IT for Statistics and Learning
2024

Assignment 12
Assigned: Thu, Feb 8, 2024
Due: before the lecture on Thu, Feb 15, 2024 T. Oechtering

Problem 12.1: *Complete the proof.* Show that for the number of possible n-types we have

$$|\mathbb{P}_n| = \binom{n+M-1}{M-1} \leq (n+1)^{M-1}$$

Problem 12.2: *Stirling's formula.* Show that for the size of $\mathcal{T}_P^n$ we have

$$\log|\mathcal{T}_P^n| = nH(P) - \frac{s(P)-1}{2}\log(2\pi n) - \frac{1}{2}\sum_{a:P(a)>0}\log P(a) - \frac{\vartheta(n,P)}{12\ln 2}s(P)$$

with $s(P)$ is the number of elements $a \in \mathcal{A}$ with $P(a) > 0$ and $0 \leq \vartheta(n,P) \leq 1$. Note that $P(a) \leq \frac{1}{n}$ if $P(a) > 0$. Use Robbins' sharpening of Stirling's formula :

$$\sqrt{2\pi}n^{n+\frac{1}{2}}\mathrm{e}^{-n+\frac{1}{12(n+1)}} \leq n! \leq \sqrt{2\pi}n^{n+\frac{1}{2}}\mathrm{e}^{-n+\frac{1}{12n}}$$

Problem 12.3: *Large deviation.* Let $\mathcal{P}$ be any set of probability distributions on $\mathcal{A}$ and let $\mathcal{P}_n$ be the set of those distributions $P \in \mathcal{P}$ which are types of sequences in $\mathcal{A}^n$. Show that for every distribution Q on $\mathcal{A}$ we have

$$\left|\frac{1}{n}\log Q^n(\{x^n : P_{x^n} \in \mathcal{P}\}) + \min_{P\in\mathcal{P}_n} D(P||Q)\right| \leq \frac{\log(n+1)}{n}|\mathcal{A}|$$

Problem 12.4: *V-shell.* Every $y^n \in \mathcal{B}^n$ in the V-shell of an $x^n \in \mathcal{A}^n$ has the same type Q where $Q(b)$ is defined as

$$Q(b) = \sum_{a\in\mathcal{A}} P_{x^n}(a)V(b|a).$$

(i) Show that $\mathcal{T}_V^n(x^n) \neq \mathcal{T}_Q^n$ even if all rows of the matrix V are equal to Q (unless x^n consists of identical elements).

(ii) Show that if $P_{x^n} = P$ then

$$(n+1)^{-|\mathcal{A}||\mathcal{B}|}2^{-nI(P,V)} \leq \frac{|\mathcal{T}_V^n(x^n)|}{|\mathcal{T}_Q^n|} \leq (n+1)^{|\mathcal{B}|}2^{-nI(P,V)}$$

where the mutual information is defined as $I(P;V) = H(Q) - H(V|P)$ with roles $P = P_X$ and $V = P_{Y|X}$.