Probability and Random Processes
2024

Assignment 5
Assigned: Thursday, February 15, 2024
Due: Thursday, February 22, 2024 M. Skoglund

In solving the problems below, given $(\Omega, \mathcal{A}, P)$ the following results can be used without proof:

For sets $A_1 \subset A_2 \subset A_3 \subset \cdots$ in $\mathcal{A}$, it holds that

$$P\left(\bigcup_n A_n\right) = \lim_n P(A_n)$$

For sets $A_1 \supset A_2 \supset A_3 \supset \cdots$ in $\mathcal{A}$, it holds that

$$P\left(\bigcap_n A_n\right) = \lim_n P(A_n)$$

Also, for any sequence $\{A_n\}$ of sets from $\mathcal{A}$, it holds that $\limsup_n P(A_n) \leq P(\{A_n \text{ i.o}\})$

Problem 5.1: Define/explain the concepts: probability space, event, probability, and independence (pairwise and mutual)

Problem 5.2: Define/explain the concepts: random variable, distribution, probability distribution function, expectation

Problem 5.3: Prove the Borel–Cantelli lemma.

Problem 5.4: Given $(\Omega, \mathcal{A}, P)$ and a sequence of random variables $\{X_n\}$, show that

$$\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = \bigcap_{m=1}^{\infty}\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{k=1}^{\infty}\{\omega : |X_{n+k}(\omega) - X_n(\omega)| < \frac{1}{m}\}\right)\right)$$

Hint: a sequence $\{a_n\}$ of real numbers converges iff it's a *Cauchy sequence*, i.e. iff for any $\varepsilon > 0$ there is an N such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$.

Problem 5.5: For mutually independent random variables $\{X_n\}$ with $E[X_n] = 0$ and $\sum_n \mathrm{Var}(X_n) < \infty$, use the result in Problem 5.4 and Kolmogorov's inequality (without proof) to show that $\sum_n X_n$ converges with probability one.

Problem 5.6: Given an iid sequence of zero-mean random variables $\{X_n\}$, let $Y_n = X_n \chi_{\{|X_n| \leq n|\}}$, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Var}[Y_n] < \infty$$

Problem 5.7: Prove the (strong) law of large numbers. You can use Lemma 1–3 from the lecture slides, the result in Problem 5.6 and also the Borel–Cantelli lemma, without proof.