# Information Theory
## Spring semester, 2025

**Assignment 1**
Assigned: Wednesday, March 19, 2025
Due: Thursday, March 27, 2025 M. Skoglund

---

**Problem 1.1:** Cover & Thomas 2.8 (p. 45)

**Problem 1.2:** Cover & Thomas 2.19 (p. 48)

**Problem 1.3:** Cover & Thomas 2.25 (p. 49)

**Problem 1.4:**
The weatherman's record in a given city is given in the table below, the numbers indicating the relative frequency of the indicated event.

| | Actual | |
|---|---|---|
| Prediction | Rain | No rain |
| Rain | 1/8 | 3/16 |
| No rain | 1/16 | 10/16 |

A clever student notices that the weatherman is right only 12/16 of the time, but could be right 13/16 of the time by always predicting no rain. The student explains the situation and applies for the weatherman's job, but the weatherman's boss (who has studied information theory) turns him down. Why?

**Problem 1.5:**
Due to intense pressure from the gambling community, KTH has developed a small handheld device, which, at a press of a button, samples the environment and predicts the outcome of the flip of an unbiased coin with 80% accuracy.

(a) What is the (average) mutual information between the actual and the predicted outcomes?

(b) A cautious gambler, unimpressed with the accuracy of the device, has decided to follow its predictions, but risk only a fixed fraction $\alpha$ of her capital on each successive coin flip. Assuming that her initial capital is $S_0$ and that every krona she bets is either lost or earns an additional krona depending on the outcome, find $S_n$, the capital of the gambler after $n$ independent coin flips. Define the growth of her capital as

$$G_n = \frac{1}{n} \log \frac{S_n}{S_0}; \qquad \left(S_n = S_0 2^{nG_n}\right).$$

Find the values of $\alpha$ that maximize the expectations $E\,S_n$ and $E\,G_n$. Compare the maximum expected growth with the mutual information from part (a), and interpret the maximizing fraction.

(c) If you had access to the KTH gadget, which value of $\alpha$ would you use and why?

**Problem 1.6:**
Consider two random variables $X$ and $Y$ with $\Pr(X = x) = p(x)$, $\Pr(Y = y) = q(y)$, and $\Pr(X = x, Y = y) = r(x, y)$. Let

$$i(x; y) = \log \frac{r(x, y)}{p(x)q(y)}$$

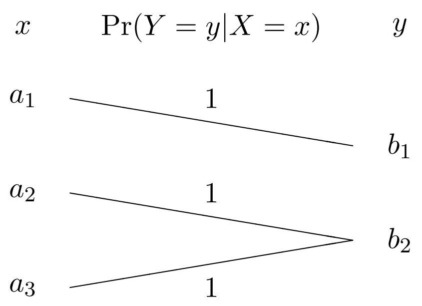
denote the *elementary mutual information* between outcomes $x$ and $y$. Then, the mutual information between $X$ and $Y$ is the expected value of the random variable $i(X; Y)$, i.e., $I(X; Y) = E\, i(X; Y)$. In this problem we are concerned with the *variance* of $i(X; Y)$.

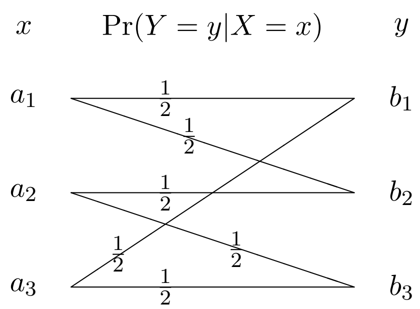
(a) Prove that $\operatorname{Var} i(X; Y) = 0$ iff there is a constant $c$ such that, for all $x, y$ with $r(x, y) > 0$,

$$r(x, y) = cp(x)q(y).$$

(b) Express $I(X; Y)$ in terms of $c$ and interpret the special case $c = 1$.

(c) For each of the channels below, find a probability assignment $p(x)$ such that $I(X; Y) > 0$ and $\operatorname{Var} i(X; Y) = 0$. Calculate $I(X; Y)$.




*Note*: Problems 5–7 are adapted from R. Gallager, *Information Theory and Reliable Communication*, Wiley 1968.