Probability and Random Processes

Lecture 1

- Lebesgue measure on the real line

Measure Size/Length of Real Sets

- I an interval of the form $[a, b]$, $[a, b)$, $(a, b]$ or (a, b), for $b \geq a$
- $\ell(I) = b - a =$ length of I
  - in particular, $\ell(I) = 0$ if $a = b$
- How do we generalize "length" to sets which are more complicated?
- For any generalization, it would be reasonable to require
  - length$(A) \geq 0$ for all A
  - length$(\emptyset) = 0$
  - length$(A) = \ell(A)$ if A is an interval
  - length$(B) =$ length(B_1) + length(B_2) if $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$

- Clue: for any open set B, this should work

$$\text{length}(B) = \sum_i \ell(I_i)$$

where $\{I_i\}$ are the open intervals that form $B \Rightarrow$ we know how to measure open sets

- Define 'length(B)' as above if B is open

$\Rightarrow$ Lebesgue outer measure, for any $A \subset \mathbb{R}$ define

$$\lambda^*(A) = \inf \text{length}(B) \text{ over all open } B \text{ such that } A \subset B$$

- Can λ^* work as the extension of length we are looking for?
  - $\lambda^*(A) \geq 0$ OK
  - $\lambda^*(A) = \ell(A)$ if A is an interval OK
  - $\lambda^*(B) = \lambda^*(B_1) + \lambda^*(B_2)$ if $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$ not OK
- It can be shown that there are disjoint sets B_1 and B_2 that do not fulfil $\lambda^*(B_1 \cup B_2) = \lambda^*(B_1) + \lambda^*(B_2)$
- However, the problem is not the definition of λ^*, it is the fact that we allow arbitrary sets $\subset \mathbb{R}$...
  - ...there is general consensus that there are sets that are not "measurable" according to any useful definition

The Banach–Tarski Paradox (1924)

- A ball in $\mathbb{R}^3$ can be decomposed into finitely many disjoint pieces which can be rearranged by rigid motions and reassembled to form two balls of the same size as the original.
  - Given any two bounded subsets A and B of $\mathbb{R}^k$, $k \geq 3$, both of which have a non-empty interior, there are partitions of A and B into a finite number of disjoint subsets, $A = A_1 \cup \cdots \cup A_N$, $B = B_1 \cup \cdots \cup B_N$, such that for each n between 1 and N, the sets A_n and B_n are congruent (equal up to translation, rotation and reflection).
  - For $k = 1, 2$ the same statement is true for countably infinite partitions instead of finite.

Lebesgue Measurable

- B_1 and B_2 need to be *sufficiently separated*, the sets in the paradox are arbitrarily intermingled
- If O is an open set such that $A \subset O$ and $B \subset O^c$, then $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$
- In particular

$$\lambda^*(A) = \lambda^*(A \cap O) + \lambda^*(A \cap O^c)$$

for all A and any open O

- A set $W \subset \mathbb{R}$ is Lebesgue measurable if

$$\lambda^*(A) = \lambda^*(A \cap W) + \lambda^*(A \cap W^c)$$

  for all A
- "Lebesgue measurable" more general than "open"
- Note that if W_1 and W_2 are Lebesgue measurable and disjoint, then with $A = W_1 \cup W_2$ we have

$$\begin{aligned}\lambda^*(W_1 \cup W_2) = \lambda^*(A) &= \lambda^*(A \cap W_1) + \lambda^*(A \cap W_1^c) \\ &= \lambda^*(W_1) + \lambda^*(W_2)\end{aligned}$$

Lebesgue Measure

- λ^* restricted to sets in $\mathcal{L} = \lambda =$ Lebesgue measure,
  - "restricted to," notation $\lambda = \lambda^*_{|\mathcal{L}}$
- $\lambda(A) =$ the most general (widely accepted) definition of "length(A)" for any $A \in \mathcal{L}$
- $\lambda(B) = \sum_i \lambda(B_i)$ if $B = \cup_i B_i$ and for all $B_i \in \mathcal{L}$, $i = 1, 2, 3, \ldots$, such that $B_i \cap B_j = \emptyset$

Lebesgue Measurable Function

- A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if the inverse image of every open set is Lebesgue measurable, i.e.,
  - $f^{-1}(O) = \{x : f(x) \in O\} \in \mathcal{L}$ for all open O

- Two functions f and g are equal Lebesgue almost everywhere, λ-a.e., if
$$\lambda(\{x : f(x) \neq g(x)\}) = 0$$
- If f is Lebesgue measurable and $g = f$ λ-a.e. then g is Lebesgue measurable

- f continuous iff the inverse image of every open set is open; sets in $\mathcal{L}$ are more general than "open" $\Rightarrow$ Lebesgue measurable functions are more general than "continuous"
- If $\{f_n\}$ are Lebesgue measurable and $f_n \to f$ pointwise then f is Lebesgue measurable
  - C.f. "continuous functions" where the class is closed under uniform but not pointwise convergence