Hybrid and Embedded Systems EL2450 - Exercise 6

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EL2450 - Exercise 6, January 29th, 2020

Content

- Lyapunov
- Sampling
- Event-triggered control

What can we use Lyapunov for? What is sampling? What is event-triggered control?

Exercise 6.1

Problem

$$\dot{x} = x + u$$
$$u = -2x$$
$$x = 5 \text{ when } t = 0$$

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(a) Show that under this control law, the closed-loop system asymptotically converges to the origin, using a Lyapunov function.

(b) Suppose $x(t)$ is sampled periodically and the controller is followed by a ZOH:

$$u(t) = -2x(kh), \qquad t \in [kh, kh + h),$$

where $k \in \mathbb{N}$, $h > 0$ is the sampling period. What is the maximal value $h_{\max}$ of h before the closed-loop system becomes unstable?

(c) Now $x(t)$ is sampled aperiodically at the sequence $\{t_k\}$, $k \in \mathbb{N}$ and the controller is followed by a ZOH:

$$u(t) = -2x(t_k), \; t \in [t_k, t_{k+1}).$$

how to design the sequence $\{t_k\}$, $k \in \mathbb{N}$ such that the closed-loop system still converges to the origin? what is the maximal sampling interval $\max_k(t_{k+1} - t_k)$ in this case?

(d) Describe how the event-triggered controller could be implemented on a digital platform and compare it with periodic controller in (b).

Exercise 6.7

Problem

$$(S): \begin{cases} \dot{x}_1(t) = 3x_1(t) + x_2(t) + u(t) \\ \dot{x}_2(t) = 5x_1(t) - 2x_2(t) + u(t) \end{cases}$$

with $[x_1, x_2]^\top = x \in \mathbb{R}^2, u \in \mathbb{R}, t \geq 0$ and initial conditions $x_1(0) = x_2(0) = 1$.

(a) Show that the system is unstable.

(b) Determine a linear state-feedback controller $u(t) = Kx(t)$ with $K = [K_1\ K_2]$, $K_1, K_2 \in \mathbb{R}$ such that the poles of the closed loop system are placed in -2 and -4.

(c) In order to implement the controller on a digital platform, the state of the system is sampled aperiodically at a sequence of time instants $\{t_k\}, k \in \mathbb{N}$, and the control signal is now given by

$$u(t) = Kx(t_k), t \in [t_k, t_{k+1}).$$

Find the closed loop equation of the system in terms of the state $x(t)$ and the state error $e(t)$, where

$$e(t) = x(t_k) - x(t), t \in [t_k, t_{k+1}).$$

(d) By using the positive definite quadratic Lyapunov function

$$V(x) = \frac{1}{2} x^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x$$

find a relation between the error $e(t)$ and the state $x(t)$ such that the system is still asymptotically stable.

Exercise 6.8

Problem

3 robots R_1, R_2 and R_3 (controlled by a integrator) want to meet at the same place. $p_i(t) \in \mathbb{R}^2$ is the position of robot R_i, and the motion of the robot is described by: $\dot{p}_i(t) = u_i(t)$. R_1 follows R_2, R_2 follows R_3, and R_3 follows R_1, as described by the following equations:

$$u_1(t) = p_2(t) - p_1(t), \quad u_2(t) = p_3(t) - p_2(t), \quad u_3(t) = p_1(t) - p_3(t).$$

Consider the state variables $x_1(t) = p_2(t) - p_1(t)$ and $x_2(t) = p_3(t) - p_2(t)$, $x(t) = [x_1(t), x_2(t)]^\top$ and $u(t) = [u_1(t), u_2(t), u_3(t)]^\top$. (The goal is reached if $x_1 = x_2 = 0$.)

(a) Find the state space representation:

$$\dot{x}(t) = Bu(t)$$

(b) Find the matrix K such that:

$$u(t) = Kx(t)$$

(c) Write the closed-loop system as:

$$\dot{x}(t) = BKx(t)$$

(d) Use the Lyapunov function:

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

to show that the robots asymptotically meet at the same place.

Exercise 6.8

Problem continued

Now, the positions are updated only on the aperiodic sampling times t_k, with $k \in \mathbb{N}$. Therefore, the control inputs becomes, for $t \in [t_k, t_{k+1})$:

$$u_1(t) = p_2(t_k) - p_1(t_k), \quad u_2(t) = p_3(t_k) - p_2(t_k), \quad u_3(t) = p_1(t_k) - p_3(t_k).$$

(e) Let $e(t) = x(t) - x(t_k)$, and write the closed-loop system for $t \in [t_k, t_{k+1})$ as a function of $x(t)$ and $e(t)$.

(f) Using the same Lyapunov function as in (d), to find a condition in the form $\|e(t)\| \leq \alpha \|x(t)\|$, with $\alpha > 0$, that guarantees that the robot asymptotically meet at the same place. Choose α as large as possible.

Hint: for a matrix $M \in \mathbb{R}^{n \times m}$, we have $||M|| = \sqrt{\lambda_{\max}(M^T M)}$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue.