Probability and Random Processes
Lecture 9

- Extensions to measures
- Product measure

Cartesian Product

- For a finite number of sets $A_1, \ldots, A_n$

\[ \times_{k=1}^n A_k = \{(a_1, \ldots, a_n) : a_k \in A_k, k = 1, \ldots, n\} \]

- notation $A^n$ if $A_1 = \cdots = A_n$

- For an arbitrarily indexed collection of sets $\{A_t\}_{t \in T}$

\[ \times_{t \in T} A_t = \{\text{functions } f \text{ from } T \text{ to } \bigcup_{t \in T} A_t : f(t) \in A_t, t \in T\} \]

- $A_t = A$ for all $t \in T$, then $A^T = \{\text{all functions from } T \text{ to } A\}$

- For a finite $T$ the two definitions are equivalent (why?)
• For a set \( \Omega \), a collection \( C \) of subsets is a **semialgebra** if
  • \( A, B \in C \Rightarrow A \cap B \in C \)
  • if \( C \in C \) then there is a pairwise disjoint and finite sequence of sets in \( C \) whose union is \( C^c \)

• If \( C_1, \ldots, C_n \) are semialgebras on \( \Omega_1, \ldots, \Omega_n \) then
  \[
  \{ C \in \times_{k=1}^n C_k : C_k \in C_k, \, k = 1, \ldots, n \}
  \]
  is a semialgebra on \( \times_{k=1}^n \Omega_k \)

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**Extension**

This is how we constructed the Lebesgue measure on \( \mathbb{R} \):

• For any \( A \subset \mathbb{R} \)
  \[
  \lambda^*(A) = \inf \left\{ \sum_n \ell(I_n) : \{I_n\} \text{ open intervals, } \bigcup_n I_n \supset A \right\}
  \]
  (where \( \ell = \text{“length of interval”} \))

• A set \( E \subset \mathbb{R} \) is Lebesgue measurable if for any \( W \subset \mathbb{R} \)
  \[
  \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c)
  \]

• The Lebesgue measurable sets \( \mathcal{L} \) form a \( \sigma \)-algebra containing all intervals

• \( \lambda = \lambda^* \) restricted to \( \mathcal{L} \) is a measure on \( \mathcal{L} \), and \( \lambda(I) = \ell(I) \) for intervals
We started with a set function $\ell$ for intervals $I \subset \mathbb{R}$

- the intervals form a semialgebra

Then we extended $\ell$ to work for any set $A \subset \mathbb{R}$

- here we used outer measure for the extension

We found a $\sigma$-algebra of measurable sets,

- based on a criterion relating to the union of disjoint sets

Finally we restricted the extension to the $\sigma$-algebra $\mathcal{L}$, to arrive at a measure space $(\mathbb{R}, \mathcal{L}, \lambda)$

Given $\Omega$ and a semialgebra $\mathcal{C}$ of subsets, assume we can find a set function $m$ on sets from $\mathcal{C}$, such that

1. if $\emptyset \in \mathcal{C}$ then $m(\emptyset) = 0$
2. if $\{C_k\}_{k=1}^n$ is a finite sequence of pairwise disjoint sets from $\mathcal{C}$ such that $\bigcup_{k=1}^n C_k \subset \mathcal{C}$, then
   \[ m\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n m(C_k) \]
3. if $C, C_1, C_2, \ldots$ are in $\mathcal{C}$ and $C \subset \bigcup_{n=1}^\infty C_n$, then
   \[ m(C) \leq \sum_{n=1}^\infty m(C_n) \]

Call such a function $m$ a pre-measure
• For a set Ω, a semialgebra C and a pre-measure m, define the set function \( \mu^* \) by

\[
\mu^*(A) = \inf \left\{ \sum_n m(C_n) : \{C_n\}_n \subset \mathcal{C}, \bigcup_n C_n \supset A \right\}
\]

Then \( \mu^* \) is called the outer measure induced by \( m \) and \( \mathcal{C} \)

• A set \( E \subset \Omega \) is \( \mu^* \)-measurable if

\[
\mu^*(W) = \mu^*(W \cap E) + \mu^*(W \cap E^c)
\]

for all \( W \in \Omega \). Let \( \mathcal{A} \) denote the class of \( \mu^* \)-measurable sets

• \( \mathcal{A} \supset \mathcal{C} \) and \( \mathcal{A} \) is a \( \sigma \)-algebra

• \( \mu = \mu^*_\mathcal{A} \) is a measure on \( \mathcal{A} \)

The Extension Theorem

1. Given a set \( \Omega \), a semialgebra \( \mathcal{C} \) of subsets and a pre-measure \( m \) on \( \mathcal{C} \). Let \( \mu^* \) be the outer measure induced by \( m \) and \( \mathcal{C} \) and \( \mathcal{A} \) the corresponding collection of \( \mu^* \)-measurable sets, then

• \( \mathcal{A} \supset \mathcal{C} \) and \( \mathcal{A} \) is a \( \sigma \)-algebra

• \( \mu = \mu^*_\mathcal{A} \) is a measure on \( \mathcal{A} \)

• \( \mu|_\mathcal{C} = m \)

Also, the resulting measure space \( (\Omega, \mathcal{A}, \mu) \) is complete

2. Let \( \mathcal{E} = \sigma(\mathcal{C}) \subset \mathcal{A} \). If there exists a sequence of sets \( \{C_n\} \) in \( \mathcal{C} \) such that

• \( \bigcup_n C_n = \Omega \), and

• \( m(C_n) < \infty \)

then the extension \( \mu^*_\mathcal{E} \) is unique,

• that is, if \( \nu \) is another measure on \( \mathcal{E} \) such that \( \nu(C) = \mu^*_\mathcal{E}(C) \) for all \( C \in \mathcal{C} \) then \( \nu = \mu^*_\mathcal{E} \) also on \( \mathcal{E} \)
• Note that $E \subset A$ in general, and $\mu^*_E$ may not be complete

• In fact, $(\Omega, A, \mu^*_A)$ is the completion of $(\Omega, E, \mu^*_E)$,
  - on $\mathbb{R}$, $\mu^*_A$ corresponds to Lebesgue measure and $\mu^*_E$ to Borel measure

• Also compare the condition in 2. to the definition of $\sigma$-finite measure:
  - Given $(\Omega, A)$ a measure $\mu$ is $\sigma$-finite if there is a sequence $\{A_i\}, A_i \in A$, such that $\cup_i A_i = \Omega$ and $\mu(A_i) < \infty$

• If the condition in 2. is fulfilled for $m$, then $\mu^*_A$ is the unique complete and $\sigma$-finite measure on $A$ that extends $m$

Extension in Standard Spaces

• Consider a (metrizable) topological space $\Omega$ and assume that $C$ is an algebra of subsets (i.e., also a semialgebra)
  - Algebra: closed under set complement and finite unions

• An algebra $C$ has the countable extension property [Gray], if for every function $m$ on $C$ such that $m(\Omega) = 1$ and
  - for any finite sequence $\{C_k\}_{k=1}^n$ of pairwise disjoint sets from $C$ we get
    $$m\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n m(C_k)$$
  then this holds also for $n = \infty$
• Any algebra on $\Omega$ is said to be \textit{standard} (according to Gray) if it has the countable extension property

• A measurable space $(\Omega, \mathcal{A})$ is standard if $\mathcal{A} = \sigma(\mathcal{C})$ for a standard $\mathcal{C}$ on $\Omega$

• If $\mathcal{E} = (\Omega, \mathcal{T})$ is \textit{Polish}, then $(\Omega, \sigma(\mathcal{E}))$ is standard

• Note that if $\mathcal{E} = (\Omega, \mathcal{T})$ is Polish, then $(\Omega, \sigma(\mathcal{E}))$ is also “standard Borel” $\Rightarrow$ for Polish spaces the two definitions of “standard” are equivalent
  • again, we take the $(\Omega, \sigma(\mathcal{E}))$ from Polish space as our default standard space

\textbf{Extension and Completion in Standard Spaces}

• For $(\Omega, \mathcal{T})$ Polish and $(\Omega, \mathcal{A})$ the corresponding standard (Borel) space, there is always an algebra $\mathcal{C}$ on $\Omega$ with the countable extension property, and such that $\mathcal{A} = \sigma(\mathcal{C})$

• Thus, for any normalized and finitely additive $m$ on $\mathcal{C}$
  1. $m$ can always be extended to a measure on $(\Omega, \mathcal{A})$
  2. the extension is unique

• Let $(\Omega, \mathcal{A}, \rho)$ be the corresponding extension ($\rho(\Omega) = 1$)

• Also let $(\Omega, \bar{\mathcal{A}}, \bar{\rho})$ be the completion. Then $(\Omega, \bar{\mathcal{A}}, \bar{\rho})$ is \textit{equivalent with probability one} to $([0,1], \mathcal{L}, \lambda)$
Product Measure Spaces

• For an arbitrary (possibly infinite/uncountable) set $T$, let $(\Omega_t, A_t)$ be measurable spaces indexed by $t \in T$
• A measurable rectangle $= \text{any set } O \subset \times_{t \in T} \Omega_t \text{ of the form}$
  $$O = \{ f \in \times_{t \in T} \Omega_t : f(t) \in A_t \text{ for all } t \in S \}$$
  where $S$ is a finite subset $S \subset T$ and $A_t \in A_t$ for all $t \in S$
• Given $T$ and $(\Omega_t, A_t), t \in T$, the smallest $\sigma$-algebra containing all measurable rectangles is called the resulting product $\sigma$-algebra
  • Example: $T = \mathbb{N}$, $\Omega_t = \mathbb{R}$, $A_t = B$ give the infinite-dimensional Borel space $(\mathbb{R}^\infty, B^\infty)$

• For a finite set $I$, of size $n$, assume that $(\Omega_i, A_i, \mu_i)$ are measure spaces indexed by $i \in I$
• Let $U = \{ \text{all measurable rectangles} \}$ corresponding to $(\Omega_i, A_i), i \in I$
• Let $\Omega = \times_i \Omega_i$ and $A = \sigma(U)$
• Define the product pre-measure $m$ by
  $$m(A) = \prod_{i} \mu_i(A_i)$$
  for any $A_i \in A_i, i \in I$, and $A = \times_i A_i \in U$
The measurable rectangles $\mathcal{U}$ form a semialgebra

The product pre-measure $m$ is a pre-measure on $\mathcal{U}$

1. Given $(\Omega_i, \mathcal{A}_i, \mu_i), \ i = 1, \ldots, n$, let $m$ be the corresponding product pre-measure. Then $m$ can be extended from $\mathcal{U}$ to a $\sigma$-algebra containing $\mathcal{A} = \sigma(\mathcal{U})$. The resulting measure $m^*$ is complete.

2. If each of the $(\Omega_i, \mathcal{A}_i, \mu_i)$’s is $\sigma$-finite then the restriction $m^*_|A$ is unique.
   - Proof: $(\Omega_i, \mathcal{A}_i, \mu_i)$ $\sigma$-finite $\Rightarrow$ condition 2. on slide 8. fulfilled

If the $(\Omega_i, \mathcal{A}_i, \mu_i)$’s are $\sigma$-finite, then the unique measure $\mu = m^*_|A$ on $(\Omega, \mathcal{A})$ is called product measure and $(\Omega, \mathcal{A}, \mu)$ is the product measure space corresponding to $(\Omega_i, \mathcal{A}_i, \mu_i), \ i = 1, \ldots, n$

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$n$-dimensional Lebesgue Measure

- Let $(\Omega_i, \mathcal{A}_i, \mu_i) = (\mathbb{R}, \mathcal{L}, \lambda)$ (Lebesgue measure on $\mathbb{R}$) for $i = 1, \ldots, n$. Note that $(\mathbb{R}, \mathcal{L}, \lambda)$ is $\sigma$-finite (why?). Let $\mu$ denote the corresponding product measure on $\mathbb{R}^n$
  - Per definition, the '$n$-dimensional Lebesgue measure' $\mu$
    constructed like this, based on 2. (on slide 16), is unique but not complete
  - Using instead the construction in 1. as the definition, we get a complete version corresponding to the completion of $\mu$
- The completion $\bar{\mu}$ of the $n$-product of Lebesgue measure is called $n$-dimensional Lebesgue measure