Γ-convergence of Oscillating Thin Obstacles

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Abstract

Consider the minimum problems of obstacle type

$$\min\left\{\int_\Omega |Du|^2 dx : u \geq \psi_\varepsilon \text{ on } P,\ u = 0 \text{ on } \partial\Omega\right\},$$

as $\varepsilon \to 0$. Here ψ_ε is a periodic function of period ε, constructed from an appropriately rescaled fixed function and $P \subset\subset \Omega \subset \mathbb{R}^n$ is a subset of the hyper-plane $\{x \in \mathbb{R}^n : x \cdot \eta = 0\}$. We assume $n \geq 3$ and that the normal η satisfies a generic condition that guarantees certain ergodic properties of the quantity

$$\#\left\{k \in \mathbb{Z}^n : P \cap \{x : |x - \varepsilon k| < \varepsilon^{n/(n-1)}\}\right\}.$$

Under these hypotheses we compute explicitly the limit functional of the obstacle problem above, which is of the type

$$H_0^1(\Omega) \ni u \mapsto \int_\Omega |Du|^2 dx + \int_P G(u) d\sigma.$$

1 Preliminaries and Main Result

1.1 Introduction of the Problem

We consider an obstacle problem in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$. The obstacle is the restriction to a hyper-plane of a rescaled, periodically extended function. The given data in the problem is

1. A domain Ω in $\mathbb{R}^n$, $n \geq 3$, i.e. a bounded, open, connected subset of $\mathbb{R}^n$.
2. A continuous function ψ with compact support in $B_{1/2} = \{x \in \mathbb{R}^n : |x| < 1/2\}$.
3. A hyper-plane $\Pi = \{x \in \mathbb{R}^n : x \cdot \eta = 0\}$ with unit normal $\eta = (\eta_1, \ldots, \eta_n)$ such that $e_n \notin \Pi \Longleftrightarrow \eta_n \neq 0$.

Note that for any $E \subset \mathbb{R}^n$, $P := E \cap \Pi$ can be represented as

$$P = \{(x', \alpha x') : x' \in H\}, \tag{1}$$

where $x' = (x_1, \ldots, x_{n-1})$, $x = (x', x_n)$,

$$H = \mathrm{proj}_{\mathbb{R}^{n-1}} P$$

and

$$\alpha = (\alpha_1, \ldots, \alpha_{n-1}), \quad \alpha_i = \frac{-\eta_i}{\eta_n}.$$

Let $Q_\varepsilon = (-\varepsilon/2, \varepsilon/2)$, and for any $k \in \mathbb{Z}^n$, let $Q_\varepsilon^k = Q_\varepsilon + \varepsilon k$. Similarly, $B_{r_\varepsilon}^k$ denotes the ball of radius r_ε and center εk, i.e. $B_{r_\varepsilon}^k = B_{r_\varepsilon} + \varepsilon k$. From ψ we construct the oscillating function ψ_ε, given by

$$\psi_\varepsilon(x) = \begin{cases} \psi(a_\varepsilon^{-1}(x - \varepsilon k)), & \text{if } x \in Q_\varepsilon^k \cap \Pi, \\ -\infty, & \text{otherwise,} \end{cases} \tag{2}$$

where

$$a_\varepsilon = \varepsilon^{n/(n-1)}. \tag{3}$$

Remark 1. *From the definition of* ψ_ε *it can be seen that* $\psi_\varepsilon(x) > -\infty$ *if and only if*

$$x \in \{a_\varepsilon\{y : \psi(y) > -\infty\} + \varepsilon k\} \cap \Pi, \text{ for some } k \in \mathbb{Z}^n.$$

For this reason it needs to be determined how often Π *intersects a neigbourhood of size comparable to* a_ε *of the lattice points* $\{\varepsilon k\}_{k\in\mathbb{Z}^n}$*. This is possible in* $n \geq 3$ *dimensions, using the theory of uniform distribution of sequences. In general, this is possible when* a_ε *is not "too small". When* $n = 2$ *we would have to choose a much smaller* a_ε*, due to the logarithmic nature of the fundamental solution of the laplacian. For this reason we cannot include the two dimensional case.*

For any Borel subset $\mathcal{B}$ of Ω and $u \in H_0^1(\Omega)$, set

$$F_{\psi_\varepsilon}(u, \mathcal{B}) = \begin{cases} 0, & \text{if } u \geq \psi_\varepsilon \text{ q.e. on } \mathcal{B}, \\ \infty, & \text{otherwise,} \end{cases} \tag{4}$$

where q.e. is short for quasi everywhere, i.e. everywhere except for a set of zero capacity. Note that $\mathcal{B} \mapsto F_{\psi_\varepsilon}(u, \mathcal{B})$ only depends on $\mathcal{B} \cap \Pi$. Our main goal is to determine the asymptotic behaviour, as $\varepsilon \to 0$, of minimizers of the functional

$$J_\varepsilon(u) = \int_\Omega |Du|^2 dx + F_{\psi_\varepsilon}(u, \mathcal{B}). \tag{5}$$

1.2 The Notion of Γ-convergence

Definition 1 (Γ-convergence)**.** *A sequence of functionals* J_ε *on a topological space* V *is said to* Γ*-converge to the functional* J_0 *if the following hold for all* $v \in V$*:*

(i) whenever $v_\varepsilon \to v$ *in* V*,*

$$J_0(v) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon),$$

(ii) there exists a sequence $\{v_\varepsilon\}_\varepsilon$ *such that* $v_\varepsilon \to v$ *in* V *and*

$$J_0(v) \geq \limsup_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon).$$

The functional J_0 *is called the* Γ*-limit of* J_ε*.*

Remark 2. *It follows easily from this definition that if J_ε Γ-converges to J_0, if $v_\varepsilon \in V$ solves $\inf_V J_\varepsilon(v) = J_\varepsilon(v_\varepsilon)$ and if $v_\varepsilon \to v_0$ in V, then $J_0(v_0) = \inf_V J_0(v)$. Indeed, $J_0(v_0) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon)$ by (i), and for any other $v \in V$, there exists according to (ii) a sequence $\{\bar{v}_\varepsilon\}_\varepsilon$ converging to v in V such that $J_0(v) \geq \limsup_{\varepsilon\to 0} J_\varepsilon(\bar{v}_\varepsilon)$. Since $J_\varepsilon(v_\varepsilon) \leq J_\varepsilon(\bar{v}_\varepsilon)$, $J_0(v_0) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) \leq \limsup_{\varepsilon\to 0} J_\varepsilon(\bar{v}_\varepsilon) \leq J_0(v)$, which proves the claim.*

Next we quote a theorem of De Giorgi, Dal Maso and Longo from [4]. It is a compactness result for quadratic functionals of obstacle type and states that there is a representation theorem for the Γ-limits of these functionals. The compactness part of the theorem is valid for obstacle functionals for which there exists a sequence $u_\varepsilon \in H_0^1(\Omega)$ such that both $J_\varepsilon(u_\varepsilon)$ and $\|u_\varepsilon\|_{H_0^1(\Omega)}$ are bounded. This will be true if we assume that the set $\mathcal{B}$ in (4) is compactly contained in Ω. For the formulation below we refer to Attouch and Picard [1].

Theorem 1 ([4])**.** *There is a rich family $\mathcal{R}$ of Borel subsets of Ω such that for every $\mathcal{B} \in \mathcal{R}$ satisfying $\mathcal{B} \subset\subset \Omega$, the sequence of functionals*

$$J_\varepsilon(u) = \int_\Omega |Du|^2 dx + F_{\psi_\varepsilon}(u, \mathcal{B}) \tag{6}$$

has a subsequence that Γ-converges to

$$J_0(u) = \int_\Omega |Du|^2 dx + \int_\mathcal{B} f(x,u) d\mu + \nu(\mathcal{B}), \tag{7}$$

where μ and ν are positive Radon measures, $\mu \in H^{-1}(\Omega)$ and $f(x,u)$ is convex and monotone non-increasing with respect to u.

Remark 3. *It may be assumed that $\nu = 0$, c.f. [1], Theorem 4.1. We refer to [1] for the definition of a* rich family of Borel sets. *However, we would like to point out that a rich family $\mathcal{R}$ of the Borel sets of Ω is dense in the Borel sets, in the sense that for any Borel sets A, B such that $\overline{A} \subset intB$, there exists $E \in \mathcal{R}$ such that $\overline{A} \subset intE \subset \overline{E} \subset intB$.*

1.3 Main Theorem

Next we define the functional that is the Γ-limit of J_ε in (5). For any $\lambda \in \mathbb{R}$, let

$$\psi^\lambda(x) = \begin{cases} \psi(x), & x \in \{P + \lambda\eta\}, \\ -\infty, & \text{otherwise}, \end{cases} \tag{8}$$

and set

$$g^\lambda(t) = \min\left\{\int_{\mathbb{R}^n} |Dv|^2 dx : v - t \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ v \geq \psi^\lambda \text{ q.e. on } \mathbb{R}^n\right\}, \tag{9}$$

where t is any real number and

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \{v \in L^{2^*}(\mathbb{R}^n) : Dv \in L^2(\mathbb{R}^n)\}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}.$$

Theorem 2. *Let $\Pi = \{x \in \mathbb{R}^n : x \cdot \eta = 0\}$. Then the following holds for a.e. $\eta \in S^{n-1}$: There is a rich family $\mathcal{R}$ of Borel subsets of Ω such that for every $\mathcal{B} \in \mathcal{R}$ satisfying $\mathcal{B} \subset\subset \Omega$, the sequence of functionals*

$$J_\varepsilon(u, \mathcal{B}) = \int_\Omega |Du|^2 dx + F_{\psi_\varepsilon}(u, \mathcal{B})$$

Γ-converges in the weak topology of $H_0^1(\Omega)$ to

$$J_0(u, \mathcal{B}) = \int_\Omega |Du|^2 dx + \int_{\Pi\cap\mathcal{B}} \left(\int g^\lambda(u(x)) d\lambda\right) d\sigma(x). \tag{10}$$

In particular, the sequence of minimizers u_ε of J_ε converges weakly in $H_0^1(\Omega)$ to the minimizer u of J_0.

On the right hand side of (10), σ denotes surface measure on Π.

1.4 Related Results

In the paper [6] a problem similar to the present one was solved. In [6] the obstacle is given by

$$\psi\chi_{\Pi_\varepsilon},$$

where ψ is a fixed function and Π_ε is the intersection between a hyper-plane Π and the set

$$\bigcup_{k\in\mathbb{Z}^n} \{a_\varepsilon T + \varepsilon k\},$$

where T is a fixed subset of the unit ball. Thus in both problems the obstacle is defined on the intersection between the hyper-plane Π and a neighborhood of size a_ε of the lattice points $\{\varepsilon k\}_{k\in\mathbb{Z}^n}$. It is a crucial part of the problem to estimate the number of lattice points at a given distance from a subset of

Π. For the necessary results in this direction, which come from the theory of uniform distribution, we refer to [6].

However, a main difference between the present problem and that of [6] is that the obstacle in (2) varies on a much smaller scale, of size a_ε. For this reason the techniques used in [6] (essentially those developed in [2]) are not fit to deal with this problem. Instead we use the methods of [3], which are more adapted to the situation at hand.

2 Proofs

We start by establishing some continuity properties of a certain approximation of the function g^λ in (9), that appears naturally in the proof of Theorem 2.

Lemma 1. *Let*

$$g_R^\lambda(t) = \min\left\{\int_{B_R} |Dv|^2 dx : v - t \in H_0^1(B_R),\ v \geq \psi^\lambda \text{ q.e. on } B_R\right\}. \quad (11)$$

Assume $|\psi| \leq A$ *and that* ψ *has modulus of continuity* ρ $(|\psi(x) - \psi(y)| \leq \rho(|x-y|))$. *Then* $\lim_{R\to\infty} g_R^\lambda(t) = g^\lambda(t)$ *and for any* $2 \leq R_0 < R_1 \leq \infty$ *and any* $\lambda \in \mathbb{R}$,

$$|g_{R_1}^\lambda(t) - g_{R_2}^\lambda(t)| \leq C(A-t)_+^2(R_0^{2-n} - R_1^{2-n}), \quad (12)$$

and

$$|g_R^{\lambda+\delta}(t) - g_R^\lambda(t)| \leq C_1(A-t)_+^2((R-\delta)^{2-n} - R^{2-n}) + C_2\rho(\delta), \quad (13)$$

where C, C_1, C_2 *depend only on* n.

Proof. We may assume $t \leq A$, for otherwise $g_R^\lambda(t) = 0$. Let K^λ and K_R^λ be the set of constraints appearing in the definition of g^λ and g_R^λ respectively. That is,

$$K^\lambda = \left\{v - t \in \mathcal{D}^{1,2}(\mathbb{R}^n),\ v \geq \psi^\lambda \text{ q.e. on } \mathbb{R}^n\right\}$$

and

$$K_R^\lambda = \left\{v - t \in H_0^1(B_R),\ v \geq \psi^\lambda \text{ q.e. on } B_R\right\}.$$

Since $K_{R_0}^\lambda \subset K_{R_1}^\lambda \subset K^\lambda$ for $R_0 < R_1$, we immediately obtain $g^\lambda(t) \leq g_{R_1}^\lambda(t) \leq g_{R_0}^\lambda(t)$. The claim $\lim_{R\to\infty} g_R^\lambda(t) = g^\lambda(t)$ follows from the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $D^{1,2}(\mathbb{R}^n)$.

Fix a smooth cut-off function ζ with compact support in B_2 such that $\zeta \equiv 1$ on B_1. Then $(A-t)\zeta + t \in K_R^\lambda$ for any $R \geq 2$, $\lambda \in \mathbb{R}$ and any $t \leq A$. Thus

$$g_R^\lambda(t) \leq (A-t)^2 \int_{B_2} |D\zeta|^2 dx \leq C(A-t)_+^2. \tag{14}$$

Let $v \in K^\lambda$ satisfy $\int_{\mathbb{R}^n} |Dv|^2 dx = g^\lambda(t)$, and let $v_R \in K_R^\lambda$ satisfy $\int_{B_R} |Dv_R|^2 dx = g_R^\lambda(t)$. To estimate $v - v_R$ we construct a barrier h that is the solution to $\Delta h = 0$ in $\mathbb{R}^n \setminus B_1$, $h - t \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ and $h = A$ on B_1. In $\mathbb{R}^n \setminus B_1$, $h - v$ is harmonic, on B_1, $h - v \geq 0$ and $h - v \to 0$ at infinity. It follows from the maximum principle that $v \leq h$ in $\mathbb{R}^n$. The function h is spherically symmetric and has the explicit expression

$$h(r) = (A-t)r^{2-n} + t,$$

for $r > 1$, where $r = |x|$. It follows that

$$v(x) \leq (A-t)R^{2-n} + t \quad \text{on } \mathbb{R}^n \setminus B_R.$$

Thus

$$\hat{v}_R = \max\left(t, v - (1-\zeta)(A-t)R^{2-n}\right)$$

belongs to K_R^λ. Hence

$$\begin{aligned} g_R^\lambda(t) &\leq \int_{B_R} |D\hat{v}_R|^2 dx \\ &\leq \int_{B_R} |Dv|^2\, dx + 2(A-t)R^{2-n} \int_{B_R} D\zeta Dv dx + ((A-t)R^{2-n})^2 \int_{B_R} |D\zeta|^2 dx \\ &\leq g^\lambda(t) + 2(A-t)R^{2-n} \|D\zeta\|_{L^2(B_R)} \sqrt{g^\lambda(t)} + ((A-t)R^{2-n})^2 \int_{B_R} |D\zeta|^2 dx. \end{aligned}$$

Hence we obtain, using (14),

$$|g^\lambda(t) - g_R^\lambda(t)| \leq C(A-t)^2 R^{2-n}. \tag{15}$$

If $2 < R_0 < R_1$, we find in a similar way that

$$v_{R_1} \leq h_{R_1} = (A-t)\frac{r^{2-n} - R_1^{2-n}}{1 - R_1^{2-n}} + t \quad \text{on } B_{R_1} \setminus B_1,$$

and that

$$\hat{v}_{R_0} = \max\left(t, v_{R_1} - (1-\zeta)(A-t)\frac{R_0^{2-n} - R_1^{2-n}}{1 - R_1^{2-n}}\right)$$

belongs to $K^\lambda_{R_0}$. From this we obtain the estimate

$$|g^\lambda_{R_1}(t) - g^\lambda_{R_2}(t)| \leq C(A-t)^2(R_0^{2-n} - R_1^{2-n}). \qquad (16)$$

Next we prove the continuity w.r.t. λ. For any $\gamma > 0$ there exists a $\delta > 0$ ($\delta = \rho^{-1}(\gamma)$) such that

$$\psi^\lambda(x + \delta\eta) - \gamma < \psi^{\lambda+\delta}(x) \leq \psi^\lambda(x + \delta\eta) + \gamma.$$

Let

$$h_R = \frac{r^{2-n} - R^{2-n}}{1 - R^{2-n}},$$

for $r = |x| > 1$, $h_R = 1$ on B_1. Let $v^\lambda_{R-\delta} \in K^\lambda_{R-\delta}$ satisfy $\int_{B_{R-\delta}} |Dv^\lambda_{R-\delta}|^2 dx = g^\lambda_{R-\delta}$. Then $w_R(x) = v^\lambda_{R-\delta}(x + \delta\eta) + \gamma h_R(x)$ belongs to $K^{\lambda+\delta}_R$. Hence,

$$\begin{aligned}
g^{\lambda+\delta}_R(t) &\leq \int_{B_R} |Dw_R|^2 dx \\
&= \int_{B_R} |Dv^\lambda_{R-\delta}(x+\delta\eta)|^2 dx + \gamma^2 \int_{B_R} |Dh_R|^2 dx + 2\gamma \int_{B_R} Dh_R Dv^\lambda_{R-\delta} dx \\
&\leq g^\lambda_R(t) + C(A-t)^2((R-\delta)^{2-n} - R^{2-n}) \\
&+ \gamma^2 \int_{B_R} |Dh_R|^2 dx + 2\gamma \|Dv^\lambda_{R-\delta}\|_{L^2(B_R)} \|Dh_R\|_{L^2(B_R)}.
\end{aligned}$$

It is easy to check that $\int_{B_R} |Dh_R|^2 dx$ is bounded uniformly in R. In fact, as $R \to \infty$, $\int_{B_R} |Dh_R|^2 dx \to \mathrm{cap}(B_1)$, the capacity of the unit ball. By interchanging the roles of $g^{\lambda+\delta}_R(t)$ and $g^\lambda_R(t)$ we obtain a lower bound on $g^{\lambda+\delta}_R(t) - g^\lambda_R(t)$. Thus for any $\gamma > 0$, we have (assuming $\gamma < 1$)

$$|g^{\lambda+\delta}_R(t) - g^\lambda_R(t)| \leq C_1(A-t)^2((R-\delta)^{2-n} - R^{2-n}) + C_2\gamma. \qquad (17)$$

□

We now turn to the

proof of Theorem 2. Let w^k_ε be the solution to

$$\min\left\{\int_{Q^k_\varepsilon} |Dw|^2 dx : w \geq \psi_\varepsilon \text{ q.e. on } Q^k_\varepsilon,\ w = t \text{ on } Q^k_\varepsilon \setminus B^k_{\varepsilon/2}\right\}. \qquad (18)$$

The following definition will be important in the sequel. In order to simplify notation we set $P = \Pi \cap \mathcal{B}$.

Definition 2. *Let λ_ε^k be the unique real number such that*

$$Q_\varepsilon^k \cap P = Q_\varepsilon \cap \{P + \lambda_\varepsilon^k \eta\} \quad (mod\ \varepsilon), \quad \text{if } Q_\varepsilon^k \cap P \neq \emptyset.$$

If $Q_\varepsilon^k \cap P = \emptyset$ we set $\lambda_\varepsilon^k = \infty$.

Let $y = x - \varepsilon k$. Then

$$y + \varepsilon k \in Q_\varepsilon^k \cap P \Longleftrightarrow y \in Q_\varepsilon \cap \{P + \lambda_\varepsilon^k \eta\}.$$

Thus

$$\int_{Q_\varepsilon^k} |Dw_\varepsilon^k|^2 dx$$
$$= \min \left\{ \int_{Q_\varepsilon} |Dw|^2 dx : w \geq \psi_\varepsilon^{\lambda_\varepsilon^k} \text{ q.e. on } Q_\varepsilon,\ w = t \text{ on } Q_\varepsilon \setminus B_{\varepsilon/2} \right\},$$

where $\psi_\varepsilon^{\lambda_\varepsilon^k}$ is ψ_ε with $P + \lambda_\varepsilon^k \eta$ in place of P. Clearly, $w_\varepsilon^k = t$ if $\psi_\varepsilon^{\lambda_\varepsilon^k} \leq t$. In particular, $w_\varepsilon^k = t$ if $Q_\varepsilon^k \cap (\Omega \cap P) = \emptyset$. Let $z = a_\varepsilon^{-1} y$. Then, noting that $a_\varepsilon z = y \in Q_\varepsilon \cap \{P + \lambda_\varepsilon^k \eta\} \Longleftrightarrow z \in Q_{\varepsilon/a_\varepsilon} \cap \{P + (\lambda_\varepsilon^k / a_\varepsilon)\eta\}$,

$$\int_{Q_\varepsilon^k} |Dw_\varepsilon^k|^2 dx = \min \left\{ a_\varepsilon^{n-2} \int_{Q_{\varepsilon/a_\varepsilon}} |Dw|^2\, dx : w \geq \psi^{\lambda_\varepsilon^k/a_\varepsilon} \text{ q.e. on } Q_{\varepsilon/a_\varepsilon}, \right.$$
$$\left. \text{and } w = t \text{ on } Q_{\varepsilon/a_\varepsilon} \setminus B_{\varepsilon/2a_\varepsilon} \right\}.$$

Let $R_\varepsilon = \varepsilon/2a_\varepsilon$. The choice of a_ε implies that $\lim_{\varepsilon\to 0} R_\varepsilon = \infty$. Since $w - t$ has its support in B_{R_ε} and $\psi^{\lambda_\varepsilon^k/a_\varepsilon} = -\infty$ outside $B_1 \subset B_{R_\varepsilon}$, we have

$$\min \left\{ a_\varepsilon^{n-2} \int_{Q_{\varepsilon/a_\varepsilon}} |Dw|^2\, dx : w \geq \psi^{\lambda_\varepsilon^k/a_\varepsilon} \text{ q.e. on } Q_{\varepsilon/a_\varepsilon}, \right.$$
$$\left. \text{and } w = t \text{ on } Q_{\varepsilon/a_\varepsilon} \setminus B_{\varepsilon/2a_\varepsilon} \right\} =$$

$$= \min \left\{ a_\varepsilon^{n-2} \int_{B_{R_\varepsilon}} |Dw|^2\, dx : w \geq \psi^{\lambda_\varepsilon^k/a_\varepsilon} \text{ q.e. on } B_{R_\varepsilon}, \right.$$
$$\left. \text{and } w - t \in H_0^1(B_{R_\varepsilon}) \right\}$$
$$= a_\varepsilon^{n-2} g_{R_\varepsilon}^{\lambda_\varepsilon^k/a_\varepsilon}(t).$$

It is clear that $\psi^{\lambda_\varepsilon^k/a_\varepsilon} \equiv -\infty$ for small enough $\varepsilon > 0$ if $a_\varepsilon = o(\lambda_\varepsilon)$. Choose $\lambda_0 < \lambda_1$ such that $B_1 \cap \{P + \lambda\eta\} = \emptyset$ if $\lambda \notin [\lambda_0, \lambda_1]$. Let $\delta > 0$ be a small number such that $\lambda_1 = \lambda_0 + M\delta$ for some positive integer M, and let $\lambda_j = \lambda_0 + j\delta$. Now set $\lambda_{\varepsilon,j} = a_\varepsilon \lambda_j$ and let

$$I_{\varepsilon,j} = \{Q_\varepsilon \cap \{P + \lambda\eta\} : \lambda_{\varepsilon,j} \leq \lambda \leq \lambda_{\varepsilon,j+1}\},$$
$$I_{\varepsilon,j}^k = \{I_{\varepsilon,j} + \varepsilon k\}, \quad k \in \mathbb{Z}^n.$$

Let $A_{\varepsilon,j}$ be the number of $k \in \mathbb{Z}^n$ for which P and $I_{\varepsilon,j}^k$ has non-empty intersection. This is precisely the number of $k = (k', k_n)$ such that εk_n and $\alpha\varepsilon k'$ belong to the same cube Q_ε^k, and $\lambda_\varepsilon^k \in I_{\varepsilon,j}$, where we use the notation in (1). Let

$$P_\varepsilon = \{k \in \mathbb{Z}^n : Q_\varepsilon^k \cap P \neq \emptyset\}.$$

Thus if

$$\mathbb{K}_{\varepsilon,j} = \{k \in P_\varepsilon : \lambda_\varepsilon^k \in I_{\varepsilon,j}\},$$

then

$$A_{\varepsilon,j} = \#\mathbb{K}_{\varepsilon,j}.$$

It was proven in [6], Lemma 5.2.2, that for a.e. $\eta \in S^{n-1}$,

$$A_{\varepsilon,j} = |P|\frac{\delta a_\varepsilon}{\varepsilon^n} + o(a_\varepsilon \varepsilon^{-n}). \tag{19}$$

To make the statement more precise we introduce

$$N_\varepsilon = \#\{k' \in \mathbb{Z}^{n-1} \cap \mathrm{proj}_{\mathbb{R}^{n-1}} \varepsilon^{-1} P\}.$$

Then, since the intersection between P and $I_{\varepsilon,j}^k$ is completely determined by the value of $\varepsilon\alpha k'$ at a point $(\varepsilon k', \alpha\varepsilon k') \in P$, we have

$$A_{\varepsilon,j} = \#\left\{k' \in \mathbb{Z}^{n-1} \cap \mathrm{proj}_{\mathbb{R}^{n-1}} \varepsilon^{-1} P : \alpha k'/\mathbb{Z} \in [p_j, p_j + \delta a_\varepsilon/(\eta_n \varepsilon)]/\mathbb{Z}\right\},$$

where p_j is chosen such that

$$P \cap I_{\varepsilon,j}^k \neq \emptyset \text{ iff } \alpha k'/\mathbb{Z} \in [p_j, p_j + \delta a_\varepsilon/(\eta_n \varepsilon)]/\mathbb{Z}.$$

Note that the distance δa_ε in η (normal) direction between two planes, corresponds to the distance $\delta a_\varepsilon/\eta_n$ in e_n direction between these planes. Using tools from the theory of uniform distribution mod 1, it can be shown that

$$\left|\frac{A_{\varepsilon,j}}{N_\varepsilon} - \frac{\delta a_\varepsilon}{\varepsilon\eta_n}\right| = o(\varepsilon^s), \quad \text{for any } s \in (0,1).$$

This implies (19) since $a_\varepsilon/\varepsilon \geq \sqrt{\varepsilon}$ for $n \geq 3$. Define w_ε by $w_\varepsilon = w_\varepsilon^k$ on Q_ε^k. Since $w_\varepsilon^k = t$ on $\partial B_{r_\varepsilon}^k$, $w_\varepsilon \in H^1(\Omega)$ and, noting that $w_\varepsilon^k \equiv t$ if $k \notin \mathbb{K}_{\varepsilon,j}$ for some j,

$$\int_\Omega |Dw_\varepsilon|^2 dx = \sum_{j=0}^{M} \sum_{k\in\mathbb{K}_{\varepsilon,j}} \int |Dw_\varepsilon^k|^2 dx \tag{20}$$

$$= \sum_{j=0}^{M} \sum_{k\in\mathbb{K}_{\varepsilon,j}} a_\varepsilon^{n-2} \left(g_{R_\varepsilon}^{\lambda_\varepsilon^k/a_\varepsilon}(t) - g_{R_\varepsilon}^{\lambda_j}(t) \right) + \sum_{j=0}^{M} a_\varepsilon^{n-2} A_{\varepsilon,j}\, g_{R_\varepsilon}^{\lambda_j}(t). \tag{21}$$

Since $|\lambda_\varepsilon^k/a_\varepsilon - \lambda_j| \leq \delta$ when $k \in \mathbb{K}_{\varepsilon,j}$, we have for such k that

$$\left| g_{R_\varepsilon}^{\lambda_\varepsilon^k/a_\varepsilon}(t) - g_{R_\varepsilon}^{\lambda_j}(t) \right| \leq C_1(A-t)_+^2((R_\varepsilon - \delta)^{2-n} - R_\varepsilon^{2-n}) + C_2\rho(\delta) =: E(\varepsilon, \delta),$$

by (13) in Lemma 1. Hence the first term in (21) is bounded by

$$\sum_{j=0}^{M} A_{\varepsilon,j} a_\varepsilon^{n-2} E(\varepsilon,\delta) \leq C \sum_{j=0}^{M} |P|\delta \frac{a_\varepsilon^{n-1}}{\varepsilon^n}\, E(\varepsilon,\delta) \leq C|P|\, E(\varepsilon,\delta), \tag{22}$$

where we used (19), the fact that $a_\varepsilon^{n-1}/\varepsilon^n = 1$ by the choice of a_ε in (3) and that $M = 1/\delta$. The right hand side of (22) clearly tends to zero as $\varepsilon, \delta \to 0$ in any order. The term $a_\varepsilon^{n-2} A_{\varepsilon,j}\, g_{R_\varepsilon}^{\lambda_j}(t)$ converges to $|P|\delta g^{\lambda_j}(t)$ as $\varepsilon \to 0$. Hence,

$$\int |Dw_\varepsilon|^2\, dx = \sum_{j=0}^{M} \sum_{k\in\mathbb{K}_{\varepsilon,j}} \int |Dw_\varepsilon^k|^2 dx = O(\rho(\delta)) + \sum_{j=0}^{M} A_{\varepsilon,j} g_{R_\varepsilon}^{\lambda_j}(t)$$

$$\to \sum_{j=0}^{M} \delta |P| g^{\lambda_j}(t),$$

as $\varepsilon \to 0$. Letting $\delta \to 0$, we obtain

$$\int_\Omega |Dw_\varepsilon|^2 dx = \sum_k \int_\Omega |Dw_\varepsilon^k|^2 dx \to |P| \int_{\lambda_0}^{\lambda_1} g^\lambda(t) d\lambda. \tag{23}$$

The next step is to show that $w_\varepsilon \rightharpoonup t$ in $H^1(\Omega)$. Since $w_\varepsilon - t \in H_0(B_{\varepsilon/2}^k)$, Poincare's inequality implies that

$$\int_{B_{\varepsilon/2}^k} |w_\varepsilon^k - t|^2 dx \leq \varepsilon \int_{B_{\varepsilon/2}^k} |Dw_\varepsilon^k|^2 dx.$$

Indeed, the Poincare constant of a ball of radius R does not exceed R. Thus

$$\int_\Omega |w_\varepsilon - t|^2 dx = \sum_k \int_{B^k_{\varepsilon/2}} |w^k_\varepsilon - t|^2 dx \tag{24}$$

$$\leq \varepsilon \sum_k \int_{B^k_{\varepsilon/2}} |Dw^k_\varepsilon|^2 dx = \varepsilon^2 \int_\Omega |Dw_\varepsilon|^2 \, dx. \tag{25}$$

By (23) $\{w_\varepsilon\}_\varepsilon$ is bounded in $H^1_0(\Omega)$ and hence has a weakly convergent subsequence. From (24)-(25) it follows that every weakly convergent subsequence must converge to t, thus the entire sequence $\{w_\varepsilon\}_\varepsilon$ converges weakly to t.

By Theorem 1, $J_\varepsilon(u) = \int_\Omega |Du|^2 dx + F_{\psi_\varepsilon}(u, \mathcal{B})$ has a subsequence that Γ-converges to a functional of the type $J_0(u) = \int_\Omega |Du|^2 dx + \int_\mathcal{B} f(x,u) d\mu$. We will prove that for each $t \in \mathbb{R}$,

$$\int_\mathcal{B} f(x,t) d\mu = \sigma(\Pi \cap \mathcal{B}) \int g^\lambda(t) d\lambda. \tag{26}$$

Let us show that the theorem follows from (26). Due to (26) and the fact that the family of sets $\mathcal{R} \ni \mathcal{B}$ is dense in the Borel subsets of Ω, $f(x,t)d\mu$ is a measure on Π, absolutely continuous w.r.t. σ. Hence $f(x,t)d\mu = h(x,t)d\sigma$ for some $h(x,t) \in L^1_{loc}(\Pi, \sigma)$. But

$$\int_{\Pi \cap \mathcal{B}} h(x,t) d\sigma = \sigma(\Pi \cap \mathcal{B}) \int g^\lambda(t) d\lambda$$

for all $t \in \mathbb{R}$ and all $\mathcal{B} \in \mathcal{R}$ implies that h is independent of x, thus $h(x,t) = h(t) = \int g^\lambda(t) d\lambda$.

We now prove (26). Choose $v \in C^\infty_c(\Omega)$ such that $v = t$ on a neigbourhood of $\mathcal{B}$. Let

$$v_\varepsilon(x) = \begin{cases} w_\varepsilon(x), & \text{if } x \in \mathcal{B}, \\ v(x), & \text{if } x \in \Omega \setminus \mathcal{B}. \end{cases} \tag{27}$$

Then clearly $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$. According to Definition 1 (i),

$$\int_\Omega |Dv|^2 dx + \int_\mathcal{B} f(u,x) d\mu = \int_{\Omega \setminus \mathcal{B}} |Dv|^2 dx + \int_\mathcal{B} f(t,x) d\mu$$

$$\leq \liminf_{\varepsilon \to 0} \int_\Omega |Dv_\varepsilon|^2 dx = \int_{\Omega \setminus \mathcal{B}} |Dv|^2 dx + \sigma(\mathcal{B} \cap \Pi) \int g^\lambda(t) d\lambda.$$

It remains to prove that

$$\int_{\mathcal{B}} f(x,t)d\mu \geq \sigma(\mathcal{B}\cap\Pi)g^\lambda(t)d\lambda. \tag{28}$$

Let z_ε be a sequence given by Definition 1 (ii), i.e. $z_\varepsilon \rightharpoonup v$ and $\limsup_\varepsilon J_\varepsilon(z_\varepsilon) \leq J_0(v)$. By (i) in the same definition, we have $\lim_{\varepsilon\to 0} J_\varepsilon(z_\varepsilon) = J_0(v)$. Since v is bounded we may assume z_ε is bounded. To see this we assume $|v| \leq C$ and claim that

$$\bar{z}_\varepsilon = \min(z_\varepsilon^+, 2C) - \min(z_\varepsilon^-, 2C) \rightharpoonup v.$$

Indeed, $\bar{z}_\varepsilon$ is uniformly bounded in $H^1(\Omega)$ and therefore has a weak limit in this space. Moreover,

$$\begin{aligned}\int_\Omega |\bar{z}_\varepsilon - v|^2 dx = \int_{\Omega\setminus\{|z_\varepsilon|>2C\}} |z_\varepsilon - v|^2 dx - \int_{\{z_\varepsilon>2C\}} |2C - v|^2 dx \\ - \int_{\{z_\varepsilon<-2C\}} |-2C - v|^2 dx.\end{aligned}$$

Since $z_\varepsilon \to v$ strongly in $L^2(\Omega)$ and

$$\int_\Omega |z_\varepsilon - v|^2 dx \geq C^2 \text{measure}(\{|z_\varepsilon| > 2C\}),$$

measure$(\{|z_\varepsilon| > 2C\}) \to 0$ and hence $\bar{z}_\varepsilon \to v$ strongly in $L^2(\Omega)$. Additionally, $\int |D\bar{z}_\varepsilon|^2 dx \leq \int |Dz_\varepsilon|^2 dx$, which implies, again by (i) in Definition 1,

$$\lim_{\varepsilon\to 0} J_\varepsilon(\bar{z}_\varepsilon) = J_0(v) = \int_{\Omega\setminus\mathcal{B}} |Dv|^2 dx + \int_{\mathcal{B}} f(t,x)d\mu.$$

Thus if we let v_ε be the function given by (27), (28) follows if we prove

$$\begin{cases} \lim_{\varepsilon\to 0} \int_\Omega |Dv_\varepsilon|^2 dx \leq \lim_{\varepsilon\to 0} \int_\Omega |Dz_\varepsilon|^2 dx, \\ \text{for all } z_\varepsilon \in H_0^1(\Omega) \text{ such that } z_\varepsilon \geq \psi_\varepsilon, \\ z_\varepsilon \rightharpoonup v \text{ and } \sup_{\varepsilon>0} \|z_\varepsilon\|_{L^\infty} < \infty. \end{cases} \tag{29}$$

By convexity of the functional $v \mapsto \int_\Omega |Dv|^2 dx$, we have

$$\int_\Omega |Dz_\varepsilon|^2 - |Dv_\varepsilon|^2 dx \geq 2\int_\Omega Dv_\varepsilon(Dz_\varepsilon - Dv_\varepsilon)dx \tag{30}$$

$$= \langle -\Delta v_\varepsilon, z_\varepsilon - v_\varepsilon\rangle = \int_{\Omega\setminus\mathcal{B}} -\Delta v(z_\varepsilon - v)dx + \sum_k \langle -\Delta w_\varepsilon^k, z_\varepsilon - w_\varepsilon^k\rangle, \tag{31}$$

where the sum is taken over

$$\{k \in \mathbb{Z}^n : \Pi \cap \mathcal{B} \subset \{a_\varepsilon\{y : \psi(y) > -\infty\} + \varepsilon k\} \ (\subset B^k_{a_\varepsilon/2})\}.$$

The first term in (31) goes to zero since v is smooth and $z_\varepsilon \rightharpoonup v$. The Laplacian of w^k_ε consists of two measures μ^k_ε and ν^k_ε such that

$$-\Delta w_\varepsilon = \mu^k_\varepsilon - \nu^k_\varepsilon,$$

where

$$\nu^k_\varepsilon(E) = -\int_{E\cap Q^k_\varepsilon} \frac{\partial w^k_\varepsilon}{\partial n} dS,$$

and

$$\operatorname{supp}\mu^k_\varepsilon \subset \{w^k_\varepsilon = \psi^\varepsilon\} \subset B^k_{a_\varepsilon}, \tag{32}$$

which follows from the fact that w^k_ε solves (18) (see [5]). From (32) and the fact that $z_\varepsilon \geq \psi_\varepsilon$ it follows that

$$\begin{aligned}\int_{Q^k_\varepsilon} (z_\varepsilon - w^k_\varepsilon) d\mu^k_\varepsilon &= \int_{Q^k_\varepsilon} (z_\varepsilon - \psi_\varepsilon) d\mu^k_\varepsilon + \int_{Q^k_\varepsilon} (\psi_\varepsilon - w^k_\varepsilon) d\mu^k_\varepsilon \\ &= \int_{Q^k_\varepsilon} (z_\varepsilon - \psi_\varepsilon) d\mu^k_\varepsilon \geq 0.\end{aligned}$$

It remains to show that

$$\lim_{\varepsilon\to 0} \sum_k \int_{Q^k_\varepsilon} (z_\varepsilon - w^k_\varepsilon) d\nu^k_\varepsilon = 0.$$

Let W^k_ε solve

$$\min\left\{\int_{Q^k_\varepsilon} |DW|^2 dx : W - t \in H^1_0(B^k_{\varepsilon/2}) \text{ and } W \geq \max\psi = A \text{ on } B^k_{a_\varepsilon}\right\}.$$

Since $W^k_\varepsilon = w^k_\varepsilon$ on $\partial B^k_{\varepsilon/2}$, $W^k_\varepsilon \geq w^k_\varepsilon$ on $B^k_{a_\varepsilon}$ and W^k_ε and w^k_ε are harmonic in $B^k_{\varepsilon/2} \setminus B^k_{a_\varepsilon}$, we get $W^k_\varepsilon \geq w^k_\varepsilon$ in $B^k_{\varepsilon/2}$ from the maximum principle, hence

$$-\frac{\partial W^k_\varepsilon}{\partial n} \geq -\frac{\partial w^k_\varepsilon}{\partial n} \text{ on } \partial B^k_{\varepsilon/2}.$$

Thus if we let

$$\hat{\nu}^k_\varepsilon(E) = \int_{\partial B^k_{\varepsilon/2}\cap E} -\frac{\partial W^k_\varepsilon}{\partial n} dS,$$

and set $\hat{\nu}_\varepsilon = \sum_k \hat{\nu}_\varepsilon^k$, $\nu_\varepsilon = \sum_k \nu_\varepsilon^k$, then $\hat{\nu}_\varepsilon \geq \nu_\varepsilon$. In [6] (see the proof of Lemma 2.0.8 therein) it was shown that

$$\lim_{\varepsilon \to 0} \int_\Omega (h_\varepsilon - h) d\hat{\nu}_\varepsilon = 0, \tag{33}$$

whenever $h_\varepsilon \rightharpoonup h$ in $H_0^1(\Omega)$ and $\sup_{\varepsilon>0} \|h_\varepsilon\|_{L^\infty} < \infty$. Since $\nu_\varepsilon \leq \hat{\nu}_\varepsilon$, it follows that (33) holds for ν_ε after writing $(h_\varepsilon - h) = (h_\varepsilon - h)_+ - (h_\varepsilon - h)_-$. This proves (29). Since the Γ-limit J_0 does not depend on the particular Γ-convergent subsequence, the entire sequence J_ε Γ-converges to J_0.

□

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