \\ &\le C\,\text{p-cap}(E_\delta)^{(d-1)/(d-p)} = C\delta^{(d-1)/(d-p)}\end{aligned}$$

(see (48) in the Appendix) and (38), we find that

$$\int_{E_\delta}|f_n-f||\nabla w_\varepsilon|^p dx \le C\delta.$$

Thus

$$\lim_{\substack{\varepsilon\to 0\\ n\to\infty}}\int_\Omega |f_n-f||\nabla w_\varepsilon|^p dx = 0.$$

Since f_n is smooth and $|\nabla w_\varepsilon|^p dx \rightharpoonup^* c_\nu d\mathcal{H}^{d-1}$,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla w_\varepsilon|^p f_n dx = c_\nu \int_\Gamma f_n d\mathcal{H}^{d-1}.$$

By the trace theorem,

$$\lim_{n\to\infty}\int_\Gamma f_n d\mathcal{H}^{d-1} = \int_\Gamma f d\mathcal{H}^{d-1},$$

which proves **H2**.

3.3. **Verifying H3.** In this section we prove that the corrector w_ε verifies **H3** which manifests the continuity of $\Delta_p w_\varepsilon$ in some weak sense.

Lemma 5. *For any $\eta \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ there holds for a.e. $\nu \in S^{d-1}$*

$$\lim_{\varepsilon\to 0}\int \eta d\mu_\varepsilon^+ = \lim_{\varepsilon\to 0}\int \eta|\nabla w_\varepsilon|^p = c_\nu \int_\Gamma \eta d\mathcal{H}^{d-1},$$

where μ_ε is given by (17)-(18).

Proof. From the divergence theorem we have

$$\begin{aligned}\int_\Omega \eta\left[|\nabla w_\varepsilon|^p - (1-w_\varepsilon)\Delta_p w_\varepsilon\right] &= -\int_\Omega \eta \mathrm{div}\left[(1-w_\varepsilon)|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon\right]\\ &= \int_\Omega (1-w_\varepsilon)|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon \nabla\eta.\end{aligned}$$

Due to weak convergence $w_\varepsilon \rightharpoonup w$ in $W^{1,p}(\Omega)$ we have $\nabla w_\varepsilon \to \nabla w$ strongly in L^q for any $q\in(1,p)$. Thus applying Vitali's theorem (see Theorem 15 in Appendix) we obtain that the last integral vanishes in the limit. Therefore

$$\lim_{\varepsilon\to 0}\int_\Omega \eta|\nabla w_\varepsilon|^p = \lim_{\varepsilon\to 0}\int_\Omega \eta(1-w_\varepsilon)\Delta_p w_\varepsilon.$$

On the other hand, recalling **H2**, we see that $\lim_{\varepsilon\to 0}\int \eta|\nabla w_\varepsilon|^p = c_\nu\int_\Gamma \eta d\mathcal{H}^{d-1}$. In order to finish the proof we have to show that $\lim_{\varepsilon\to 0}\int_\Omega \eta(1-w_\varepsilon)\Delta_p w_\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega \eta d\mu_\varepsilon^+$.

We decompose $\Delta_p w_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^-$ and note that by construction $\mathrm{supp}\mu_\varepsilon^- \subset \Gamma_\varepsilon$, see (17)-(18). Because $1-w_\varepsilon = 0$ on Γ_ε, we infer that

$$\int_\Omega \eta(1-w_\varepsilon)\Delta_p w_\varepsilon = \int_\Omega \eta(1-w_\varepsilon)d\mu_\varepsilon^+ = \int_{\mathrm{supp}\mu_\varepsilon^+} \eta(1-w_\varepsilon)d\mu_\varepsilon^+.$$

But $\mathrm{supp}\mu_\varepsilon^+ \subset \partial B^k_{\varepsilon/2}, k\in\mathbb{Z}, B^k_{\varepsilon/2}\cap\Gamma \neq \emptyset$. Because $1-w_\varepsilon = 1$ on $\mathrm{supp}\mu_\varepsilon^+$, we conclude that

$$\int_\Omega \eta(1-w_\varepsilon)\Delta_p w_\varepsilon = \int_{\mathrm{supp}\mu_\varepsilon^+}\eta d\mu_\varepsilon^+ = \int_\Omega \eta d\mu_\varepsilon^+.$$

□

Lemma 6. *Let* $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $h_\varepsilon \in W^{1,p}(\Omega) \cap L^\infty(\Omega), h_\varepsilon = 0$ *on* Γ_ε, $h_\varepsilon \rightharpoonup h$ *in* $W^{1,p}(\Omega)$. *Then for a.e.* $\nu \in S^{d-1}$

$$\lim_{\varepsilon \to 0} \int_\Omega \phi |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla h_\varepsilon = c_\nu \int_\Gamma \phi h d\mathcal{H}^{d-1}.$$

Proof. Clearly

$$\int_\Omega \phi |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla h_\varepsilon = \int_\Omega |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla (\phi h_\varepsilon) - \int_\Omega h_\varepsilon |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla \phi.$$

The last integral vanishes in the limit thanks to Vitali's theorem, see Theorem 15 in Appendix.

Noting that $\phi h_\varepsilon = 0$ on $\text{supp}\mu_\varepsilon^-$, $\phi h_\varepsilon \in W_0^{1,p}(\Omega)$ and after partial integration we get

$$\begin{aligned}
\int_\Omega |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla (\phi h_\varepsilon) &= -\int_\Omega \phi h_\varepsilon \Delta_p w_\varepsilon \\
&= \int_\Omega \phi h_\varepsilon d\mu_\varepsilon^+ \\
&= \int_\Omega \phi (h_\varepsilon - h) d\mu_\varepsilon^+ + \int_\Omega \phi h d\mu_\varepsilon^+.
\end{aligned}$$

Applying Lemma 5 we see that $\lim\limits_{\varepsilon \to 0} \int_\Omega \phi h d\mu_\varepsilon^+ = c_\nu \int_\Gamma \phi h d\mathcal{H}^{d-1}$. Thus it remains to show that $\int_\Omega \phi(h_\varepsilon - h) d\mu_\varepsilon^+$ vanishes in the limit. The latter follows from the refined Egoroff's theorem and quasi-uniform convergence.

Indeed, let $\delta > 0$ be a fixed, small number. From Theorem 12 we have that

$$\int_\Omega \phi (h_\varepsilon - h) d\mu_\varepsilon^+ = \int_{\Omega \setminus E_\delta} \phi (h_\varepsilon - h) d\mu_\varepsilon^+ + \int_{E_\delta} \phi (h_\varepsilon - h) d\mu_\varepsilon^+.$$

Since on $\Omega \setminus E_\delta$ we have uniform convergence, it follows that the first integral on the right vanishes in the limit.

As for the remaining integral we have

$$\int_{E_\delta} \phi (h_\varepsilon - h) d\mu_\varepsilon^+ \leq \sup_\Omega |h_\varepsilon - h| \int_{E_\delta} |\phi| d\mu_\varepsilon^+.$$

We will show that

$$\int_{E_\delta} |\phi| d\mu_\varepsilon^+ \longrightarrow 0.$$

For any nonnegative $\chi \in W^{1,p}(\Omega) \cap C(\Omega)$ such that $\chi(x) = 1$ for $x \in E_\delta$ we have

$$\int_{E_\delta} |\phi| d\mu_\varepsilon^+ \leq \int_{E_\delta} |\phi| \chi d\mu_\varepsilon^+ \longrightarrow c_\nu \int_\Gamma |\phi| \chi d\mathcal{H}^{d-1}$$

where the convergence follows from Lemma 5 because $|\phi|\chi \in W_0^{1,p}(\Omega)$. Now we choose a sequence of continuous functions χ_n such that $\chi_n \uparrow \chi_{E_\delta}$ where χ_{E_δ} is the characteristic function of E_δ.

Thus to finish the proof it remains to show that $\int_{\Gamma\cap E_\delta} |\phi| d\mathcal{H}^{d-1} = 0$. To see this we use (48) and show that $\mathcal{H}^\beta(E_\delta) < \infty$ for some $\beta < d-1$.

Denote $\beta = d - q + t_1 > d - q$ for some $t_1 > 0$ to be fixed below. To fulfill the requirement of Theorem 12 we take $q = p - t_2 < p$ for some $t_2 > 0$.

If we demand $d - q + t_1 < d - 1$ then we get $t_1 + t_2 < p - 1$. For instance we could take $t_1 = t_2 = \frac{p-1}{3}$. □

Corollary 7. *Let $\phi_\varepsilon \in W_0^{1,p}(\Omega)$ such that $\|\phi_\varepsilon\|_{L^\infty(\Omega)} \le C$, $\phi_\varepsilon = 0$ on Γ_ε and $\phi_\varepsilon \rightharpoonup \phi$ in $W_0^{1,p}(\Omega)$ then for a.e. $\nu \in S^{d-1}$*

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle \to \langle c_\nu \mathcal{H}^{d-1} \llcorner \Gamma, \phi \rangle = c_\nu \int_\Gamma \phi d\mathcal{H}^{d-1}.$$

Proof. Since by assumption $\phi_\varepsilon \in W_0^{1,p}(\Omega)$ we obtain

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle = \int_\Omega |\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon \nabla \phi_\varepsilon.$$

After using the decomposition $\mu_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^-$ and using the same reasoning as in the proof of Lemma 6 we infer

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle = \int_\Omega \phi_\varepsilon d\mu_\varepsilon^+ = \int_\Omega \phi d\mu_\varepsilon^+ + \int_\Omega (\phi_\varepsilon - \phi) d\mu_\varepsilon^+.$$

Then the rest of the proof follows that of Lemma 6. □

4. Lower semicontinuity

In this section we prove lower semicontinuity of $p-$Dirichlet energy, Lemma 8. Our proof is a refinement of Theorem 3.1 [1].

Lemma 8. *Let $v_\varepsilon \in W_0^{1,p}(\Omega)$ such that $\|v_\varepsilon\|_{L^\infty(\Omega)} \le C$, $v_\varepsilon \rightharpoonup v$ and $v_\varepsilon = 0$ on Γ_ε. Then for a.e. $\nu \in S^{d-1}$ we have*

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla v_\varepsilon|^p \ge \int_\Omega |\nabla v|^p + c_\nu \int_\Gamma |v|^p d\mathcal{H}^{d-1}.$$

In order to prove this lemma we have to establish a number of auxiliary results, see Section 3 [1] (see also the discussion following Theorem 2.1 in [7]).

Proof. We use the well-known convexity estimate

$$|\xi|^p \ge |\eta|^p + p|\eta|^{p-2}\eta(\xi - \eta), \quad \forall \xi, \eta \in \mathbb{R}^d. \tag{39}$$

This inequality follows from Taylor's expansion

$$|\xi|^p = |\eta|^p + p|\eta|^{p-2}\eta(\xi-\eta) + \frac{p}{2}(|\eta^*|^{p-2}Id + (p-2)|\eta^*|^{p-4}\eta^*\otimes\eta^*)(\xi-\eta)(\xi-\eta).$$

Since the matrix $|\eta^*|^{p-2}Id + (p-2)|\eta^*|^{p-4}\eta^*\otimes\eta^*$ is positive definite for $p > 1$, we obtain (39).

Taking $z_\varepsilon = 1 - w_\varepsilon$ we then obtain

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon|^p &\geq \int_\Omega |\nabla(z_\varepsilon v)|^p + p\int_\Omega |\nabla(vz_\varepsilon)|^{p-2}\nabla(z_\varepsilon v)(\nabla(v_\varepsilon - z_\varepsilon v)) \\
&= \underbrace{\int_\Omega |v\nabla z_\varepsilon + z_\varepsilon\nabla v|^p - |v\nabla z_\varepsilon|^p}_{I_1(\varepsilon)} + \underbrace{\int_\Omega |v|^p|\nabla z_\varepsilon|^p}_{I_2(\varepsilon)} \\
&\quad \underbrace{+\, p\int_\Omega |\nabla(vz_\varepsilon)|^{p-2}\nabla(z_\varepsilon v)\nabla(v_\varepsilon - z_\varepsilon v)}_{I_3(\varepsilon)}.
\end{aligned}$$

Note that, by **H2** we have that $0 \leq z_\varepsilon \leq 1, |\nabla z_\varepsilon| = |\nabla w_\varepsilon|$ hence

$$I_2(\varepsilon) \longrightarrow \int_\Omega |v|^p d\mu = c_\nu \int_\Gamma |v|^p d\mathcal{H}^{d-1}.$$

In order to deal with $I_1(\varepsilon)$ we recall Vitali's theorem, see Theorem 15 in the appendix. From the mean value theorem we have

$$\left||v\nabla z_\varepsilon + z_\varepsilon\nabla v|^p - |v\nabla z_\varepsilon|^p\right| \leq p\left[|v\nabla z_\varepsilon + z_\varepsilon\nabla v|^{p-1} + |v\nabla z_\varepsilon|^{p-1}\right]|z_\varepsilon\nabla v|.$$

In order to apply Vitali's theorem we need to show that the functions

$$h_\varepsilon = \left[|v\nabla z_\varepsilon + z_\varepsilon\nabla v|^{p-1} + |v\nabla z_\varepsilon|^{p-1}\right]|z_\varepsilon\nabla v|$$

are equi-integrable. This follows from the property **H1** that $0 \leq z_\varepsilon \leq 1$ and $z_\varepsilon \rightharpoonup 1$ in $W^{1,p}(\Omega)$. Next applying Hölder's inequality we have

$$\begin{aligned}
\int_E \left||v\nabla z_\varepsilon + z_\varepsilon\nabla v|^p - |v\nabla z_\varepsilon|^p\right| &\leq p\int_E \left[|v\nabla z_\varepsilon + z_\varepsilon\nabla v|^{p-1} + |v\nabla z_\varepsilon|^{p-1}\right]|z_\varepsilon\nabla v| \\
&\leq Cp\left(\int_E |\nabla v|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Now the equiintegrability follows from Lebesgue's absolute continuity of $|\nabla v|^p$. Hence we can apply Vitali's theorem to the functions $|v\nabla z_\varepsilon + z_\varepsilon\nabla v|^p - |v\nabla z_\varepsilon|^p$.

From a.e. convergence (at least for a subsequence implied by weak convergence in $W^{1,p}(\Omega)$) we infer that $v\nabla z_\varepsilon + z_\varepsilon\nabla v \to \nabla v$ a.e. in Ω. Therefore

$$I_1(\varepsilon) \longrightarrow \int_\Omega |\nabla v|^p.$$

To finish the proof it remains to estimate $I_3(\varepsilon)$. We slightly reformulate our task. Given $\phi_\varepsilon \in W_0^{1,p}(\Omega)$, $\|\phi_\varepsilon\|_{L^\infty(\Omega)} \leq C$ such that $\phi_\varepsilon = 0$ on Γ_ε and $\phi_\varepsilon \rightharpoonup \phi$ in $W_0^{1,2}(\Omega)$. Then

$$\int_\Omega |\nabla(z_\varepsilon v)|^{p-2}\nabla(z_\varepsilon v)\nabla\phi_\varepsilon \longrightarrow \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\phi + \int_\Omega |v|^{p-2}v\phi d\mu. \tag{40}$$

Clearly (40) is enough to establish the desired convergence because one can take $\phi_\varepsilon = v_\varepsilon - vz_\varepsilon$. Note that in our case $\phi_\varepsilon \to 0$ because $z_\varepsilon \rightharpoonup 1$ in $W^{1,p}(\Omega)$, i.e. $\phi = 0$ implying that $\lim\limits_{\varepsilon\to 0} I_3(\varepsilon) = 0$.

Recall the well-known estimates [1] (1.2)-(1.5)

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq \begin{cases} (p-1)(|\xi|^{p-2}+|\eta|^{p-2})|\xi-\eta|, & \text{if } p\geq 2 \\ 2^{2-p}|\xi-\eta|^{p-1} & \text{if } 1<p\leq 2. \end{cases} \tag{41}$$

Decompose the vector-field as follows

$$\begin{aligned} |\nabla(z_\varepsilon v)|^{p-2}\nabla(z_\varepsilon v) &= \underbrace{|\nabla(z_\varepsilon v)|^{p-2}\nabla(z_\varepsilon v) - |v\nabla z_\varepsilon|^{p-2}v\nabla z_\varepsilon}_{J_1} \\ &\quad + \underbrace{|v\nabla z_\varepsilon|^{p-2}v\nabla z_\varepsilon}_{J_2}. \end{aligned}$$

We want to show that Vitali's theorem applies to $J_1\nabla\phi_\varepsilon$. Indeed, by (41)

$$|J_1| \leq \begin{cases} (p-1)(|\nabla(vz_\varepsilon)|^{p-2}+|v\nabla z_\varepsilon|^{p-2})|z_\varepsilon\nabla v|, & \text{if } p\geq 2 \\ 2^{2-p}|z_\varepsilon\nabla v|^{p-1} & \text{if } 1<p\leq 2. \end{cases}$$

Using $0 \leq z_\varepsilon \leq 1$ we see that $|J_1| \in L^{p'}(\Omega)$ if $1 < p \leq 2$ with conjugate $p' = \frac{p}{p-1}$. Thus the equiintegrability condition is fulfilled for $J_1\nabla\phi_\varepsilon$. As for the other case $p > 2$ from Hölder's inequality we have

$$\int_E (|\nabla z_\varepsilon|^{p-2}|\nabla\phi_\varepsilon|)^{p'} \leq \int_E \left[\frac{1}{\alpha}|\nabla\phi_\varepsilon|^{\alpha p'} + \frac{1}{\alpha'}|\nabla z_\varepsilon|^{p'\alpha'(p-2)}\right]$$

for any measurable set E and choose $\alpha = p-1 > 1$. Thus we see that for any $p > 1$ the functions $J_1\nabla\phi_\varepsilon$ are equiintegrable. Therefore from Vitali's theorem we obtain that

$$\int_\Omega J_1\nabla\phi_\varepsilon \longrightarrow \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\phi.$$

As for the remaining J_2 we have

$$\begin{aligned}\int_\Omega J_2\nabla\phi_\varepsilon &= \int_\Omega |v|^{p-2}v|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon \\ &= \int_\Omega (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon - \int_\Omega (v^-)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon\end{aligned}$$

It is obvious that $v^\pm \in W_0^{1,p}(\Omega)$. Let $\delta > 0$ and split the integral for v^+ as follows:

$$\begin{aligned}\int_\Omega (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon &= \int\limits_{\{v^+\le\delta\}} + \int\limits_{\{v^+>\delta\}} (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon \\ &= O(\delta^{p-1}) + \int\limits_{\{v^+>\delta\}} (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon\end{aligned}$$

Notice that $\int\limits_{\{v^+\le\delta\}} (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon = O(\delta^{p-1})$ because we have uniform bounds for $\|\nabla z_\varepsilon\|_{L^p(\Omega)}, \|\nabla\phi_\varepsilon\|_{L^p(\Omega)}$ thanks to weak convergence.

Next we deal with the remaining integral

$$\begin{aligned}\int\limits_{\{v^+>\delta\}} (v^+)^{p-1}|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon &= \int_\Omega \max[(v^+)^{p-1},\delta^{p-1}]|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon + \\ &\quad -\delta^{p-1}\int\limits_{\{v^+\le\delta\}} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon \\ &= \int_\Omega V_\delta^+|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon + \\ &\quad -\delta^{p-1}\int\limits_{\{v^+\le\delta\}} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon \\ &\quad +\delta^{p-1}\int\limits_{\Omega} |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon \\ &= \int_\Omega V_\delta^+|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon + O(\delta^{p-1})\end{aligned}$$

where $V_\delta^+ = \max[(v^+)^{p-1},\delta^{p-1}] - \delta^{p-1}$. Notice that $V_\delta^+ \in W_0^{1,p}(\Omega)$. Thus employing Green's identity we conclude

$$\int_\Omega V_\delta^+|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla\phi_\varepsilon = \int_\Omega |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla(V_\delta^+\phi_\varepsilon) - \int_\Omega \phi_\varepsilon|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\nabla V_\delta^+.$$

Since $|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \in L^{p'}(\Omega)$, with uniformly bounded $L^{p'}$ norm, we can apply Vitali's theorem again to conclude that the second integral on the right hand side vanishes in the limit.

Finally

$$\begin{aligned}\int_\Omega |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \nabla (V_\delta^+\phi_\varepsilon) &= -\langle \Delta_p z_\varepsilon, V_\delta^+\phi_\varepsilon\rangle \\ &\longrightarrow -\langle \mu, V_\delta^+\phi\rangle\end{aligned}$$

where the last line follows from **H3**. Sending δ to zero the proof of (40) follows. □

Corollary 9. *Let $E \subset \Omega$ be an open set such that $\Gamma \cap \Omega \subset E$ and*

$$\mathrm{supp} w_\varepsilon \cap \Omega \subset E,$$

for small enough $\varepsilon > 0$. Then Lemma 8 holds with E in place of Ω. That is, if $v_\varepsilon \in W_0^{1,p}(\Omega)$, $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C$, $v_\varepsilon \rightharpoonup v$ and $v_\varepsilon = 0$ on Γ_ε, then we have

$$\liminf_{\varepsilon\to 0} \int_E |\nabla v_\varepsilon|^p \geq \int_E |\nabla v|^p + c_\nu \int_\Gamma |v|^p d\mathcal{H}^{d-1} \tag{42}$$

Proof. The proof is identical to that of Lemma 8 as soon as we have verified that

(i) If $\phi_\varepsilon \in W_0^{1,p}(\Omega)$ and $\phi_\varepsilon = 0$ on Γ_ε, then

$$\int_E |\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon \nabla \phi_\varepsilon = \langle -\Delta w_\varepsilon, \phi_\varepsilon\rangle_E.$$

That is, the term involving $\int_{\partial E}$ vanishes when integrating by parts.

(ii) For any $f \in W_0^{1,p}(\Omega)$ we have

$$\lim_{\varepsilon\to 0} \int_E |\nabla w_\varepsilon|^p f dx = c_\nu \int_\Gamma f d\mathcal{H}^{d-1}.$$

To prove (i) we only need to note that the boundary of E consists of $\partial E \cap \partial\Omega$, where $\phi_\varepsilon = 0$, and $\partial E \cap \Omega$, where $w_\varepsilon = 0$.

(ii) follows if we consider smooth approximations χ_n of χ_E, such that $\lim_{n\to\infty} \chi_n = \chi_E$. Replace f by $f\chi_n$, then take $n \to \infty$ and apply **H2**. □

Corollary 10. *Let u_ε be the solution to (5) and suppose $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$. Then for a.e. $\nu \in S^{d-1}$*

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^p \geq \int_\Omega |\nabla u|^p + c_\nu \int_\Gamma ((\psi - u)^+)^p$$

Proof. Let u_ε be the solution to (5) and write $\psi - u_\varepsilon = (\psi - u_\varepsilon)^+ - (\psi - u_\varepsilon)^-$. It is a well-known fact that that the solution to the obstacle problem is bounded by the obstacle itself in L^∞-norm, i.e.

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|\psi_\varepsilon\|_{L^\infty(\Omega)} \leq \|\psi\|_{L^\infty(\Omega)}.$$

Thus $(\psi - u_\varepsilon)^+$ satisfies the hypothesis of Lemma 8. Let E be a set of small measure satisfying the hypothesis of Corollary 9. Applying (42) to $(\psi - u_\varepsilon)^+$ and using the usual lower semicontinuity of the norm on $|\nabla(\psi - u_\varepsilon)^-|$, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_E |\nabla(\psi - u_\varepsilon)|^p dx &= \liminf_{\varepsilon\to 0}\int_E |\nabla(\psi - u_\varepsilon)^+|^p dx + \\ &\quad + \liminf_{\varepsilon\to 0}\int_E |\nabla(\psi - u_\varepsilon)^-|^p dx \\ &\geq \int_E |\nabla(\psi - u_\varepsilon)^+|^p dx + c_\nu \int_\Gamma ((\psi - u)^+)^p + \\ &\quad + \int_E |\nabla(\psi - u)^-|^p dx \\ &= \int_E |\nabla(\psi - u)|^p dx + c_\nu \int_\Gamma ((\psi - u)^+)^p.\end{aligned}$$

Write

$$\int_\Omega |\nabla u_\varepsilon|^p dx = \int_{\Omega\setminus E} |\nabla u_\varepsilon|^p dx + \int_E |\nabla u_\varepsilon|^p dx = I_1 + I_2.$$

For I_1 we apply the usual lower semicontinuity of the norm:

$$\liminf_{\varepsilon\to 0}\int_{\Omega\setminus E} |\nabla u_\varepsilon|^p dx \geq \int_{\Omega\setminus E} |\nabla u|^p dx.$$

For I_2, Young's inequality implies

$$\begin{aligned}-\int_E \nabla u_\varepsilon \nabla(\psi - u_\varepsilon)|\nabla(\psi - u_\varepsilon)|^{p-2} dx &\leq \frac{1}{p}\int_E |\nabla u_\varepsilon|^p dx + \\ &\quad + \frac{p-1}{p}\int_E |\nabla(\psi - u_\varepsilon)|^p dx.\end{aligned} \tag{43}$$

For (43) we have the following lower bound:

$$\begin{aligned}-\int_E \nabla u_\varepsilon \nabla(\psi - u_\varepsilon)|\nabla(\psi - u_\varepsilon)|^{p-2} dx &= \\ &= \int_E |\nabla(\psi - u_\varepsilon)|^p dx - \\ &\quad - \int_E \nabla\psi \nabla(\psi - u_\varepsilon)|\nabla(\psi - u_\varepsilon)|^{p-2} dx \\ &\geq \int_E |\nabla(\psi - u_\varepsilon)|^p dx \\ &\quad - \left(\int_E |\nabla\psi|^p dx\right)^{1/p}\left(\int_E |\nabla(\psi - u_\varepsilon)|^p dx\right)^{1/(p-1)} \\ &\geq \int_E |\nabla(\psi - u_\varepsilon)|^p dx - C\left(\int_E |\nabla\psi|^p dx\right)^{1/p}. \qquad (44)\end{aligned}$$

Combining (43) and (44), we get

$$\frac{1}{p}\int_E |\nabla u_\varepsilon|^p dx \geq \frac{1}{p}\int_E |\nabla(\psi - u_\varepsilon)|^p dx - C\left(\int_E |\nabla\psi|^p dx\right)^{1/p}.$$

Consequently, we infer that

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p dx &\geq \liminf_{\varepsilon\to 0}\left[\int_{\Omega\setminus E}|\nabla u_\varepsilon|^p dx + \int_E |\nabla(\psi - u_\varepsilon)|^p dx\right] - \\
&\quad - C\left(\int_E |\nabla\psi|^p dx\right)^{1/p} \\
&\geq \int_{\Omega\setminus E}|\nabla u|^p dx + c_\nu\int_\Gamma |(\psi - u)^+|^p d\mathcal{H}^{d-1} + \\
&\quad + \int_E |\nabla(\psi - u)|^p dx - C\left(\int_E |\nabla\psi|^p dx\right)^{1/p}.
\end{aligned}$$

Since this holds for any $E \subset \Omega$ with $E\cap\Gamma = \Omega\cap\Gamma$, we can make $|E| \to 0$, proving our claim.

□

Lemma 11. *Let u_ε be the solution of* (5) *then for a.e. $\nu \in S^{d-1}$*

$$\limsup_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p - \sigma u_\varepsilon \leq \inf_{W_0^{1,p}(\Omega)}\int_\Omega |\nabla v|^p - \sigma v + c_\nu\int_\Gamma ((\psi - v)^+)^p$$

Proof. Let $v \in C_c^\infty(\Omega)$. Then the function $v_\varepsilon := v + w_\varepsilon(\psi - v)^+$ belongs to the class of admissible functions $\mathcal{O}_\varepsilon$ since $w_\varepsilon = 1$ on Γ_ε. Let u_ε be the solution to (5). By definition we have

$$\int_\Omega |\nabla u_\varepsilon|^p - \sigma u_\varepsilon dx \leq \int_\Omega |\nabla v_\varepsilon|^p - \sigma v_\varepsilon dx.$$

Since $v_\varepsilon \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ we have $\lim_{\varepsilon\to 0}\int_\Omega \sigma v_\varepsilon dx = \int_\Omega \sigma v dx$. Assume p is an integer. Then

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon|^p dx &= \int_\Omega |\nabla v + w_\varepsilon\nabla(\psi - v)^+ + (\psi - v)^+\nabla w_\varepsilon|^p dx \\
&\leq \int_\Omega [|\nabla v + w_\varepsilon\nabla(\psi - v)^+| + |(\psi - v)^+\nabla w_\varepsilon|]^p dx
\end{aligned}$$

Besides $|\nabla v + w_\varepsilon\nabla(\psi - v)^+|^p + |(\psi - v)^+)\nabla w_\varepsilon|^p$, the expression within the brackets consists of the terms

$$\binom{p}{k}|\nabla v + w_\varepsilon\nabla(\psi - v)^+)|^k|(\psi - v)^+\nabla w_\varepsilon]^{p-k}, \quad 1 \leq k \leq p-1.$$

As we have seen previously, the integral of these terms vanish, by Vitali's theorem and Hölder's inequality. Using the strong convergence $w_\varepsilon \to 0$ in $L^p(\Omega)$,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla v + w_\varepsilon\nabla(\psi - v)^+|^p dx = \int_\Omega |\nabla v|^p dx,$$

and by Lemma 5,

$$\lim_{\varepsilon\to 0}\int_\Omega |(\psi-v)^+\nabla w_\varepsilon|^p dx = c_\nu\int_\Gamma |(\psi-v)^+|^p d\mathcal{H}^{d-1}.$$

If p is not an integer, let m be the integer part of p. Then

$$\begin{aligned}\int_\Omega |\nabla v_\varepsilon|^p dx &= \int_\Omega |\nabla v + w_\varepsilon\nabla(\psi-v)^+ + (\psi-v)^+\nabla w_\varepsilon|^p dx\\ &\le \int_\Omega \left[|\nabla v + w_\varepsilon\nabla(\psi-v)^+| + |(\psi-v)^+\nabla w_\varepsilon|\right]^p\end{aligned}$$

Consequently, from the binomial theorem

$$\begin{aligned}\int_\Omega |\nabla v_\varepsilon|^p dx &\le \int_\Omega [|\nabla v + w_\varepsilon\nabla(\psi-v)^+| + |(\psi-v)^+\nabla w_\varepsilon|]^m \times\\ &\quad \times[|\nabla v + w_\varepsilon\nabla(\psi-v)^+| + |(\psi-v)^+\nabla w_\varepsilon|]^{p-m}dx.\end{aligned} \tag{45}$$

Since $0<p-m<1$,

$$\begin{aligned}[|\nabla v + w_\varepsilon\nabla(\psi-v)^+| + |(\psi-v)^+\nabla w_\varepsilon|]^{p-m} &\le |\nabla v + w_\varepsilon\nabla(\psi-v)^+|^{p-m} +\\ &\quad +|(\psi-v)^+\nabla w_\varepsilon|^{p-m}.\end{aligned}$$

Hence the left hand side of (45) consists of integrals

$$\int_\Omega |\nabla v + w_\varepsilon\nabla(\psi-v)^+|^p + |(\psi-v)^+)\nabla w_\varepsilon|^p$$

and terms of the form

$$\int_\Omega \binom{m}{k}|\nabla v + w_\varepsilon\nabla(\psi-v)^+|^{k+p-m}|(\psi-v)^+)\nabla w_\varepsilon|^{m-k}dx+ \tag{46}$$

$$+\int_\Omega \binom{m}{k}|\nabla v + w_\varepsilon\nabla(\psi-v)^+|^{m-k}|(\psi-v)^+)\nabla w_\varepsilon|^{k+p-m}dx, \tag{47}$$

where $0\le k\le m-1$. Again by Vitali's theorem and Hölder's inequality, the integrals (46)-(47) vanish in the limit $\varepsilon\to 0$. □

4.0.1. *Proof of Theorem A.*

Proof. Let u_ε be the solution to (5). Then $\|u_\varepsilon\|_{W^{1,p}(\Omega)}\le C$, so for a subsequence, $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$. The proof of Theorem A now follows from Corollary 10 and Lemma 11. □

5. Appendix

5.1. **$p-$capacity and quasi-uniform convergence.** In this section we recall some well-known facts from capacity theory, used in this paper.

Theorem 12. *(Refinement of Egoroff's theorem) Let $1<p<d$ and $\phi_\varepsilon\in W^{1,p}(\Omega)$, $\phi_\varepsilon\rightharpoonup\phi\in W^{1,p}(\Omega)$. Then for any small $\delta>0$ and $1<q<p$ there is a relatively closed set E_δ such that (at least for a subsequence) $\phi_\varepsilon\to\phi$ uniformly in $\Omega\setminus E_\delta$ and* q-cap$(E_\delta)<\delta$.

For a proof see [5] Theorem 2.3 page 10. We also note that our assumption (A_1) on the Lipschitz regularity of $\partial\Omega$ is necessary in order to apply this theorem.

Definition 13. *We say that $u_n \to u$ quasi-uniformly on Ω if for every $\delta > 0$ there exists a set $E \subset \Omega$ such that* p-cap$(E) < \delta$ *and $u_n \to u$ uniformly on $\Omega \setminus E$.*

Theorem 14. *Assume $u_n \to u$ strongly in $W^{1,p}(\Omega)$ and that $\partial\Omega$ is Lipschitz. Then u_n has a subsequence for which $u_n \to u$ quasi-uniformly in Ω.*

We refer to [5] p.8 for this result.

Recall the following estimate for the $q-$capacity and Hausdorff measure, see [8] Corollary 2, page 203:

$$\mathcal{H}^\beta(E) \le C(d)\,[\text{q-cap}(E)]^{\frac{\beta}{d-q}}, \quad \beta > d-q. \tag{48}$$

Here $C(d)$ is a dimensional constant.

Finally we recall Vitali's theorem [11] Chapter 6, page 133:

Theorem 15. *Let $h_n \in L^1(\Omega)$, where $|\Omega| < \infty$, $\sup_n \int_\Omega |h_n| < \infty$, and the limit $\lim\limits_{n\to\infty} h_n(x)$ exists pointwise and is finite a.e. in Ω, and assume that $\{h_n\}$ is equiintegrable, i.e. for any $\gamma > 0$ there is $\delta > 0$ such that for any measurable set E with $|E| < \delta$ we have*

$$\int_E |h_n| < \gamma.$$

Then $h \overset{\text{def}}{=} \lim\limits_{n\to\infty} h_n$ is in $L^1(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega h_n = \int_\Omega \lim_{n\to\infty} h_n.$$

5.2. **Uniform Distribution.** In this section we record some facts about the distribution modulo 1 of sequences of the type

$$\{n\alpha\}_{n=1}^N, \quad \alpha \in \mathbb{R}.$$

We also show what this implies for the multidimensional sequences of the type

$$\left\{ \sum_{i=1}^m n_i\alpha_i : 1 \le n_i \le N_i \right\}.$$

Definition 16. *Let $\{x_n\}_{n=1}^N$ be a sequence of real numbers. For any interval $I = (a, b]$ such that $0 < a < b \le 1$, let*

$$A = A(I) = \#\{1 \le n \le N : x_n \in I \pmod 1\}.$$

The discrepancy of $\{x_n\}_{n=1}^N$ *is defined as*

$$D = \sup_{I\subset(0,1]} \left| \frac{A}{N} - |I| \right|.$$

We shall need the following well-known result, see [6] Chapter 2, Exercise 3.13, or [2] Theorem 1.72.

Theorem 17. *Consider* $\{x_n\}_{n=1}^N$ *with* $x_n = n\alpha$*. Then for a.e.* $\alpha \in \mathbb{R}$*, we have*

$$D = O(N^{\delta-1}),$$

for any $\delta > 0$*.*

This result can be sharpened in the sense that the factor N^δ in front of $1/N$ may be replaces by factors of $\log N$. This is, however, not necessary for the purposes of this paper.

Now let $\mathcal{N} = \{n = (n_1, \ldots, n_m) : 1 \leq n_i \leq N_i,\ i = 1, \ldots, m\}$ and let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. We want to estimate the discrepancy of scalar products

$$\{n\alpha\}_{n\in\mathcal{N}}. \tag{49}$$

Let $N = \prod_{i=1}^m N_i$ and, for a given interval $I \subset \mathbb{R}$, let

$$A = \#\{n \in \mathcal{N} : n\alpha \in I \pmod 1\}.$$

Then

$$D = \sup_I \left| \frac{A}{N} - |I| \right|.$$

Corollary 18. *Let* D_i *be the discrepancy of* $\{n_i\alpha_i\}_{n_i=1}^{N_i}$*. Then*

$$D \leq \min_{1\leq i\leq m} D_i.$$

In particular, for a.e. $\alpha \in \mathbb{R}^m$ *we have*

$$D = O((\min_i N_i)^{\delta-1}),$$

for any $\delta > 0$*.*

Corollary 18 can be deduced from a result in [6], but we record a proof here because of its simplicity.

Proof. We write

$$\left|\frac{A}{N}-|I|\right| = \left|\sum_{n_j=1,\, j\neq i}^{N_j} \frac{\#\{n_i : n_i\alpha_i + \sum_{j\neq i} k_j\alpha_j \in I \pmod 1\}}{N_i \prod_{j\neq i} N_j} - |I|\right|$$

$$\leq \sum_{n_j=1,\, j\neq i}^{N_j} \left|\frac{\#\{n_i : n_i\alpha_i + \sum_{j\neq i} k_j\alpha_j \in I \pmod 1\}}{N_i \prod_{j\neq i} N_j} - \frac{|I|}{\prod_{j\neq i} N_j}\right|$$

$$\leq \sum_{n_j=1,\, j\neq i}^{N_j} \frac{D_i}{\prod_{j\neq i} N_j} = D_i.$$

By Theorem 17 we have

$$D_i = O(N_i^{\delta-1}), \tag{50}$$

for a.e. $\alpha_i \in \mathbb{R}$. □

Proposition 19. *For any $\nu \in S_\lambda = \{\nu \in S^{d-1} : \nu_d \geq \lambda > 0\}$, let $\alpha = (\nu_1/\nu_d, \dots, \nu_{d-1}/\nu_d)$ and consider the sequence*

$$\left\{\alpha k = \sum_{i=1}^{d-1} \alpha_i k_i : 1 \leq k_i \leq N_i,\ i = 1, \dots, d-1\right\}. \tag{51}$$

Let D be the discrepancy of the sequence in (51). Then for a.e. $\nu \in S_\lambda$ we have

$$D = O((\min_i N_i)^{\delta-1}),$$

for any $\delta > 0$. In particular, using the notation preceding Corollary 18, we have

$$\left|\frac{A}{N}-|I|\right| = O((\min_i N_i)^{\delta-1}), \quad I \subset [0,1],$$

for any $\delta > 0$.

Proof. Since $\nu_d \geq \lambda > 0$ for all $\nu \in S_\lambda$, there exists $c_0 > 0$ such that for any subset $B \subset S_\lambda$ we have

$$\mathcal{H}^{d-1}(B) \leq c_0 \mathcal{H}^{d-1}(B'). \tag{52}$$

Let $\mathcal{B}$ be the set of $\alpha \in \mathbb{R}^{d-1}$ such that (50) does not hold for any $i = 1, \dots, d-1$. We need to estimate the measure of the set

$$B = \{\nu \in S_\lambda : \alpha = (\nu_1/\nu_d, \dots, \nu_{d-1}/\nu_d) \in \mathcal{B}\}.$$

Define

$$\Phi : \begin{cases} \{x \in \mathbb{R}^{d-1} : |x| < 1\} \to \mathbb{R}^{d-1}, \\ x \to \frac{x}{\sqrt{1-|x|^2}}. \end{cases}$$

Then

$$B' = \{x \in S'_\lambda : \alpha = \Phi(x) \in \mathcal{B}\}.$$

Since $\Phi : S'_\lambda \to \Phi(S'_\lambda)$ is a diffeomorphism and $\mathcal{H}^{d-1}(\mathcal{B}) = 0$,

$$\begin{aligned}\mathcal{H}^{d-1}(B') &= \int_{\Phi^{-1}(\mathcal{B}\cap\Phi(S'_\lambda))} dy \\ &= \int_{\mathcal{B}\cap\Phi(S'_\lambda)} J_{\Phi^{-1}}(x)dx = 0.\end{aligned}$$

Now the claim follows from (52).

□

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Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, King's Buildings, Mayfield Road, EH9 3JZ, Edinburgh, Scotland, UK
E-mail address: aram.karakhanyan@ed.ac.uk

Department of Mathematics, Kungliga Tekniska Högskolan, Lindstedtsvägen 25, SE-100 44 Stockholm, Sweden
E-mail address: stromqv@math.kth.se