Probability and Random Processes

Lecture 10

- Random processes
- Kolmogorov's extension theorem
- Random sequences and waveforms

Random Objects

- A probability space $(\Omega, \mathcal{A}, P)$ and a measurable space $(E, \mathcal{E})$
- A measurable transformation $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$, is a random
  - variable if $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B})$
  - vector if $(E, \mathcal{E}) = (\mathbb{R}^n, \mathcal{B}^n)$
  - sequence if $(E, \mathcal{E}) = (\mathbb{R}^\infty, \mathcal{B}^\infty)$
  - object, in general

More on Product Spaces

- $(E, \mathcal{E})$ a measurable space and T an arbitrary parameter set
- $E^T = \{$ all mappings from T to E $\}$
- A measurable rectangle $\{f \in E^T : f(t) \in A_t$ for all $t \in S\}$ where S is a finite subset $S \subset T$ and $A_t \in \mathcal{E}$ for all $t \in S$
- For $\mathcal{U} = \{$ all measurable rectangles $\}$, let $\mathcal{E}^T = \sigma(\mathcal{U})$
- For $t \in T$, define $\pi_t : E^T \to E$ to be the evaluation map

$$\pi_t(f) = f(t), \quad \text{for any } f \in E^T$$

- Then it holds that $\mathcal{E}^T = \sigma(\{\pi_t : t \in T\})$ i.e., $\mathcal{E}^T$ is the smallest σ-algebra such that all

$$\pi_t : (E^T, \mathcal{E}^T) \to (E, \mathcal{E}), \;\; t \in T$$

  are measurable

- For $S \subset T$ define the restriction map $\pi_S : E^T \to E^S$, via $\pi_S(f) = f|_S$
- For a finite $S \subset T$ and $A_S \in \mathcal{E}^S$, a subset $F \subset E^T$ is a measurable cylinder if it has the form $F = \pi_S^{-1}(A_S)$, i.e.

$$F = \{f \in E^T : \pi_S(f) \in A_S, \;\pi_{T \setminus S}(f) \in E^{T \setminus S}\} = A_S \times E^{T \setminus S}$$

- Then it holds that $\mathcal{E}^T = \sigma(\{$ all measurable cylinders $\})$
- A measurable σ-cylinder is a measurable cylinder where the set $S \subset T$ is possibly infinite but countable
- Then we also have $\mathcal{E}^T = \{$ all measurable σ-cylinders $\}$,
  - even when T is uncountable, membership $f \in A \in \mathcal{E}^T$ imposes restrictions on the values $f(t)$ only for countably many t's

Random Processes

Given $(\Omega, \mathcal{A}, P)$

- Random process, definition 1: a collection $\{X_t : t \in T\}$ where for each t, X_t is a random object $X_t : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$,

$$X_t : \Omega \to E, \quad X_t^{-1} : \mathcal{E} \to \mathcal{A}$$

  for each t, X_t maps ω into a value $X_t(\omega) \in E$
- Random process, definition 2: a random object $X : (\Omega, \mathcal{A}) \to (E^T, \mathcal{E}^T)$

$$X : \Omega \to E^T, \quad X^{-1} : \mathcal{E}^T \to \mathcal{A}$$

  X maps each ω into a function $X_t(\omega) \in E^T$

Extension Results

- Based on definition 2, the process distribution μ_X is the distribution of the random object X, that is,

$$\mu_X(A) = P(\{\omega : X_t(\omega) \in A\}), \quad A \in \mathcal{E}^T$$

- For a subset $S \subset T$, restricting the process to S means that $f(t) = X_t(\omega)$ is restricted to $t \in S$, $\pi_S(f) = f|_S$, with corresponding marginal distribution $\mu_{X|S}$ on $(E^S, \mathcal{E}^S)$

- Assume that $(E, \mathcal{E}, \mu_t)$ are probability spaces for each $t \in S$, where S is a finite subset $S \subset T$, and let $(E^S, \mathcal{E}^S, \mu^S)$ be the corresponding product measure space
- Even in the case of an uncountable T, $(E^S, \mathcal{E}^S, \mu^S)$ can be extended to the full space $(E^T, \mathcal{E}^T, \mu_X)$, in the sense that there exists a unique μ_X such that

$$\mu_{X|S}(A) = \mu^S(A)$$

for all $A \in \mathcal{E}^S$ and any finite $S \subset T$
- Proof: The cylinder sets are a semialgebra that generates $\mathcal{E}^T$; a finite product of probability measures is a pre-measure on the cylinders; our previous extension result for product measure can then be extended to a countable S; finally, the fact that $\mathcal{E}^T$ is the class of σ-cylinders can be used to extend to the full class $\mathcal{E}^T$

- Remember from the definition of product measure, that $(E^S, \mathcal{E}^S, \mu^S)$ corresponds to a process with independent values $X_t(\omega)$, $t \in S$
- Hence we now know how to construct memoryless processes, even for an uncountable T, based on marginal distributions for each finite S
- How about completely general μ_X's?
- First result, uniqueness in the general case: for any $\mu_X^{(1)}$ and $\mu_X^{(2)}$ on $(E^T, \mathcal{E}^T)$, if

$$\mu_{X|S}^{(1)}(A) = \mu_{X|S}^{(2)}(A)$$

for all finite $S \subset T$ and $A \in \mathcal{E}^S$, then $\mu_X^{(1)} = \mu_X^{(2)}$
- That is, the finite-dimensional marginal distributions uniquely determine the process distribution, if it exists

Existence: Kolmogorov's Extension Theorem

- A marginal distribution $\mu_{X|S}$, for any finite $S \subset T$, is consistent if $\mu_{X|S}$ implies $\mu_{X|V}$ for all $V \subset S$
  - of no concern for product measure, i.e., memoryless marginals...(why?)
- Extension Theorem: For a given process X from $(\Omega, \mathcal{A})$ to $(E^T, \mathcal{E}^T)$, assume that a consistent distribution $\mu_{X|S}$ is specified for any finite subset $S \subset T$. If $(E, \mathcal{E})$ is standard, then a unique process distribution μ_X exists on $(E^T, \mathcal{E}^T)$ that agrees with $\mu_{X|S}$ for all finite $S \subset T$
- Additional structure is necessary; the result does not hold for all possible $(E, \mathcal{E})$

Discrete-time Real-valued Random Process

- Given $(\Omega, \mathcal{A}, P)$, let $E = \mathbb{R}$, $\mathcal{E} = \mathcal{B}$, and interpret T as "time"
- If $T = \mathbb{Z}$ or $\mathbb{N}$, then X is a random sequence or a discrete-time random process, that is $\{X_t\}_{t\in T}$ is a countable collection of random variables
- $(E, \mathcal{E})$ is standard

$\Rightarrow$ Any set of distributions for all random vectors that can be formed by restricting to $S = \{t_1, t_2, \dots, t_m\}$ can be extended to a unique process distribution

Continuous-time Real-valued Random Process

- Given $(\Omega, \mathcal{A}, P)$, let $E = \mathbb{R}$, $\mathcal{E} = \mathcal{B}$, and interpret T as "time"
- If $T = \mathbb{R}$ or $\mathbb{R}^+$, then X is a random waveform or a continuous-time random process, that is $\{X_t\}_{t \in T}$ is an uncountable collection of random variables
- $(E, \mathcal{E})$ is standard, so consistent finite-dimensional marginals can be extended to a unique process distribution on $(E^T, \mathcal{E}^T)$

Finite-energy Waveforms

- Introduce the L_2 norm

$$\|g\| = \left(\int |g(t)|^2 dt \right)^{1/2}$$

  and let $\mathcal{L}_2 = \{$ Lebesgue measurable f such that $\|f\|^2 < \infty\}$
- Equipped with the inner product

$$\langle f, g \rangle = \int f g dt$$

  $\mathcal{L}_2$ is then a separable Hilbert space (with $\|f\| = (\langle f, f \rangle)^{1/2}$)
- With topology $\mathcal{T}$ determined by the metric $\rho(f, g) = \|f - g\|$ the space $\mathcal{A} = (\mathcal{L}_2, \mathcal{T})$ is Polish and $(\mathcal{L}_2, \sigma(\mathcal{A}))$ is standard
- The resulting space $(\mathcal{L}_2, \sigma(\mathcal{A}))$ is a model for random finite-energy waveforms

Continuous Waveforms

- For a closed interval $T \subset \mathbb{R}$, let $C(T) = \{\text{ all continuous functions } f : T \to \mathbb{R}\}$
- For $g, f \in C(T)$, define the metric

$$\rho(f, g) = \sup\{|f(t) - g(t)| : t \in T\}$$

- With topology $\mathcal{T}$ determined by ρ, $\mathcal{A} = (C(T), \mathcal{T})$ is Polish and $(C(T), \sigma(\mathcal{A}))$ is standard
- The resulting space $(C(T), \sigma(\mathcal{A}))$ is a model for continuous waveforms on T

Gaussian Processes

- Let $T = \mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ or $\mathbb{N}$
- For any finite $S \subset T$, of size n, let $E^S = \mathbb{R}^n$ and $\mathcal{E}^S$ the corresponding Borel sets
- Define $\mu_{X|S}$ on $(E^S, \mathcal{E}^S)$ to be the finite Borel measure with density

$$f_n(x^n) = \frac{1}{\sqrt{(2\pi)^n |V_n|}} \exp\left(-\frac{1}{2}(x^n - m^n)V_n^{-1}(x^n - m^n)'\right)$$

with respect to n-dimensional Lebesgue measure, where V_n is a positive-definite $n \times n$ matrix and $m^n \in \mathbb{R}^n$

Discrete time

- For $T = \mathbb{Z}$ or $\mathbb{N}$, the distributions specified by (m^n, V_n) for all finite n uniquely determine a Gaussian sequence $\{X_t\}$ with process distribution μ_X
- μ_X is uniquely specified by knowing

$$m(t) = E[X_t], \quad V(k,l) = E[(X_k - m(k))(X_l - m(l))]$$

  for all $t, k, l \in T$

Continuous time

- For $T = \mathbb{R}$ or $\mathbb{R}^+$, the distributions specified by (m^n, V_n) for all finite n uniquely determine a Gaussian waveform $\{X_t\}$ with process distribution μ_X, specified by

$$m(t) = E[X_t], \quad V(s,u) = E[(X_s - m(s))(X_u - m(u))]$$

  for all $t, s, u \in T$
- Here we need

$$\int V(t,t)dt < \infty$$

  to get finite-energy waveforms (with probability one)

Brownian Motion

- Given $(\Omega, \mathcal{A}, P)$ and $C(T)$ = the class of continuous waveforms on $T = [0, \tau]$ for $\tau > 0$
- There is a probability space $(C(T), \mathcal{E}^T, \mu_X)$ such that
  - For $X_t \in C(T)$, $X_0(\omega) = 0$ for all $\omega \in \Omega$
  - For every $0 \le s \le t \le \tau$, $Y(t,s) = X_t - X_s \sim \mathcal{N}(0, t-s)$. Also $Y(t,s)$ and X_u are independent for all $0 \le u \le s$
  - $\mathcal{E}^T = \sigma(\mathcal{A})$ on slide 13
  - μ_X is unique
- μ_X = the Wiener measure (usually for $T = [0, \infty)$)
- Consequently, X_t is a Gaussian waveform with $m(t) = 0$ and $V(s,u) = \min(s,u)$, and $X_t(\omega)$ is continuous on $[0,\tau]$ for all $\omega \in \Omega$
- The realizations X_t are non-differentiable Lebesgue a.e., for all $\omega \in \Omega$,
  - the derivative "$\frac{d}{dt}X_t$" is Gaussian "white noise"

- Starting from a Gaussian process on $(\mathbb{R}^S, \mathcal{B}^S)$, $S \subset T = [0,\tau]$ and finite, with $m(t) = 0$ and $V(s,u) = \min(s,u)$ for t, s, u in S, and then using the extension theorem cannot work, because $C(T)$ is not in
$$\mathcal{B}^T = \sigma(\{\text{ measurable rectangles with sides in } \mathcal{B}\,\})$$
- Given $(E^T, \mathcal{E}^T, \mu)$ and $G \subset E^T$ (but possibly $G \notin \mathcal{E}^T$)
- For any $E \subset E^T$ let $\mu^*(E) = \inf\{\mu(E') : E \subset E',\ E' \in \mathcal{E}^T\}$
- If $\mu^*(G) = 1$ then $(G, \mathcal{G}, \mu^*)$ with $\mathcal{G} = \{G \cap E : E \in \mathcal{E}^T\}$ is a process with all sample paths in G
- For $G = C(T)$, $E^T = \mathbb{R}^T$, $\mathcal{E}^T = \mathcal{B}^T$ and $(\mathbb{R}^T, \mathcal{B}^T, \mu)$ Gaussian with $m(t) = 0$ and $V(t,s) = \min(t,s)$, we have $\mu^*(G) = 1$ and the resulting space $(G, \mathcal{G}, \mu^*)$ is Brownian motion, with μ^* = the Wiener measure