Autumn 2011 O. Sisask

Extremal Combinatorics examples sheet 4

There are also exercises in the notes; some of these are included below and some are not, but you should attempt all exercises to ensure a thorough understanding of the course material. The examples sheets are unassessed, but you are welcome to hand in your attempts for feedback.

1. Prove the upper-shadow form of the Kruskal-Katona theorem stated in lectures.

2. Show that if $\mathcal{A} \subseteq \binom{X}{r}$ is a set system and $\mathcal{C} \subseteq \binom{X}{r}$ is an initial segment of colex of size $|\mathcal{C}| = |\mathcal{A}|$ then $|\partial^t \mathcal{A}| \geqslant |\partial^t \mathcal{C}|$ whenever $1 \leqslant t \leqslant r$. Conclude that if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial^t \mathcal{A}| \geqslant \binom{k}{r-t}$.

3. Show that an initial segment $\mathcal{C}$ of the cube order on $\mathcal{P}(X)$ is i-compressed for each $i \in X$ – that is, $C_i(\mathcal{C}) = \mathcal{C}$ for each i.

4. For a graph G and a subset $S \subseteq V(G)$ we define the *t-neighbourhood* of S to be
$$N^t(S) = \{x \in V(G) : d(x, S) \leqslant t\},$$
where $d(x, S) = \min_{y \in S} d(x, y)$ is the number of edges in a shortest path between x and a vertex in S. Show that if $\mathcal{A} \subseteq V(Q_n) = \mathcal{P}(X)$ has
$$|\mathcal{A}| \geqslant \sum_{i=0}^{r} \binom{n}{i}$$
then, for any $1 \leqslant t \leqslant n - r$,
$$|N^t(\mathcal{A})| \geqslant \sum_{i=0}^{r+t} \binom{n}{i}.$$

5. Show that an initial segment of the binary order on $\mathcal{P}(X)$ is i-binary-compressed (in the natural sense) for each $i \in X$.

Please let me know if you have any comments or corrections.

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