Quantum

Lecture 2

- Dirac notation
- Hilbert space quantum mechanics

Dirac Notation

A Hilbert space $\mathcal{H}$, with inner product $\langle \cdot\,,\cdot \rangle = g(\cdot\,,\cdot)$

Elements in $\mathcal{H}$ are denoted $|x\rangle$, kets

Elements in $\mathcal{H}^*$ are denoted $\langle x|$, bras

$\langle x| \in \mathcal{H}^*$ iff

$$\langle x|(|x\rangle) = g(|y\rangle, |x\rangle)$$

for some $|y\rangle \in \mathcal{H}$ for all $|x\rangle \in \mathcal{H}$

$\Rightarrow$ for $|z\rangle \in \mathcal{H}$ the corresponding bra is $\langle z|(\cdot) = g(z, \cdot)$

Hence the notation $\langle x|y\rangle$ ("bra(c)ket") means *both/either*

mapping $|y\rangle$ to $\langle x|(|y\rangle)$ and/or

carrying out the inner product $g(|x\rangle, |y\rangle) = \langle |x\rangle, |y\rangle\rangle$

Linear operators O act on kets, notation $O(|x\rangle) = O|x\rangle$

The outer product $|x\rangle\langle y|$ between $|x\rangle$ and $|y\rangle$ is the linear operator L that solves $L|z\rangle = g(|y\rangle, |z\rangle)|x\rangle = \langle y|z\rangle|x\rangle$

For compact self-adjoint operators O we have

$$O|x\rangle = \sum_i \lambda_i P_i(|x\rangle)$$

where $\{\lambda_i\}$ are the (distinct) eigenvalues and P_i is the projection onto

$$\{|x\rangle : O|x\rangle = \lambda_i|x\rangle\}$$

That is, $P_i(|x\rangle) = \sum_j \langle x|u_{ij}\rangle|u_{ij}\rangle$ over all orthonormal eigenvectors $|u_{ij}\rangle$ corresponding to the ith eigenvalue λ_i

Since $\langle x|u_{ij}\rangle|u_{ij}\rangle = |u_{ij}\rangle\langle u_{ij}|(|x\rangle) = |u_{ij}\rangle\langle u_{ij}|x\rangle$ (where $\langle u_{ij}|x\rangle = \langle x|u_{ij}\rangle$ becase O is self-adjoint) we get

$$P_i = \sum_j |u_{ij}\rangle\langle u_{ij}|$$

Because the $|u_{ij}\rangle$'s form an orthonormal basis we can write

$$O(\cdot) = \sum_i \lambda_i P_i(\cdot) = \sum_i \lambda_i \sum_j |u_{ij}\rangle\langle u_{ij}|(\cdot), \quad |x\rangle = \sum_{ij} a_{ij}|u_{ij}\rangle$$

to get

$$O|x\rangle = \sum_i \lambda_i \sum_j a_{ij}|u_{ij}\rangle$$

and

$$g(|x\rangle, O|x\rangle) = \langle x|O|x\rangle = \sum_i \lambda_i \sum_j |\langle x|u_{ij}\rangle|^2 = \sum_i \lambda_i \sum_j |a_{ij}|^2$$

Notation for tensor product, $x \otimes y = |x\rangle|y\rangle = |xy\rangle$ (more in Lec. 3)

For operators O and T, we have the composition OT defined via $OT|x\rangle = O(T(|x\rangle))$

The Hilbert–Schmidt inner product (O, T) between operators O and T is obtained as $(O, T) = \mathrm{Tr}(O^* T)$

The commutator between O and T is $[O, T] = OT - TO$,
if $[O, T] = 0$ the operators O and T commute

Similarly, the anti-commutator is $\{O, T\} = OT + TO$,
if $\{O, T\} = 0$ the operators O and T anti-commute

The Postulates of Hilbert Space Quantum Mechanics

Postulate 1: The state of any isolated quantum system is fully characterized by a state vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, the state space

$|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathcal{H}$ are considered to represent the same quantum mechanical state if $|\psi_2\rangle = \alpha|\psi_1\rangle$ for some $\alpha \in \mathbb{C}$. We will implicitly assume that $\||\psi\rangle\| = 1$ always

Postulate 2: The time-evolution of any closed quantum system is fully described by a unitary linear mapping. That is, if the state is $|\psi_1\rangle$ at time t_1, then the state at time t_2 is $|\psi_2\rangle = U|\psi_1\rangle$ where U is unitary and depends only on (t_1, t_2)

The evolution of the state $|\psi(t)\rangle$ characterizing a closed quantum system evolving in continuous time is described by the Schrödinger equation

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

where $\hbar$ is Planck's constant and where H is a fixed self-adjoint operator known as the Hamiltonian

For continuous-time systems, the validity of the Schrödinger equation can be verified to imply Postulate 2

Postulate 3: An isolated quantum system can interact with the outside world by measurement. Any measurement that can be performed is characterized by a set of linear operators $\{M_n\}$, where the index n refers to different outcomes of the experiment

The measurement operators satisfy the completeness condition

$$\sum_n M_n^* M_n = I$$

where I is the unity operator ($I|x\rangle = |x\rangle$ for any $|x\rangle \in \mathcal{H}$)

If an isolated system is in state $|\psi\rangle$ immediately before measurement, then the probability that result n is observed is

$$p(n) = \langle\psi|E_n|\psi\rangle$$

where $\{E_n\}$ are the POVM elements, $E_n = M_n^* M_n$

After observing result n, the new state is $(p(n))^{-1/2} M_n|\psi\rangle$

The only way to obtain information about the state $|\psi\rangle$ of a quantum system is by measurement. Two states $|\psi_1\rangle$ and $|\psi_2\rangle$ can only be distinguished "with probability one" iff $\langle\psi_1|\psi_2\rangle = 0$

Projective measurements: The special case of a projective measurement is fully characterized by a compact linear self-adjoint operator M, with eigen-decomposition

$$M = \sum_i \lambda_i P_i$$

(where P_i projects onto $\{|x\rangle : M|x\rangle = \lambda_i|x\rangle, \langle x|x\rangle = 1\}$)

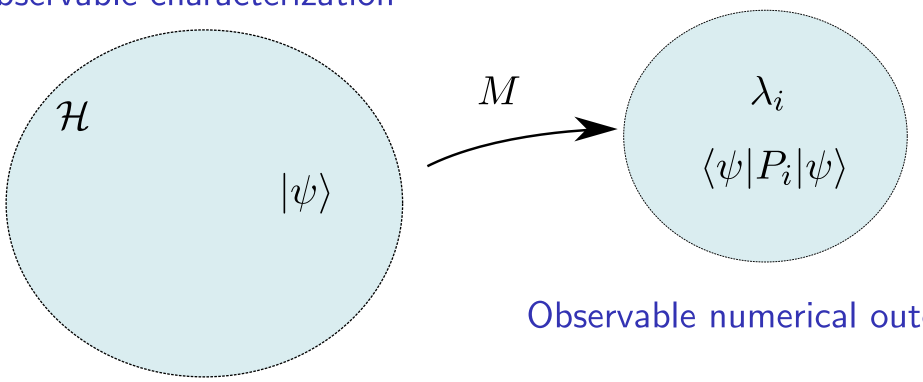
The possible (real, numerical) outcomes of the measurement are the eigenvalues $\{\lambda_i\}$, occurring with probabilities

$$p(i) = \langle\psi|P_i|\psi\rangle$$

Similarly, the expected outcome of the measurement is

$$\langle M\rangle = \sum_i p(i)\lambda_i = \langle\psi|M|\psi\rangle$$

Unobservable characterization



Observable numerical outcome

The system is in state $|\psi\rangle$. The value of $|\psi\rangle$ can be unknown, or known in the case where the system was prepared in this state (or as $|\psi_0\rangle$ and then evolved to $|\psi\rangle$ according to Schrödinger)

When measured, the state $|\psi\rangle$ collapses to an eigen-state/space of the measurement, $|\psi\rangle \to P_i|\psi\rangle$

There is no way the state can be observed without collapsing

Uncertainty relation [Heisenberg/Robertson]:
For (projective) measurements A and B, let $\Delta A = A - \langle A \rangle I$, $\Delta B = B - \langle B \rangle I$, then for a given state $|\psi\rangle$

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

where $\langle (\Delta A)^2 \rangle = \langle \psi | (\Delta A)^2 | \psi \rangle$, $\langle (\Delta B)^2 \rangle = \langle \psi | (\Delta B)^2 | \psi \rangle$
and $\langle [A, B] \rangle = \langle \psi | [A, B] | \psi \rangle$

Qubits

Assume a quantum system is fully described by a two-dimensional space $\mathcal{H}$. The state $|\psi\rangle \in \mathcal{H}$ is then called a quantum bit or qubit

Given a projective measurement M on $\mathcal{H}$ with eigenvalues $\{\lambda_0, \lambda_1\}$ and corresponding eigenvectors $|0\rangle$ and $|1\rangle$ we can write any state as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

and the measurement as $M = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$

The outcome of the measurement is either "$|0\rangle$" with numerical value λ_0 or "$|1\rangle$" with value λ_1

λ_0 is measured with probability $\langle \psi | 0 \rangle \langle 0 | \psi \rangle = |\alpha|^2$

and λ_1 with probability $\langle \psi | 1 \rangle \langle 1 | \psi \rangle = |\beta|^2$

Bloch sphere representation

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$