Topological Spaces

- How do we measure “closeness” for objects in abstract spaces?
- Consider \( \mathbb{R} \) and the collection \( \mathcal{O} \) of open intervals, or more generally open sets
- \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( b \) if \( f(x) \) is close to \( f(b) \) for all \( x \) sufficiently close to \( b \)
  \[ \iff \quad \text{for each } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \]
  \[ f(x) \in (f(b) - \varepsilon, f(b) + \varepsilon) \text{ for all } x \in (b - \delta, b + \delta) \]
  \[ \iff \quad \text{for each } O_1 \in \mathcal{O} \text{ containing } f(b), \text{ there is a set } O_2 \in \mathcal{O} \]
  \[ \text{containing } b \text{ such that } f(x) \in O_1 \text{ for all } x \in O_2 \]
  \[ \iff \quad f^{-1}(O) \in \mathcal{O} \text{ for all } O \in \mathcal{O} \]
- Hence, the class of open sets appears to be fundamental in making statements about “closeness” and “limits”
Fundamental properties of sets in $\mathcal{O}$ (on the real line):
- $\mathbb{R}$ and $\emptyset$ are in $\mathcal{O}$
- if $A$ and $B$ are in $\mathcal{O}$ then so is $A \cap B$
- if $\{O_i\}$ are all open, then so is $\bigcup_i O_i$

⇒ a characterization of “open sets” in the general case

For a given nonempty set $\Omega$, a class $\mathcal{T}$ of subsets is a topology on $\Omega$ if
1. $\Omega, \emptyset \in \mathcal{T}$
2. $O_1, O_2 \in \mathcal{T} \Rightarrow O_1 \cap O_2 \in \mathcal{T}$
3. $S \subseteq \mathcal{T} \Rightarrow \bigcup_{O \in S} O \in \mathcal{T}$

The pair $(\Omega, \mathcal{T})$ is a topological space and the sets in $\mathcal{T}$ are called $\mathcal{T}$-open, or simply open

Continuous and Borel Measurable Functions

- Let $A = (\Omega, \mathcal{T})$ and $B = (\Lambda, \mathcal{S})$ be topological spaces, then a function $f : \Omega \to \Lambda$ is continuous if $O \in \mathcal{S} \Rightarrow f^{-1}(O) \in \mathcal{T}$
- Given $A = (\Omega, \mathcal{T})$, the $\sigma$-algebra generated by $\mathcal{T}$ is the Borel $\sigma$-algebra on $(\Omega, \mathcal{T})$, notation $\sigma(A)$
- $(\Omega, \sigma(A))$ is the (measurable) Borel space corresponding to $A = (\Omega, \mathcal{T})$
- Given $A = (\Omega, \mathcal{T})$ and $B = (\Lambda, \mathcal{S})$, a function $f : \Omega \to \Lambda$ is Borel measurable if $O \in \sigma(B) \Rightarrow f^{-1}(O) \in \sigma(A)$
  - usually the default for “measurable function” is “Borel measurable”
Metric Spaces

• For a given set $\Omega$, a function $\rho : \Omega \times \Omega \rightarrow \mathbb{R}$ is a metric if for all $x, y, z \in \Omega$
  1. $\rho(x, y) \geq 0$ with $= 0$ only if $x = y$
  2. $\rho(x, y) = \rho(y, x)$
  3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

• The pair $(\Omega, \rho)$ is a metric space

Metric Spaces as Topological Spaces

• Given $(\Omega, \rho)$, the set $B_r(x) = \{y \in \Omega : \rho(x, y) < r\}$ is called the open ball of radius $r$ centered at $x$

• A set $O$ in $\Omega$ is open if for any $x \in O$ there is an $r$ such that $B_r(x) \subset O$,

$\Rightarrow$ defines a topology $T_\rho$ on $\Omega$; the topology induced by $\rho$

• Two metrics $\rho_1$ and $\rho_2$ are equivalent if $T_{\rho_1} = T_{\rho_2}$

• $(\Omega, T)$ is metrizable if there is a metric $\rho$ such that $T = T_\rho$

• Example: $(\mathbb{R}^n, T)$ with $T = T_\rho$ using $\rho(x, y) = \|x - y\|$ (ordinary Euclidean distance)
  - for $\mathbb{R}^n$ we always assume this topology
Sequences and Completeness

- A topological space \((\Omega, \mathcal{T})\) and a sequence \(\{x_n\}, x_n \in \Omega\)
- The sequence converges to \(x \in \Omega\) if
  - for each \(O \in \mathcal{T}\) such that \(x \in O\) there is an \(N\) such that \(x_n \in O\) for all \(n \geq N\)
- In a metric space \((\Omega, \rho)\), a sequence \(\{x_n\}\)
  - is a Cauchy sequence if for each \(\varepsilon > 0\) there is an \(N\) such that \(\rho(x_n, x_m) < \varepsilon\) for all \(n, m \geq N\)
  - converges to a point \(x\) if \(\lim_n \rho(x_n, x) = 0\)
- \((\Omega, \rho)\) is complete if all Cauchy sequences converge to a point in \(\Omega\)
- \((\Omega, \mathcal{T})\) is completely metrizable if there is a complete metric space \((\Gamma, \rho)\) and a 1-to-1 mapping between \((\Omega, \mathcal{T})\) and \((\Gamma, \mathcal{T}_\rho)\) that is continuous in both directions

Limit Points, Closure

- A topological space \((\Omega, \mathcal{T})\). Given a set \(E \subset \Omega\), a point \(x \in \Omega\) is a limit point of \(E\) if \(O \cap E \neq \emptyset\) for all \(O \in \mathcal{T}\) with \(x \in O\)
- The set of all limit points of \(E = \) the closure of \(E\), notation \(\overline{E}\)
- A set \(E\) is closed if \(E^c\) is open
- \(\overline{E}\) is the smallest closed set that contains \(E\)
Separability

- A set $E$ is dense in $\Omega$ if $\overline{E} = \Omega$
  - c.f. the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$
- A topological space $(\Omega, T)$ is separable if there is a countable set $E \subset \Omega$ such that $\overline{E} = \Omega$
  - c.f. $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and $\mathbb{R} = \overline{\mathbb{Q}}$
- $(\Omega, T)$ is a Polish space if it is completely metrizable and separable

Compactness

- Given a set $E$, a collection $S$ of sets is a covering of $E$ if $E \subset \bigcup_{S \in S} S$
- Given $E$, if $S$ is a covering of $E$ and $S' \subset S$ is also a covering, then $S'$ is a subcovering
- In $(\Omega, T)$ a covering $S$ is open if $S \subset T$
- Given $(\Omega, T)$, a subset $E \subset \Omega$ is compact if every open covering of $E$ has a finite subcovering
  - $E \subset \mathbb{R}^n$ is compact $\iff$ $E$ is closed and bounded
- $(\Omega, T)$ is compact if $\Omega$ is compact
  - $\mathbb{R}^n$ is not compact
Standard Spaces

Three kinds of “standard” (probability) spaces

- **Standard Borel spaces**: Borel equivalence to \(([0, 1], \mathcal{B}([0, 1]))\)
- **Standard spaces as defined by Gray**: The “countable extension property” (next lecture...)
- **Lebesgue spaces**: Isomorphic to a mixture of \(([0, 1], \mathcal{L}([0, 1]), \lambda)\) and a countable space

Standard Borel Spaces

- Two measurable spaces \((\Omega, \mathcal{A})\) and \((\Gamma, \mathcal{G})\) are equivalent if there is a 1-to-1 mapping between them that is measurable in both directions
- If \((\Omega, \mathcal{A})\) and \((\Gamma, \mathcal{G})\) are Borel spaces corresponding to topologies on \(\Omega\) and \(\Gamma\), then they are called Borel equivalent if they are equivalent
- A standard Borel space is a measurable space that is Borel equivalent to either \(([0, 1], \mathcal{B})\) or a subspace of \(([0, 1], \mathcal{B})\), where \(\mathcal{B} = \mathcal{B}([0, 1])\) are the Borel subsets of \([0, 1]\), i.e. the smallest \(\sigma\)-algebra that contains all the open intervals in \([0, 1]\)
• Uncountable standard Borel ⇒ Borel equivalent to \([0, 1], \mathcal{B}\)

• Hence, by “subspace” \((Ω, \mathcal{A})\) we need only consider
  1. \(Ω ⊂ [0, 1]\) is finite, and \(\mathcal{A} = \mathcal{P}(Ω) ⊂ \mathcal{B}\)
     (= the power set = collection of all subsets)
  2. \(Ω ⊂ [0, 1]\) is countable, and again \(\mathcal{A} = \mathcal{P}(Ω) ⊂ \mathcal{B}\)

• If \(E = (Ω, \mathcal{T})\) is Polish, then \((Ω, \sigma(E))\) is standard Borel
  • sometimes used as the definition of “standard Borel”
  • this case will be our default “standard” space

Isomorphic Probability Spaces

Two probability spaces \((Ω, \mathcal{A}, P)\) and \((Γ, \mathcal{G}, Q)\) are

• isomorphic if
  1. \((Ω, \mathcal{A})\) and \((Γ, \mathcal{G})\) are equivalent, with 1-to-1 mapping \(φ\)
  2. For all \(A ∈ \mathcal{A}\), \(P(A) = Q(φ(A))\)
  3. For all \(G ∈ \mathcal{G}\), \(Q(G) = P(φ^{-1}(G))\)

• isomorphic mod 0 if
  1. \((Ω, \mathcal{A}, P)\) and \((Γ, \mathcal{G}, Q)\) are not isomorphic
  2. there are sets \(A_0 ∈ \mathcal{A}, G_0 ∈ \mathcal{G}\), with \(P(A_0) = Q(G_0) = 0\)
  3. \((Ω, \mathcal{A}, P)\) and \((Γ, \mathcal{G}, Q)\) are isomorphic when restricted to points in \(Ω \setminus A_0\) and \(Γ \setminus G_0\)
Lebesgue Spaces

- \((\Omega, A, P)\) is a Lebesgue (probability) space if \(P\) is a probability measure of the form \(\alpha P_1 + (1 - \alpha) P_2\), \(\alpha \in [0, 1]\), and
  1. \((\Omega, A, P)\) is complete
  2. \(P_1\) has no atoms and \((\Omega, A, P_1)\) is isomorphic mod 0 to \(([0, 1], L([0, 1]), \lambda)\)
  3. There are a countable number of points \(\omega_i \in \Omega\), such that with \(p_i = P(\{\omega_i\})\) we have \(P_2(A) = \sum_{i: \omega_i \in A} p_i\) for all \(A \in A\)

Some Standard Borel Spaces

- Any finite set
- The rational numbers, and the irrational numbers
- \((\mathbb{R}^n, B^n)\) (with \(B^n = \) the Borel sets \(\subset \mathbb{R}^n\))
- Separable Hilbert spaces, i.e., Hilbert spaces which admit a countable basis; for example the space of square-integrable functions with inner product
  \[
  \langle f, g \rangle = \int fg \, dx
  \]
  and metric \(\rho(f, g) = (\langle f - g, f - g \rangle)^{1/2}\)

Most abstractions corresponding to real-world phenomena result in standard Borel spaces \(\Rightarrow\) one can almost always work with \(([0, 1], L, \lambda)\) or \(([0, 1], B, \lambda|_B)\), plus a finite/countable space.