Probability and Random Processes
2024

Assignment 7
Assigned: Thursday, March 7, 2024
Due: Thursday, March 14, 2024 M. Skoglund

Problem 7.1: Define and explain the concept conditional probability; conditioning on a σ-algebra and a random variable, respectively.

Problem 7.2: Define and explain the concept conditional expectation and relate it to conditional probability.

Problem 7.3: Define and explain Lebesgue decomposition of measure.

Problem 7.4: Define and explain continuous and discrete measure, and state a decomposition result in terms of such. For finite Borel measure, relate to properties of the distribution function.

Problem 7.5: Given $(\Omega, \mathcal{A}, P)$ and a random variable X, let $P(E|X) = P(E|\sigma(X))$ for $E \in \mathcal{A}$. Establish that there is a Borel measurable function $g(x) : \mathbb{R} \to \mathbb{R}^+$ such that $P(E|X) = g(X)$, P-a.e., that is, as a function of $\omega \in \Omega$, $P(E|X)$ has the form $\omega \to X(\omega) \to g(X(\omega))$ (P-a.e.). For $x \in \mathbb{R}$ and $E \in \mathcal{A}$, let $P(E|X = x) = g(x)$; the conditional probability of E given $X = x$. Then, for any $E \in \mathcal{A}$ and $B \in \mathcal{B}$, prove that

$$P(\{X \in B\} \cap E) = \int_B P(E|X = x) d\mu_X$$

Problem 7.6: Assume that X and Y are discrete random variables with joint pmf $p(x, y)$. Define

$$p(y|x) = \begin{cases} p(x,y)/p(x), & p(x) > 0 \\ 0, & \text{o.w.} \end{cases}$$

Show that

$$P(\{Y = y\}|X = x) = p(y|x)$$

where $P(\{Y = y\}|X = x)$ is defined as in problem 7.5. Also determine $P(\{Y = y\}|X)$.

Problem 7.7: Assume that X and Y are jointly absolutely continuous random variables with joint pdf $f(x, y)$. Define

$$f(y|x) = \begin{cases} f(x,y)/f(x), & f(x) > 0 \\ 0, & \text{o.w.} \end{cases}$$

Show that for each $B \in \mathcal{B}$

$$P(\{Y \in B\}|X = x) = \int_B f(y|x) dy$$

where $P(\{Y \in B\}|X = x)$ is defined as in problem 7.5. Also determine $P(\{Y \in B\}|X)$.

Problem 7.8: Given $(\Omega, \mathcal{A}, P)$ and a random variable X such that $E(X^2) < \infty$. Given $\mathcal{G} \subset \mathcal{A}$ let $\mathcal{F} = \{\text{ all } \mathcal{G}\text{-measurable functions }\}$. Show that $\hat{Y}(\omega) = E[X|\mathcal{G}]$ is a P-a.e. unique solution to the problem: find $Y \in \mathcal{F}$ to minimize $E[(X - Y)^2]$.