Asymptotic and universal spectral estimates with applications in many-body quantum mechanics and spectral shape optimization

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Abstract

This thesis consists of eight papers primarily concerned with the quantitative study of the spectrum of certain differential operators. The majority of the results split into two categories. On the one hand Papers B–E concern questions of a spectral-geometric nature, namely, the relation of the geometry of a region in d-dimensional Euclidean space to the spectrum of the associated Dirichlet Laplace operator. On the other hand Papers G and H concern kinetic energy inequalities arising in many-particle systems in quantum mechanics.

Paper A falls outside the realm of spectral theory. Instead the paper is devoted to a question in convex geometry. More precisely, the main result of the paper concerns a lower bound for the perimeter of inner parallel bodies of a convex set. However, as is demonstrated in Paper B the result of Paper A can be very useful when studying the Dirichlet Laplacian in a convex domain.

In Paper B we revisit an argument of Geisinger, Laptev, and Weidl for proving improved Berezin–Li–Yau inequalities. In this setting the results of Paper A allow us to prove a two-term Berezin–Li–Yau inequality for the Dirichlet Laplace operator in convex domains. Importantly, the inequality exhibits the correct geometric behaviour in the semiclassical limit.

Papers C and D concern shape optimization problems for the eigenvalues of Laplace operators. The aim of both papers is to understand the asymptotic shape of domains which in a semiclassical limit optimize eigenvalues, or eigenvalue means, of the Dirichlet or Neumann Laplace operator among classes of domains with fixed measure. Paper F concerns a related problem but where the optimization takes place among a one-parameter family of Schrödinger operators instead of among Laplace operators in different domains. The main ingredients in the analysis of the semiclassical shape optimization problems in Papers C, D, and F are combinations of asymptotic and universal spectral estimates. For the shape optimization problem studied in Paper C, such estimates are provided by the results in Papers B and E.

Paper E concerns semiclassical spectral asymptotics for the Dirichlet Laplacian in rough domains. The main result is a two-term asymptotic expansion for sums of eigenvalues in domains with Lipschitz boundary.

The topic of Paper G is lower bounds for the ground-state energy of the homogeneous gas of R-extended anyons. The main result is a non-trivial lower bound for the energy per particle in the thermodynamic limit.

Finally, Paper H deals with a general strategy for proving Lieb–Thirring inequalities for many-body systems in quantum mechanics. In particular, the results extend the Lieb–Thirring inequality for the kinetic energy given by the fractional Laplace operator from the Hilbert space of antisymmetric (fermionic) wave functions to wave functions which vanish on the k-particle coincidence set, assuming that the order of the operator is sufficiently large.

Sammanfattning

Denna avhandling utgörs av åtta artiklar vars huvudsakliga tema är kvantitativa resultat om spektrumet av differentialoperatorer. Merparten av resultaten faller i en av två kategorier. Å ena sidan handlar Artikel B till och med E om frågor av spektralgeometrisk karaktär, specifikt relationen mellan formen av ett område i d-dimensionellt Euklidiskt rum och spektrumet av den associerade Dirichlet-Laplaceoperatorn. Å andra sidan handlar Artiklarna G och H om begränsningar för den kinetiska energin av mångpartikelsystem inom kvantmekanik.

Artikel A faller utanför spektralteori och handlar istället om ett problem inom konvex geometri. Mer precist, handlar artikeln om en undre begränsning för måttet av ytan av de inre parallela kropparna av ett konvext område. Hur som helst är artikelns huvudresultatet användbart för att studera Dirichlet-Laplaceoperatorn i konvexa områden, vilket demonstreras i Artikel B.

I Artikel B återvänder vi till ett argument av Geisinger, Laptev, och Weidl för att bevisa förbättrade Berezin–Li–Yau-olikheter. I detta sammanhang tillåter resultaten i Artikel A oss att bevisa en tvåterms-Berezin–Li–Yau-olikhet för Dirichlet-Laplaceoperatorn på konvexa områden. Av stor vikt för de tillämpningar vi har i åtanke är att oliheten uppvisar korrekt geometriskt beteende i den semiklassiska gränsen

I Artiklarna C och D studeras geometriska optimeringsproblem för egenvärden av Laplaceoperatorer. Specifikt handlar båda artiklarna om den asymptotiska formen av de områden som i en semiklassisk gräns optimerar egenvärden, eller medelvärden av egenvärden, av Dirichlet- eller Neumann-Laplaceoperatorn inom klasser av områden med fixerat mått. Artikel F behandlar ett likartat optimeringsproblem men där optimieringen sker över en en-parameter familj av Schrödingeroperatorer istället för bland Laplaceoperatorer i olika områden. De huvudsakliga ingredienserna i analysen av de semiklassiska optimeringsproblem som studeras i Artiklarna C, D och F är kombinationer av asymptotiska och universella spektraluppskattningar. För det problem som studeras i Artikel C bevisas spektraluppskattningar av denna form i Artikel B och Artikel E.

Artikel E handlar om semiklassisk asymptotik för Dirichlet-Laplacianen i icke-reguljära områden. Huvudresultatet är en asymptotisk utveckling till andra ordning för summan av egenvärden i områden med Lipschitz rand.

Temat för Artikel G är undre begränsningar för grundtillståndsenergin av den homogena gasen av R-utvidgade anyoner. Huvudresultatet av artikeln är en icke-trivial undre begränsning för energin per partikel i den termodynamiska gränsen.

Slutligen handlar Artikel H om en generell metod för att bevisa Lieb–Thirring-olikheter för mångpartikelsystem i kvantmekanik. Som en tillämpning av den generella metoden utvidgas Lieb–Thirring-olikheten för den fraktionella Laplaceoperatorn från Hilbertrummet av antisymmetriska (fermionska) vågfunktioner till vågfunktioner som försvinner på k-partikeldiagonaler, under antagandet att ordningen av operatorn är tillräckligt hög.

Contents

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Part I

Introduction and summary

1 Introduction

How is the sound of a drum or a bell affected by its shape?
How many integer lattice points are there in a disk of radius r centred at the origin?
How does the distribution of heat in an object evolve with the passage of time?
Why is quantum mechanical matter stable?

It is far from apparent that the mathematics involved in investigating these questions are even remotely related. Nonetheless, all of the questions can be naturally formulated in the language of spectral theory. The results contained in this thesis concern, in one way or other, questions of a spectral-theoretical nature. Some of these questions are motivated mainly by intrinsic mathematical interest, while others arise in the theory of many-particle systems in quantum mechanics.

This first chapter is intended as a brief and informal introduction to the kind of questions that will be discussed in the remainder of the thesis. Our aim is to keep the discussion as non-technical as possible. Precise definitions will be left until Chapter 2.

From strings and drums via blackbody radiation to quantum mechanics

To understand the relationship between the geometry and spectral properties of an operator is among the oldest and arguably one of the most fascinating problems in mathematical analysis. Already the Pythagoreans understood the effect that changing the length of a string had on the tone that it produced. The natural higher-dimensional generalization, which is a long-standing problem of spectral theory, is to understand the relationship between the shape of a domain $\Omega \subset \mathbb{R}^d$ and the eigenvalues of the associated Dirichlet Laplace operator. Indeed, these eigenvalues are in one-to-one correspondence with the oscillating frequencies of a vibrating membrane whose boundary is held fixed. In two dimensions the picture to have in mind is that of the fundamental frequency and overtones of a drum with drumhead in the shape of $\Omega \subset \mathbb{R}^2$. In three or higher dimensions the corresponding analogy is more difficult to visualize. However, already in the case of dimension two understanding the relation between the shape of Ω and the corresponding eigenvalues (or frequencies) is an extremely difficult problem. In fact, it is only in very special cases that the eigenvalues can be completely determined.

The correspondence between the frequencies of normal modes of oscillating systems and the eigenvalues of differential operators was first studied during the late 18-th century. Important contributions were made by a number of mathematicians and physicists, notably d'Alembert, Fourier, and Laplace. In particular, they considered oscillating membranes and vibrating strings but also more general oscillations of elastic bodies. Moreover, the theory that was developed extends to the more subtle oscillations of light and radiation present in the theory of electromagnetism. As a result the study of spectral properties of differential operators was found incredibly important.

Hearing the shape of a drum

In applications of spectral theory one is most often concerned with what is called the direct problem, that is, to determine the spectrum of a given operator. Naturally one can turn this question around and ask whether if given the spectrum, one can determine the operator. A problem of this type will be referred to as an inverse problem. In the setting of vibrating two-dimensional membranes one such problem was popularized by Kac in his celebrated 1966 lecture *Can one hear the shape of a drum?* See also Kac' paper [52] for the lecture in written form. Formulated in terms of spectral theory, the task is to reconstruct a domain $\Omega \subset \mathbb{R}^2$ from the eigenvalues of the associated Dirichlet Laplacian.

Not long before Kac' lecture it had been observed by Milnor [82] that there exist two different 16-dimensional tori whose Laplacian eigenvalues coincide. Consequently, one cannot distinguish these two manifolds knowing only their respective spectrum. The problem in its two-dimensional form did not see a solution until almost 30 years later when Gordon, Webb, and Wolpert [37] found a manner in which to apply ingenious ideas of Sunada [95] and Bérard [3] to construct pairs of different planar polygons whose Dirichlet Laplacians have identical eigenvalues. For an example of two such polygons see Figure 1.1. Two domains with this property are called *isospectral*. Consequently, the answer to Kac' question is negative *no, one cannot hear the shape of a drum.*

Even though the result of Gordon, Webb, and Wolpert conclusively answers the question posed in Kac' lecture this is fortunately not the end of the story. For instance, it is natural to ask what geometric properties of Ω can be deduced from the associated eigenvalues. Indeed, if one considers the isospectral domains in Figure 1.1 one can easily verify that they have the same area, the same perimeter, and the same number of corners with a given angle. Are these quantities observable from the associated eigenvalues? Already at the time of Kac' lecture it had been known for more than 50 years that one can hear the size of a drum. Indeed, this follows from a conjecture made by Rayleigh already in 1900 and proved twelve years later by Weyl [101]. Specifically, the conjecture of Rayleigh states that the number of eigenvalues of the Dirichlet Laplace operator on $\Omega \subset \mathbb{R}^d$ less than λ grows in

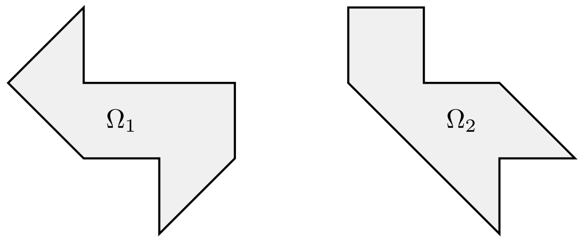

Figure 1.1: Two isospectral and non-congruent polygons constructed in [37].

proportion to the measure of Ω as λ tends to infinity, up to comparatively small errors [87].

Part of the results of this thesis concern recovering geometric properties of domains which solve certain geometric extremal problems in terms of the eigenvalues of an associated Laplace operator. In a sense, the question asked is whether the knowledge that the domain solves a spectral shape optimization problem yields sufficient additional information to solve the inverse spectral problem.

Blackbody radiation and the ultraviolet catastrophe

The conjecture of Rayleigh was, however, not motivated by the oscillations of vibrating drums, but by the theory of electromagnetic radiation. Specifically, Rayleigh was analysing how the intensity of the radiation emitted by a blackbody at fixed temperature depends on the radiation wavelength. To determine the relation he needed to count the number of standing electromagnetic waves at a certain frequency. In essence, this problem is the same as counting the number of normal modes with a given frequency for a vibrating membrane. Through explicit calculations Rayleigh was able to deduce that for a blackbody with the shape of a box this number can be well approximated in terms of the volume of the blackbody as the frequency becomes large. Based on calculations in this special case Rayleigh conjectured that the approximation was valid for blackbodies of arbitrary shape. This is precisely the conjecture alluded to above.

Although the law derived by Rayleigh succeeds in accurately describing the radiation in the regime of large wavelengths, it fails to do so when the wavelength becomes small. The law predicts that the intensity diverges to infinity as the wavelength goes to zero when, in fact, the intensity should tend to zero in this limit. The failure of the law is not due to any error in Rayleigh's derivation, but stems from fundamental problems in the laws of classical mechanics. Ultimately a law that accurately reproduced experimental data was proposed by Planck, which importantly avoided the so-called *ultraviolet catastrophe* present in the result of Rayleigh. The derivation of Planck relied on the revolutionary assumption that electromagnetic radiation could only be absorbed or emitted in certain discrete quantities proportional to its frequency. This assumption is one of the fundamental building blocks in the theory that was to grow out of Planck's ideas, namely, *quantum mechanics*.

Quantum mechanics and the semiclassical limit

It is quite remarkable that the considerations of Rayleigh remain important in the quantum mechanics that emerged from the failure of the classical mechanics in which his analysis was based. Indeed, the differential operators that describe the vibrating membrane play a central role in quantum mechanics. In the Schrödinger picture of quantum mechanics the time evolution of a quantum state is determined in terms of such operators. Moreover, the energy levels of a particle, or a system of particles, obeying the laws of quantum mechanics are given by the eigenvalues of these operators.

According to quantum mechanics the quantities in which electromagnetic radiation can be absorbed or emitted are proportional to its frequency. The proportionality constant is known as Planck's constant and usually denoted by h. By formally letting h tend to zero the theory approaches, to a certain extent, that described by classical mechanics. Here our interest will often be directed towards questions concerning this so-called *semiclassical limit*. In the language of spectral theory such analysis typically corresponds to understanding the asymptotic distribution of eigenvalues. In particular, the conjecture by Rayleigh discussed earlier is a problem of this character. The remarkable work done by Weyl in his proof of Rayleigh's conjecture importantly marks the starting point for the development of mathematical tools dedicated to analysing the semiclassical limit. The resulting field of mathematics, which goes under the name *semiclassical analysis*, is today an important and highly active branch of spectral theory. Moreover, the theory has been found highly useful also in other areas of mathematics and physics.

The results of this thesis fall into one of two categories, each of which is to some extent related to the semiclassical limit of quantum mechanical systems. The first category concerns problems in the intersection of spectral theory and geometry. Here our interest lies in geometric extremal problems for eigenvalues and their behaviour in a semiclassical limit. The second category features bounds for the kinetic energy of certain many-particle quantum mechanical systems with focus on the behaviour of the energy as the number of particles tends to infinity. Although the two sets of problems might appear fairly different, we will see that the mathematics involved in the analysis has a common basis and many of the techniques and ideas can naturally be translated between the two topics.

2 Background

In this chapter we take a step back and recall some basic concepts and fundamental theorems which lay the foundation of the spectral theory relevant for the thesis.

The main topic of this thesis concerns the quantitative study of the spectrum of certain differential or pseudodifferential operators in $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is either the full space or a bounded subset with Lipschitz boundary. The aim of Section 2.1 is to give the reader not familiar with spectral theory a brief introduction to the basics of the subject. The reader who already feels comfortable with spectral theory for self-adjoint operators in Hilbert spaces can move on to Section 2.2 directly.

2.1 Spectral theory for self-adjoint operators

We begin by giving a quick background on spectral theory for self-adjoint operators in Hilbert spaces. As this is a very broad subject it is impossible to cover even the smallest portion without leaving completely the scope of this thesis. Here we restrict ourselves to the bare minimum of what is necessary to understand the results obtained in the thesis. For the same reason it is assumed that the reader has some familiarity with basic Sobolev space theory and functional analysis. Our presentation largely follows the lecture notes of Lundholm [72] but adapted to the context at hand. For more comprehensive introductions to spectral theory of self-adjoint operators the reader is referred to [10, 24, 88].

Self-adjoint operators

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{H}}$ and norm $\|\cdot\|_{\mathscr{H}}$.[(1)] An *operator* L in $\mathscr{H}$ is a linear mapping $L\colon D(L) \to \mathscr{H}$, where $D(L) \subseteq \mathscr{H}$ is the *domain* of L. Here we always work with operators whose domain is dense in $\mathscr{H}$. For an operator L define its *operator norm*

$$\|L\| = \sup\bigl\{\|Lu\|_{\mathscr{H}} : u \in D(L),\ \|u\|_{\mathscr{H}} = 1\bigr\}.$$

(1) Here $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{H}}$ is conjugate-linear in the fist argument and linear in the second. Other conventions appear in the literature but this is the most common within mathematical physics.

If $\|L\| < \infty$ the operator L is said to be *bounded* and otherwise *unbounded*. The set of densely defined operators in $\mathscr{H}$ is denoted by $\mathcal{L}(\mathscr{H})$ and the subset of bounded operators by $\mathcal{B}(\mathscr{H})$. Without loss of generality assume that $D(L) = \mathscr{H}$ for any $L \in \mathcal{B}(\mathscr{H})$.

An operator $L \in \mathcal{L}(\mathscr{H})$ is called *closed* if its graph,

$$\big\{(u, v) \in D(L) \times \mathscr{H} : v = Lu\big\},$$

is a closed subspace of $\mathscr{H} \times \mathscr{H}$. Moreover, $L \in \mathcal{L}(\mathscr{H})$ is said to be *closable* if there exists a closed extension of L, i.e. a closed operator $\hat{L}\colon D(\hat{L}) \to \mathscr{H}$ such that $D(\hat{L}) \supset D(L)$ and $\hat{L}|_{D(L)} = L$. Every closable operator $L \in \mathcal{L}(\mathscr{H})$ has a smallest closed extension, namely, its *closure* $\bar{L}$.

For $L \in \mathcal{L}(\mathscr{H})$ define its *adjoint* L^* as the operator with domain

$$D(L^*) = \Big\{u \in \mathscr{H} : \sup_{v \in D(L),\, \|v\|_{\mathscr{H}} = 1} |\langle u, Lv\rangle_{\mathscr{H}}| < \infty\Big\}$$

such that the formula $\langle L^* u, v\rangle_{\mathscr{H}} = \langle u, Lv\rangle_{\mathscr{H}}$ holds for all $u \in D(L^*)$ and $v \in D(L)$. That this indeed defines a unique operator is a consequence of Hahn–Banach's theorem and the Riesz representation theorem.

An operator $L \in \mathcal{L}(\mathscr{H})$ is called *symmetric* (or *hermitian*) if the associated sesquilinear form

$$\begin{aligned} q\colon D(L) \times D(L) &\to \mathbb{C} \\ (u, v) &\mapsto \langle u, Lv\rangle_{\mathscr{H}} \end{aligned} \tag{2.1}$$

is such that $q(u, v) = \overline{q(v, u)}$ for all $u, v \in D(L)$. Every symmetric operator is closable.

An operator $L \in \mathcal{L}(\mathscr{H})$ is called *self-adjoint* if $L = L^*$, i.e. if L is symmetric and $D(L) = D(L^*)$. An operator $L \in \mathcal{L}(\mathscr{H})$ is called *essentially self-adjoint* if it is symmetric and has a unique self-adjoint extension, in which case $\bar{L} = L^*$. In particular, any bounded densely defined symmetric operator is essentially self-adjoint.

In the context of this thesis we are mainly concerned with symmetric operators which are bounded from below, $\langle u, Lu\rangle_{\mathscr{H}} \geq -c\|u\|^2_{\mathscr{H}}$. Such operators always have self-adjoint extensions and, moreover, there exists a somewhat distinguished extension called the *Friedrichs extension* which we define shortly.

Quadratic forms and the Friedrichs extension

As a consequence of the Reisz representation theorem there is a one-to-one correspondence between bounded sesquilinear forms on $\mathscr{H} \times \mathscr{H}$ and $\mathcal{B}(\mathscr{H})$ (see e.g. [10, Theorem 2.4.6]). By the polarization identity it suffices to consider the *quadratic form of* L defined by $u \mapsto q(u, u) = \langle u, Lu\rangle_{\mathscr{H}}$. As the risk of confusion is minimal, we use the same notation for both a quadratic form and its associated sesquilinear form. Also for unbounded operators it can be both useful and convenient to study the associated quadratic form. In fact, all operators encountered in this thesis are defined through a quadratic form.

A quadratic form $q\colon D(q) \to \mathbb{C}$, $D(q) \subseteq \mathscr{H}$, is called *hermitian* if $q\colon D(q) \to \mathbb{R}$, *non-negative* if $q(u) \geq 0$, and *positive* if $q(u) > 0$ for all $u \in D(q) \setminus \{0\}$. For two hermitian forms q_1, q_2 we say that $q_1 \geq q_2$ if $D(q_1) \subseteq D(q_2)$ and $q_1(u) \geq q_2(u)$ for all $u \in D(q_1)$. A hermitian form q is *semi-bounded (from below)* if there exists a constant $c \in \mathbb{R}$ such that $q(u) \geq -c\|u\|_{\mathscr{H}}^2$ for all $u \in D(q)$, i.e. if $q \geq -c$ where the right-hand side should be interpreted as the quadratic form $u \mapsto -c\|u\|_{\mathscr{H}}^2$.

To any semi-bounded form $q \geq -c$ we can associate the positive form $\tilde{q}(u) = q(u) + (1+c)\|u\|_{\mathscr{H}}^2$. A semi-bounded form q is called *closed* if $D(q)$ is complete with respect to the norm $\|u\|_q = \sqrt{\tilde{q}(u)}$. A form q is called *closable* if it has a closed extension $\hat{q}$, i.e. there exists a closed form $\hat{q}$ such that $D(q) \subset D(\hat{q})$ and $\hat{q}|_{D(q)} = q$. The smallest such extension $\bar{q}$ is called the *closure of* q.

An operator $L \in \mathcal{L}(\mathscr{H})$ is symmetric if q in (2.1) is hermitian. We can thus define a partial ordering on the set of symmetric operators by comparing their quadratic forms. A symmetric operator $L \in \mathcal{L}(\mathscr{H})$ is called *non-negative* or *semi-bounded (from below)* if q is non-negative or semi-bounded, respectively.

One of the main reasons to go through quadratic forms when working with symmetric operators is the following theorem

Theorem 2.1 (see e.g. [10]). *If $L \in \mathcal{L}(\mathscr{H})$ is self-adjoint and semi-bounded, then there exists a unique closed quadratic form q_L such that $D(L) \subseteq D(q_L)$ and*

$$\langle u, Lv \rangle_{\mathscr{H}} = q_L(u, v), \qquad \textit{for all } u \in D(q_L),\ v \in D(L).$$

Conversely, if $q\colon D(q) \to \mathbb{R}$, $D(q) \subseteq \mathscr{H}$, is a densely defined closed quadratic form, then q is the quadratic form of a unique self-adjoint operator.

Moreover, the operator and the associated quadratic form satisfy the same greatest lower bound

$$\inf_{u \in D(L)} \frac{\langle u, Lu \rangle_{\mathscr{H}}}{\|u\|_{\mathscr{H}}^2} = \inf_{u \in D(q_L)} \frac{q_L(u)}{\|u\|_{\mathscr{H}}^2}.$$

The key observation of the theorem is that, in contrast to operators, a form cannot be closed while failing to represent a unique self-adjoint operator.

Let $L \in \mathcal{L}(\mathscr{H})$ be self-adjoint and semi-bounded. By Theorem 2.1 there exists a unique closed quadratic form q_L associated to L. The domain of this quadratic form $D(q_L)$ is called the *form domain of* L.

We are now ready to state the Friedrichs extension theorem.

Theorem 2.2 (The Friedrichs extension; see e.g. [88, Theorem X.23]). *Let L be a semi-bounded symmetric operator. Then q defined by* (2.1) *is a closable quadratic form and its closure $\bar{q}$ is the quadratic form of a unique self-adjoint operator $\hat{L}$ called the Friedrichs extension of L. $\hat{L}$ is a semi-bounded extension of L, with the same lower bound as that of L. Furthermore, $\hat{L}$ is the largest among all self-adjoint extensions of L and the only one such that $D(\hat{L}) \subseteq D(\bar{q})$.*

The spectrum

The purpose of this section is to define the spectrum of an operator $L \in \mathcal{L}(\mathscr{H})$ and recall a number of important results in the setting of self-adjoint operators. In particular, we discuss a splitting of the spectrum of a self-adjoint operator into two disjoint parts: the discrete and essential spectrum.

Definition 2.3 (Spectrum of a linear operator)**.** For $L \in \mathcal{L}(\mathscr{H})$ we say that $\lambda \in \mathbb{C}$ is in $\sigma(L)$ *the spectrum of* L if the operator

$$L - \lambda \mathrm{Id}_{\mathscr{H}}$$

fails to have a bounded inverse.

If $(u, \lambda) \in (\mathscr{H} \setminus \{0\}) \times \mathbb{C}$ is such that $(L - \lambda \mathrm{Id}_{\mathscr{H}})u = 0$ then λ is called an *eigenvalue of* L and u a corresponding *eigenvector* (or *eigenfunction*). The subspace of all eigenvectors corresponding to an eigenvalue λ is called the *eigenspace associated to* λ. The dimension of the eigenspace, i.e. $\dim \ker(L - \lambda \mathrm{Id}_{\mathscr{H}})$, is called the *(geometric) multiplicity* of the eigenvalue λ.

A closed symmetric operator L is self-adjoint if and only if its spectrum is a subset of the real line. In fact, the spectrum of a closed symmetric operator consists of either all of $\mathbb{C}$, the set $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq 0\}$, the set $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \leq 0\}$, or a subset of the real line (see e.g. [88, Theorem X.1]).

Definition 2.4 (Essential and discrete spectrum)**.** Let $L \in \mathcal{L}(\mathscr{H})$ be self-adjoint. The *discrete spectrum of* L denoted by $\sigma_d(L)$ is defined as the set of isolated eigenvalues of L with finite multiplicity. Moreover, $\sigma_e(L)$ the *essential spectrum of* L is defined by $\sigma_e(L) = \sigma(L) \setminus \sigma_d(L)$.

In the applications considered here interest is mainly focused on the bottom of the spectrum, $\inf \sigma(L)$, or the discrete spectrum. The following theorem provides a very useful variational characterization of eigenvalues below the essential spectrum.

Theorem 2.5 (The min-max principle; see e.g. [88, Theorem XIII.1])**.** *Assume that* $L \in \mathcal{L}(\mathscr{H})$ *is self-adjoint and semi-bounded from below. For* $k \in \mathbb{N}$ *define* $\lambda_k(L)$ *by*

$$\lambda_k(L) = \sup_{\substack{H \subset D(L) \\ \dim H \leq k-1}} \inf_{u \in H^\perp \setminus \{0\}} \frac{\langle u, Lu \rangle_{\mathscr{H}}}{\|u\|_{\mathscr{H}}^2} = \inf_{\substack{H \subset D(L) \\ \dim H \geq k}} \sup_{u \in H \setminus \{0\}} \frac{\langle u, Lu \rangle_{\mathscr{H}}}{\|u\|_{\mathscr{H}}^2}.$$

Then for fixed k*, one of two cases occur*:

(i) *The operator* L *has (at least)* k *eigenvalues counted with multiplicity below* $\inf \sigma_e(L)$*, and* $\lambda_k(L)$ *is the* k*-th such eigenvalue.*

(ii) $\lambda_k(L) = \inf \sigma_e(L)$*, in this case* $\lambda_{k'}(L) = \lambda_k(L)$*, for all* $k' \geq k$*, and* L *has at most* $k-1$ *eigenvalues counted with multiplicity below* $\lambda_k(L)$*.*

Moreover, given a symmetric semi-bounded operator L the λ_k obtained through the variational procedure in Theorem 2.5 coincide with those of the Friedrichs extension of L. In particular,

$$\inf_{u \in D(L)} \frac{\langle u, Lu \rangle_{\mathscr{H}}}{\|u\|^2_{\mathscr{H}}}$$

is the bottom of the spectrum of the Friedrichs extension of L. Similarly, if one replaces $\langle u, Lu \rangle_{\mathscr{H}}$ by $q_L(u)$, then the subspaces of the operator domain $D(L)$ can be replaced by subspaces of the form domain $D(q_L)$ without further altering the statement (see [88, Theorem XIII.2]).

For the applications we have in mind here Theorem 2.5 provides an indispensable tool when analysing the spectrum of the operators we are interested in.

Let $L \in \mathcal{L}(\mathscr{H})$ be self-adjoint and semi-bounded from below with either purely discrete spectrum or a number of eigenvalues below $\sigma_e(L)$. For such L we write $\{\lambda_k(L)\}_{k \geq 1}$ for the increasingly ordered sequence of eigenvalues (finite or infinite) counted with multiplicity taken into account. When there is no risk of confusion we write simply λ_k and leave the dependence on L implicit.

The spectrum of Laplace and Schrödinger operators

With the concepts discussed above in hand we are ready to recall a number of well-known results concerning the spectrum of the Schrödinger operator

$$H = -\Delta + V \tag{2.2}$$

defined as a self-adjoint operator in $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ and the *(electric) scalar potential* $V\colon \Omega \to \mathbb{R}$ are assumed to be sufficiently regular.

The largest part of the thesis (Papers B–E) is concerned with the simplest possible case, when $\Omega \subset \mathbb{R}^d$ is bounded and V is identically zero. That is, when H is a Laplace operator $-\Delta$ in $L^2(\Omega)$. Specifically, our interest lies in understanding spectral properties of such operators in terms of the geometry of Ω. Paper F concerns problems related to the operator H with $V(x) = |x|^2$ in $L^2(\mathbb{R}^2)$. In both settings the spectrum of the operator is purely discrete, and the focus of the papers is towards the behaviour of $\lambda_k(H)$ in the limit as k tends to infinity.

In Paper G a generalization of H to the setting of many-body quantum mechanics is considered for which $V \equiv 0$ but the Laplacian is replaced by a magnetic operator $(-i\nabla + \mathbf{A})^2$ with a non-trivial *magnetic vector potential* $\mathbf{A}\colon \Omega \to \mathbb{R}^d$. However, a substantial part of the analysis is based on proving that the quadratic form of the operator can be bounded from below by that of a non-magnetic Schrödinger operator. The setting in Paper H is similarly many-body quantum mechanics but the operator of concern is not the differential operator in (2.2) but the fractional Laplace operator (see (5.7) below). In both of these papers the main interest lies in the behaviour of the infimum of the spectrum (the ground-state energy) as the number of particles tends to infinity.

Since the magnetic potential considered in Paper G is too singular to be covered by standard results, the present discussion is restricted to the case of the non-magnetic operator (2.2). However, much of what is discussed carries over under mild conditions on the magnetic potential (see e.g. [28, 29]). For instance, the results on the semi-boundedness of $H = -\Delta + V$ carries over to $(-i\nabla + \mathbf{A})^2 + V$ when $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ by Theorem 2.5 and the *diamagnetic inequality* [67, Theorem 7.21]:

$$\big|\nabla|u(x)|\big| \le |(-i\nabla + \mathbf{A}(x))u(x)|, \quad \text{for almost every } x \in \mathbb{R}^d.$$

For further reference, let q_H denote the quadratic form associated to the differential expression H in (2.2),

$$q_H(u) = \int_\Omega \big(|\nabla u(x)|^2\,dx + V(x)|u(x)|^2\big)\,dx. \tag{2.3}$$

Schrödinger operators on $\mathbb{R}^d$

If $\Omega = \mathbb{R}^d$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and the form q_H with $D(q_H) = C_0^\infty(\mathbb{R}^d)$ is semi-bounded, then it is closable (see e.g. [80]). Thus if this is the case Theorem 2.1 provides us with a self-adjoint operator in $L^2(\mathbb{R}^d)$. Defining $x_\pm = (|x| \pm x)/2$[(2)] we have

Theorem 2.6 (see e.g [67])**.** *Assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and, for some $\varepsilon > 0$,*

$$V_- \in \begin{cases} L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), & \text{if } d \ge 3, \\ L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2), & \text{if } d = 2, \\ L^1(\mathbb{R}) + L^\infty(\mathbb{R}), & \text{if } d = 1. \end{cases}$$

Then the quadratic form q_H defined in (2.3) *is semi-bounded from below. Thus it is the quadratic form of a self-adjoint operator H in $L^2(\mathbb{R}^d)$.*

Furthermore, if

$$|\{x \in \mathbb{R}^d : V(x) < -\delta\}| < \infty, \quad \text{for all } \delta > 0, \tag{2.4}$$

then $\sigma_e(H) \subseteq [0, \infty)$.

By replacing V in the last part of the theorem by $V - c$ one obtains a similar statement for potentials V such that $\liminf_{|x|\to\infty} V(x) = c$ (or the equivalent of the weaker condition in (2.4)).

An almost direct corollary of Theorem 2.6 is the following well-known result when the potential is unbounded at infinity.

Corollary 2.7. *Let V satisfy the assumptions in the first part of Theorem 2.6. If $\lim_{|x|\to\infty} V(x) = \infty$ then $\sigma(H)$ consists of an infinite number of eigenvalues of finite multiplicity, accumulating only at infinity.*

The condition on V in Corollary 2.7 is sufficient but far from necessary for the spectrum of H to be discrete. For weaker criterion and a detailed discussion of this topic see [80].

(2) Note that x_+ and x_-, the positive resp. negative part of x, are both non-negative.

Laplace and Schrödinger operators in domains

If the open set Ω is not the whole space $\mathbb{R}^d$ we consider two different self-adjoint realizations of H: the Dirichlet and the Neumann realization. Assume that $V \in L^2_{\mathrm{loc}}(\Omega)$ and that Ω is bounded and has Lipschitz boundary.

The Dirichlet operator: Under the assumptions above $-\Delta + V$ is well defined as a symmetric operator with domain $C_0^\infty(\Omega)$. If q_H is semi-bounded from below Theorem 2.2 yields a self-adjoint operator in $L^2(\Omega)$. This operator is the Dirichlet realization of the Schrödinger operator H. If the potential V is regular (say bounded) the form domain of the operator is $H_0^1(\Omega)$.

The Neumann operator: The Neumann realization is obtained similarly but starting from the quadratic form q_H with $D(q_H) = C^\infty(\overline{\Omega})$. If this quadratic form is semi-bounded and closable an application of Theorem 2.1 yields a self-adjoint operator. This operator is the Neumann realization of the Schrödinger operator H. If the potential V is regular (say bounded) these assumptions are fulfilled and the form domain of the operator is $H^1(\Omega)$.

As mentioned above the majority of this thesis concerns the simplest possible case of (2.2), namely, when V is identically zero. The resulting operators are the Dirichlet and Neumann Laplace operators. To distinguish between the operators we write $-\Delta_\Omega^{\mathcal{D}}$ for the Dirichlet realization and $-\Delta_\Omega^{\mathcal{N}}$ for the Neumann realization.

For later purposes we recall that the Laplace operator obeys the following scaling relation $\sigma(\Delta_{t\Omega}^{\mathcal{D}/\mathcal{N}}) = t^{-2}\sigma(\Delta_\Omega^{\mathcal{D}/\mathcal{N}})$, for $t > 0$.

The following theorem can be found in almost any textbook on spectral theory, see for instance [10, 67].

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^d$ be open and $|\Omega| < \infty$. Then $\sigma(-\Delta_\Omega^{\mathcal{D}})$ is discrete and consists of positive eigenvalues of finite multiplicity, accumulating only at infinity.*

For the Neumann Laplacian things are more complicated. Indeed, the statement obtained by simply replacing $-\Delta_\Omega^{\mathcal{D}}$ by $-\Delta_\Omega^{\mathcal{N}}$ in Theorem 2.8 is false. In fact, a remarkable result of Hempel, Seco, and Simon [44] states that given any closed set $S \subset [0,\infty)$ there exists a bounded open and connected set $\Omega \subset \mathbb{R}^d$ for which $\sigma_e(-\Delta_\Omega^{\mathcal{N}}) = S$. For precise conditions ensuring that the spectrum of $-\Delta_\Omega^{\mathcal{N}}$ is discrete we refer to [80]. For our purposes the following will be sufficient

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary. Then $\sigma(-\Delta_\Omega^{\mathcal{N}})$ is discrete and consists of non-negative eigenvalues of finite multiplicity, accumulating only at infinity.*

These theorems follow from the compactness of the embedding of the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ into $L^2(\Omega)$ combined with the following result.

Theorem 2.10 (see e.g. [10, Theorem 10.1.5])**.** *The spectrum of a semi-bounded self-adjoint operator $L \in \mathcal{L}(\mathscr{H})$ is discrete if and only if the embedding of the form domain $D(q_L) \hookrightarrow \mathscr{H}$ is compact.*

Theorems 2.8 and 2.9 extend with only small changes to the Dirichlet and Neumann realizations of the Schrödinger operator $H = -\Delta + V$ in $L^2(\Omega)$, $\Omega \subsetneq \mathbb{R}^d$, if V is sufficiently regular. Naturally, if V is non-positive the spectrum can become negative. That the spectrum of H is discrete if Ω satisfies the assumptions of Theorems 2.8 or 2.9 follows from Theorem 2.10 and the fact that the form domain of H coincides with that of the corresponding Laplacian. That the form domain remains the same can be verified under fairly weak assumptions on V but this will not be of any large importance here.

2.2 Weyl's law and semiclassical asymptotics

In this section we turn our attention to the asymptotic behaviour of the spectrum of the Schrödinger operator $H = -h^2\Delta + V$, $h > 0$, in the semiclassical limit $h \to 0^+$. Again our focus is on Laplace operators in domains and Schrödinger operators in the full space $\mathbb{R}^d$.

Laplace operators

In the case of Laplace operators in a bounded domain $\Omega \subset \mathbb{R}^d$ the dependence on the parameter h becomes trivial. The corresponding limit to consider is the asymptotic growth of $\lambda_k(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}})$ as $k \to \infty$. However, instead of considering the asymptotic behaviour of the eigenvalues directly it is convenient to consider the eigenvalue counting function

$$N(\lambda; -\Delta_\Omega^{\mathcal{D}/\mathcal{N}}) = \#\bigl\{k \in \mathbb{N} : \lambda_k(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}}) < \lambda\bigr\}, \quad \lambda > 0,$$

in the limit $\lambda \to \infty$. Note that $N(\lambda; -\Delta_\Omega^{\mathcal{D}/\mathcal{N}})$ is equal to the number of negative eigenvalues of $-h^2\Delta_\Omega^{\mathcal{D}/\mathcal{N}} - 1$, by setting $h = 1/\sqrt{\lambda}$.

The first result in this direction is the Weyl law [101] stating that

$$N(\lambda; -\Delta_\Omega^{\mathcal{D}/\mathcal{N}}) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{d/2} + o(\lambda^{d/2}), \quad \text{as } \lambda \to \infty. \tag{2.5}$$

Here and in what follows ω_d denotes the volume of the d-dimensional unit ball. Both Weyl's original proof and subsequent ones typically assume the boundedness of Ω and some regularity of its boundary. That (2.5) holds for the Dirichlet Laplacian without any regularity assumptions was shown by Rozenblum [90] (see also [9]). For the Neumann case, see e.g. [17, 19]. Note that in the Neumann case additional assumptions on Ω are necessary since otherwise the spectrum can fail to be discrete.

It was conjectured by Weyl [102] that the asymptotic formula (2.5) could be refined. Specifically, he conjectured the validity of the two-term asymptotic expansion

$$N(\lambda; -\Delta_\Omega^{\mathcal{D}/\mathcal{N}}) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{d/2} \pm \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}) \tag{2.6}$$

as $\lambda \to \infty$, where the second term comes with a plus in the Neumann case and a minus in the Dirichlet case. Here and in what follows $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.

Using detailed microlocal analysis, Ivrii in 1980 proved that if the boundary of Ω is smooth and the measure(3) of all periodic geodesic billiards is zero then (2.6) holds [51]. Although one can hope to reduce the regularity assumptions in Ivrii's result the assumption that the measure of all periodic billiards is zero appears necessary. Indeed, it is known to be necessary for the corresponding result on manifolds (see e.g. [92]). However, it is believed that the billiard assumption is true for any smooth domain in $\mathbb{R}^d$, but so far this has only been proved in special cases [98].

In the applications that we have in mind the a priori assumptions on the domain Ω are minimal, and in particular much weaker than those of Ivrii. The domains considered here typically arise as a solution of a spectral shape optimization problem for which a priori no regularity is known (see Chapter 3). Moreover, it will often be important for us to obtain precise bounds for the remainder term in asymptotic expansions, not only for a single domain but within a family of domains. For such purposes the techniques used by Ivrii in his proof of Weyl's conjecture are unfortunately not very applicable.

Prior to Ivrii's proof of (2.6), and in support of Weyl's conjecture, refined asymptotic expansions were obtained for certain smooth functions of the eigenvalues instead of the counting function. Indeed, one of the main sources of difficulties in proving (2.6) is the discontinuity of the counting function. An important example of such asymptotic formulae is the short-time limit for the trace of the heat kernel associated to $-\Delta_\Omega^{\mathcal{D}/\mathcal{N}}$:

$$\mathrm{Tr}\big(e^{t\Delta_\Omega^{\mathcal{D}/\mathcal{N}}}\big) = \sum_{k\geq 1} e^{-t\lambda_k} = (4\pi t)^{-d/2}\Big(|\Omega| \pm \frac{\sqrt{\pi}}{2}\mathcal{H}^{d-1}(\partial\Omega)t^{1/2} + o(t^{1/2})\Big), \qquad (2.7)$$

as $t \to 0^+$. Which is valid under appropriate assumptions on Ω (see e.g. [11]). As in (2.6) the second term comes with a plus in the Neumann case and with a minus in the Dirichlet case.

In this thesis we often encounter a certain family of regularizations of the counting function. For $\gamma \geq 0$ and $\lambda \geq 0$, define the *Riesz mean*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}} - \lambda)_-^\gamma = \sum_{\lambda_k < \lambda} (\lambda - \lambda_k)^\gamma. \qquad (2.8)$$

In a certain sense the Riesz means interpolate between the counting function and the trace of the heat kernel. For $\gamma = 0$, (2.8) is precisely the counting function. Setting $\gamma = t\lambda$, with $t > 0$, and dividing (2.8) by λ^γ, one in the limit $\lambda \to \infty$ obtains the trace of the heat kernel $\mathrm{Tr}(e^{t\Delta_\Omega^{\mathcal{D}/\mathcal{N}}})$.

(3) In terms of the natural measure on the co-tangent bundle.

The corresponding two-term asymptotic expansion in the limit $\lambda \to \infty$ reads

$$\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}} - \lambda)_-^\gamma = L_{\gamma,d}^{\mathrm{cl}}|\Omega|\lambda^{\gamma+d/2} \pm \frac{L_{\gamma,d-1}^{\mathrm{cl}}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{\gamma+(d-1)/2} + o(\lambda^{\gamma+(d-1)/2}). \tag{2.9}$$

Here and in what follows $L_{\gamma,d}^{\mathrm{cl}}$ denotes the semiclassical Lieb–Thirring constant

$$L_{\gamma,d}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+d/2+1)}.$$

Since $L_{0,d}^{\mathrm{cl}} = \omega_d/(2\pi)^d$ the expansion (2.9) for $\gamma = 0$ matches that in (2.6). Note that (2.6) implies (2.9) which, in turn, implies (2.7). However, the reverse implications are false.

In Paper E the asymptotic expansion (2.9) is proved to be valid for the Dirichlet Laplacian and $\gamma = 1$ under the assumption that $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary.

From the definition it is rather intuitive that the Riesz means can be considered as regularizations of the counting function. Before moving on we recall the Aizenman–Lieb indentity [1] which clarifies in which sense this intuition is valid. For $0 \le \gamma < \gamma'$ and $\lambda \ge 0$ it holds that

$$\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}} - \lambda)_-^{\gamma'} = B(1+\gamma, \gamma'-\gamma)^{-1}\int_0^\infty \tau^{\gamma'-\gamma-1}\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}/\mathcal{N}} - (\lambda-\tau))_-^\gamma\, d\tau. \tag{2.10}$$

Here B denotes the Euler Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

for x, y with positive real part. That is, one can write any Riesz mean as an integral of Riesz means of lower order. In particular, if one can compute the counting function one can also compute any Riesz mean. This identity will be used frequently in several of the papers included in the thesis.

Returning briefly to the question of Kac [52] mentioned in Chapter 1 we note that the validity of any of the two-term asymptotic expansions discussed in this section implies that one can hear the perimeter of a drum.

Schrödinger operators

Consider the Schrödinger operator $H = -h^2\Delta + V$ in $L^2(\mathbb{R}^d)$. If the spectrum of H is purely discrete one can naturally consider the asymptotics of the corresponding counting function or Riesz means as we did for the Laplace operator above. Such will be the case for the problem considered in Paper F.

However, in applications to quantum mechanics it is often the case that V decays at infinity and $\sigma_e(H) = [0, \infty)$. Our interest is then directed towards the number and moments of the negative eigenvalues in the limit $h \to 0^+$. Note that this can be equivalently formulated in terms of the asymptotics for the eigenvalues of $-\Delta + \beta V$ in the limit of large coupling constant $\beta \to \infty$.

Let $\{\lambda_k(h, V)\}_{k\geq 1}$ denote the negative eigenvalues of $H = -h^2\Delta + V$ ordered increasingly and counted with multiplicity. Let $N(h; V)$ denote the number of such eigenvalues.

Under appropriate conditions on V it holds that

$$N(h; V) = \frac{\omega_d}{(2\pi)^d} \int_{\mathbb{R}^d} V(x)_-^{d/2}\, dx\, h^{-d} + o(h^{-d}), \quad \text{as } h \to 0^+. \tag{2.11}$$

The expansion (2.11) was first proved by Birman in [7] for Schrödinger operators in bounded domains, or potentials V having compact support. The result has since seen a large number of extensions, see e.g. [67, 88]. If $d \geq 3$ the asymptotic formula is valid for all V such that the right-hand side is finite. In one or two dimensions the problem is more subtle and additional assumptions are necessary [8].

Formally, one obtains (2.5) from (2.11) by setting

$$V(x) = \begin{cases} -1, & \text{for } x \in \Omega, \\ \infty, & \text{for } x \in \Omega^c. \end{cases}$$

For the moments of the negative eigenvalues of H it holds under appropriate assumptions on V that, for $\gamma \geq 0$ and as $h \to 0^+$,

$$\operatorname{Tr} H_-^\gamma = \sum_{k\geq 1} |\lambda_k(h, V)|^\gamma = L_{\gamma,d}^{\mathrm{cl}} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\, dx\, h^{-d} + o(h^{-d}). \tag{2.12}$$

By rewriting the integral in (2.11) (or more generally (2.12)) as an integral over the classical phase-space,

$$L_{\gamma,d}^{\mathrm{cl}} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\, dx\, h^{-d} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 + V(x))_-^\gamma \frac{dx\, d\xi}{(2\pi h)^d},$$

the expansion can be given the semiclassical interpretation that each quantum state occupies a phase-space volume of size $(2\pi h)^d$.

2.3 Universal spectral inequalities

In the last section of this chapter our focus shifts from the asymptotic estimates discussed in the previous section to universal bounds. That is, inequalities valid for all values of the spectral parameter and not only asymptotically.

Our discussion concerns the deep fact that the leading order term in the asymptotic expansions of Section 2.2 in certain cases provides a valid bound for the spectral quantity in question, possibly up to a multiplicative universal constant. Again our discussion is restricted to the case of Laplace operators in domains and Schrödinger operators in $\mathbb{R}^d$.

The first two results concern the Dirichlet and Neumann Laplacian, respectively, and go back to work of Berezin [4] (see also [63]). For the Dirichlet Laplacian an equivalent result was obtained by Li and Yau [62]. Similarly, Kröger proved an equivalent inequality in the case of the Neumann Laplacian [55]. For the inequalities as stated here see [56].

Theorem 2.11 (The Berezin–Li–Yau inequality). *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure. Then the following inequality holds*

$$\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_- \leq L_{1,d}^{\mathrm{cl}} |\Omega| \lambda^{1+d/2} \quad \textit{for all } \lambda \geq 0.$$

Theorem 2.12 (The Kröger inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Then the following inequality holds*

$$\operatorname{Tr}(-\Delta_\Omega^{\mathcal{N}} - \lambda)_- \geq L_{1,d}^{\mathrm{cl}} |\Omega| \lambda^{1+d/2} \quad \textit{for all } \lambda \geq 0.$$

That is, for the Dirichlet Laplace operator the leading term in the asymptotic expansion in (2.9) serves as an upper bound for the trace, while for the Neumann realization the same quantity constitutes a valid lower bound. Consequently, the constants in both inequalities are sharp.

The Aizenman–Lieb identity (2.10) implies that Theorems 2.11 and 2.12 generalize to higher order Riesz means; for all $\gamma \geq 1$ and $\lambda \geq 0$,

$$\begin{aligned} \operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma &\leq L_{\gamma,d}^{\mathrm{cl}} |\Omega| \lambda^{\gamma+d/2}, \\ \operatorname{Tr}(-\Delta_\Omega^{\mathcal{N}} - \lambda)_-^\gamma &\geq L_{\gamma,d}^{\mathrm{cl}} |\Omega| \lambda^{\gamma+d/2}. \end{aligned} \tag{2.13}$$

Again we emphasize that the term in the right-hand side of either inequality in (2.13) is the leading order term in the asymptotic expansion (2.9).

As a further corollary of Theorems 2.11 and 2.12 there exist inequalities of the form (2.13) also for $\gamma < 1$, however, not necessarily with the semiclassical constant $L_{\gamma,d}^{\mathrm{cl}}$. It follows from results of Li–Yau [62] and Kröger [55] that

$$N(\lambda; -\Delta_\Omega^{\mathcal{D}}) \leq \Big(\frac{2+d}{d}\Big)^{d/2} \frac{\omega_d}{(2\pi)^d} |\Omega| \lambda^{d/2} \quad \text{and} \quad N(\lambda; -\Delta_\Omega^{\mathcal{N}}) \geq \frac{d}{2+d} \frac{\omega_d}{(2\pi)^d} |\Omega| \lambda^{d/2}, \tag{2.14}$$

for all $\lambda \geq 0$ (as stated here see [56]). An important and long-standing conjecture going back to Pólya [85] is that both inequalities remain valid if the right-hand side is replaced by the leading order Weyl term (2.5), i.e. with the factors $((2+d)/d)^{d/2}$ and $d/(2+d)$ removed.

The main result of Paper B is an improvement of the first inequality in (2.13) for $\gamma \geq 3/2$ when Ω is convex. Similar improvements of Theorems 2.11 and 2.12 will be important ingredients in Papers C and D.

Turning our attention to Schrödinger operators we have

Theorem 2.13 (The Lieb–Thirring inequality)**.** *Fix $d \in \mathbb{N}$ and let γ satisfy*

$$\begin{aligned} &\gamma \geq 1/2 && \text{if } d = 1, \\ &\gamma > 0 && \text{if } d = 2, \\ &\gamma \geq 0 && \text{if } d \geq 3. \end{aligned}$$

There exists a positive constant $L_{\gamma,d}$ such that for any V, with $V_- \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\operatorname{Tr}(-\Delta + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\, dx. \tag{2.15}$$

For $\gamma = 0$ the left-hand side of (2.15) should be interpreted as the number of negative eigenvalues of $-\Delta + V$. We emphasize that if applied to $-h^2\Delta - V$ the right-hand side of the inequality in Theorem 2.13 coincides with that in (2.12), up to a multiplicative constant.

For $\gamma > \max\{1 - d/2, 0\}$, Theorem 2.13 was obtained by Lieb and Thirring [70]. The endpoint case $\gamma = 0$ and $d \geq 3$ was obtained independently by Cwikel [20], Lieb [65], and Rozenblum [89, 91]. The inequality in this case is called the Cwikel–Lieb–Rozenblum inequality. The final endpoint case $\gamma = 1/2$ and $d = 1$ was proved by Weidl [99].

For $\gamma = 0$, $d = 2$ or $\gamma \in [0, 1/2)$, $d = 1$ the inequality is false. This follows from the existence of at least one negative eigenvalue as soon as the potential V fails to be non-negative and the behaviour of such eigenvalues in the weak coupling limit [93].

Although the question for which γ, d the Lieb–Thirring inequality is valid has been completely answered, the question of determining the sharp constants $L_{\gamma,d}$ remains an intriguing open problem which has received a great deal of interest. From the asymptotic expansion (2.12) it is clear that any constant for which the inequality (2.15) is valid must satisfy $L_{\gamma,d} \geq L^{\mathrm{cl}}_{\gamma,d}$. In fact, it was conjectured by Lieb and Thirring [70] that $L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d}$ for certain combinations of γ and d. For $d \geq 1$ and $\gamma \geq 3/2$ the conjecture is known to be true [1, 58, 70]. For $d = 1$ and $\gamma < 3/2$ or $d \geq 2$ and $\gamma < 1$ it is known that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} > 1$ [41, 42, 70]. Moreover, Hundertmark, Lieb, and Thomas [49] proved that $L_{1/2,1} = 2L^{\mathrm{cl}}_{1/2,1}$. For the best currently known constants we refer to [32, 48, 65].

The inequality in Theorem 2.13 with $\gamma = 1$ was a crucial ingredient in Lieb and Thirring's proof of the stability of quantum mechanical matter [66, 69]. The inequality used in their proof is rather a dual version of the inequality which provides a lower bound for the quantum-mechanical kinetic energy of a many-body quantum state in terms of its one-particle density (see Theorem 4.1). Kinetic energy inequalities of this form will be the topic of Papers G and H.

3 Spectral shape optimization

The present chapter is intended as a very brief introduction to the theory of spectral shape optimization. This will be the main topic of both Papers C and D, and the aim of the chapter is to provide some motivation and background for the questions considered there.

A shape optimization problem is a variational problem where given a class of sets $\mathcal{A}$, called the *admissible class*, and a *cost functional* $\mathcal{F}\colon \mathcal{A} \to \mathbb{R}$ one wishes to solve the optimization problem

$$\inf\{\mathcal{F}(\Omega) : \Omega \in \mathcal{A}\}. \tag{3.1}$$

A set in $\mathcal{A}$ is called *extremal* or *optimal* for (3.1) if it realizes the infimum.

Problems of this form arise naturally in a large number of applications, particularly in engineering sciences but also elsewhere. Questions that the theory aims to address are for instance:

- What is the best shape for the wings of an aeroplane?
- How should one shape an iron beam to obtain maximal rigidity?
- What shape should an object moving through a fluid have to minimize the resistance?

Here we consider shape optimization problems where the cost functional $\mathcal{F}$ is given in terms of the eigenvalues of a differential operator. A problem of this type will here be referred to as a *spectral shape optimization problem.* During recent years this has been a very active field of research and it would be impossible for us to give a complete review of the existing results and literature. Our short discussion is limited to what is necessary to understand the main motivation and ideas underlying the results obtained as a part of this thesis. For the current state of art the reader is referred to [45].

Problems of this type have a long history. Already in 1877 Rayleigh [86] conjectured that the first eigenvalue of the Dirichlet Laplacian in a planar domain of unit area, was minimized by the disk. In the 1920's this conjecture was proved independently by Faber [27] in $\mathbb{R}^2$ and Krahn [53, 54] in all dimensions: among all domains of unit measure the first eigenvalue of the Dirichlet Laplacian is minimized by the ball. Krahn also proved that the second eigenvalue of the Dirichlet Laplacian is minimized by the disjoint union of two balls of equal measure [54].

Naturally, one may ask the corresponding question for the Neumann Laplacian. In this case the problem to consider is to maximize the eigenvalues. Since the first eigenvalue of the Neumann Laplacian is always zero, the first interesting problem is the maximization of the second eigenvalue. Here Szegő [96] and Weinberger [100] showed that the ball is again optimal. The problem of maximizing the second non-trivial Neumann eigenvalue was settled only recently by Bucur and Henrot [14]. As in the Dirichlet case the optimal domain is the disjoint union of two balls of equal measure.

For the problem of minimizing $\lambda_k(-\Delta_\Omega^\mathcal{D})$, with $k \geq 3$, among domains of fixed measure there was until recently very little known. Even the most basic questions such as existence of minimizers remained unanswered. For the relaxed problem of minimizing $\lambda_k(-\Delta_\Omega^\mathcal{D})$ within the larger class of quasi-open sets[(1)] the existence of minimizers was recently settled [12, 81]. However, if such a minimizer is in fact open remains an important unsolved problem. One might hope that the lack of theoretical results in this area is a result of missing some key idea or simply a lack of techniques. However, numerical computations suggest that the minimizers for larger k may look rather wild and need not have any natural symmetries (see e.g. [45]). Ultimately this makes a precise characterization unlikely to be possible.

To the author's knowledge the Neumann problem for higher eigenvalues remains completely open. A major difficulty in proving existence results corresponding to those in the Dirichlet case lies in that while the form domain of the Dirichlet Laplacian $H_0^1(\Omega)$ embeds into the larger function space $H^1(\mathbb{R}^d)$ (by extension by zero), this is not the case for the Neumann problem.

Shape optimization in the semiclassical limit

For our discussion it is convenient to reformulate the optimization of eigenvalues in terms of the counting function. For $\lambda > 0$ and a class of admissible domains $\mathcal{A}$ consider the shape optimization problems

$$\sup\{N(\lambda; -\Delta_\Omega^\mathcal{D}) : \Omega \in \mathcal{A},\ |\Omega| = 1\} \tag{3.2}$$

and

$$\inf\{N(\lambda; -\Delta_\Omega^\mathcal{N}) : \Omega \in \mathcal{A},\ |\Omega| = 1\}. \tag{3.3}$$

Clearly if one can solve one of these problems for all λ one can solve the corresponding problem formulated for the eigenvalues. In fact, it suffices to solve (3.2), or (3.3), for the discrete set of λ given by

$$\lambda_k^\mathcal{D}(\mathcal{A}) = \inf\{\lambda_k(-\Delta_\Omega^\mathcal{D}) : \Omega \in \mathcal{A},\ |\Omega| = 1\},$$

resp.

$$\lambda_k^\mathcal{N}(\mathcal{A}) = \sup\{\lambda_k(-\Delta_\Omega^\mathcal{N}) : \Omega \in \mathcal{A},\ |\Omega| = 1\}.$$

[(1)] A set $\Omega \subset \mathbb{R}^d$ is quasi-open if it is open up to a set of arbitrarily small capacity [16].

Assuming that at least one such domain exists, denote by $\Omega_\lambda^{\mathcal{D}}(\mathcal{A})$, resp. $\Omega_\lambda^{\mathcal{N}}(\mathcal{A})$, any solution of the shape optimization problem (3.2), resp. (3.3). That is, any domain which realizes the supremum or infimum, respectively. All statements made here concerning $\Omega_\lambda^{\mathcal{D}/\mathcal{N}}(\mathcal{A})$ are to be interpreted as valid independently of the choice of extremal domain. Depending on the class of admissible domains $\mathcal{A}$ the existence of such domains can pose a very difficult question.

In the results of this thesis our interest will not be towards answering questions regarding the existence of extremal domains of (3.2), (3.3), or related shape optimization problems. Rather our focus is directed towards the behaviour of the extremal domains in the semiclassical limit. Heuristically, the reason one might expect the optimizing domains to exhibit some structure in this limit is the refined Weyl law (2.6). While the leading term in the asymptotics of $N(\lambda; -\Delta_\Omega^{\mathcal{D}})$ as $\lambda \to \infty$ is fixed due to the constraint $|\Omega| = 1$, maximizing the second term leads to minimizing $\mathcal{H}^{d-1}(\partial\Omega)$ among $\Omega \in \mathcal{A}$ under the constraint $|\Omega| = 1$. If the ball of unit measure is in $\mathcal{A}$ then the isoperimetric inequality implies that it provides the unique solution to this problem. The analogous argument applies to the Neumann case by remembering that the sign of the second term in the asymptotics is reversed. A major difficulty in making this heuristic argument rigorous is that one requires the asymptotic expansion (2.6) not only for a fixed domain Ω, but for the family of domains $\Omega_\lambda^{\mathcal{D}/\mathcal{N}}(\mathcal{A})$ with a priori no information concerning their geometry.

The first result in this direction was obtained in 2013 by Antunes and Freitas [2]. Their result concerns the shape optimization problem in (3.2) with

$$\mathcal{A} = \big\{\Omega \subset \mathbb{R}^2 : \Omega = (0, a) \times (0, a^{-1}),\ a \geq 1\big\}, \tag{3.4}$$

i.e. the class of unit area rectangles. The main result of [2] is that in the semiclassical limit any optimal rectangle will converge to the unit square, in the sense that the corresponding side-lengths converge to 1. Note that the unit square is the unique solution to the isoperimetric problem in the class $\mathcal{A}$, and hence the result agrees with our heuristic argument.

Since the strategy developed by Antunes and Freitas is employed in Papers C, D, and F we recall it in some detail. The strategy is split into several steps. Although the strategy is applicable in a variety of settings, for simplicity, we formulate it here only for the problem considered in [2].

Within the class of admissible domains $\mathcal{A}$ in (3.4) it is not difficult to prove the existence of an optimal domain for the shape optimization problem in (3.2) for any fixed $\lambda \geq 0$. Consequently it makes sense to consider the asymptotic behaviour of optimal rectangles in the limit $\lambda \to \infty$.

Let $\lambda_k(a)$, for $a \geq 1$, denote the k-th eigenvalue of the Dirichlet Laplacian in the rectangle $R_a = (0, a) \times (0, a^{-1})$. Let $R(\lambda)$ denote any rectangle realizing the supremum in the shape optimization problem and let $a(\lambda)$ denote the corresponding side-length, so that $R(\lambda) = R_{a(\lambda)}$.

Step 1: (A priori bound for the rate of degeneracy) The goal of the first step is to prove that the geometry of extremal domains cannot degenerate arbitrarily fast.

Such non-degeneracy will be important in the next step of the proof but is often rather easily obtained. For the problem in [2] it is suffices to note that $\lambda_1(a) = \pi^2(a^2+a^{-2})$ is greater than λ if $a \geq \sqrt{\lambda}/\pi$. Consequently, the counting function $N(\lambda; \Delta^{\mathcal{D}}_{R_a})$ is zero if $a \geq \sqrt{\lambda}/\pi$. Noting that $N(\lambda; -\Delta^{\mathcal{D}}_{R_1}) \geq 1$ for $\lambda > \lambda_1(1) = 2\pi^2$ one concludes that $a(\lambda) < \sqrt{\lambda}/\pi$ for all $\lambda > 2\pi^2$. (This bound is also important when proving the existence of optimal rectangles for fixed $\lambda \geq 0$.)

Step 2: (Reducing the class of admissible domains) The goal of the second step is to prove that it suffices to consider the optimization in a compact subset of $\mathcal{A}$. In essence, the aim is to bootstrap the a priori bound from Step 1 to conclude that the geometry does not degenerate. This is achieved by proving spectral estimates reproducing correctly the first term of the Weyl asymptotics, and up to a multiplicative constant also the second term.

There exist positive constants c_1, c_2, b_0 such that for any unit area rectangle R_a, with $a \geq 1$, the inequality

$$N(\lambda; -\Delta^{\mathcal{D}}_{R_a}) \leq L^{\mathrm{cl}}_{0,2}\lambda - c_1 b\, \mathcal{H}^1(\partial R_a)\lambda^{1/2} + c_2 b^2 a^2, \tag{3.5}$$

holds for all $\lambda \geq 0$ and $b \in [0, b_0]$ (see [36, Lemma 2.1]). Note that the first term in (3.5) matches that in (2.6). Moreover, up to a multiplicative constant, the same holds for the second term which is crucial for the proof.

By optimality of $R(\lambda)$ and (3.5),

$$N(\lambda; -\Delta^{\mathcal{D}}_{R_1}) \leq N(\lambda; -\Delta^{\mathcal{D}}_{R(\lambda)}) \leq L^{\mathrm{cl}}_{0,2}\lambda - c_1 b\, \mathcal{H}^1(\partial R(\lambda))\lambda^{1/2} + c_2 b^2 a(\lambda)^2.$$

By (2.6) with $\Omega = R_1$, this inequality implies that

$$L^{\mathrm{cl}}_{0,2}\lambda - \frac{L^{\mathrm{cl}}_{0,1}}{4}\mathcal{H}^1(\partial R_1)\lambda^{1/2} + o(\lambda^{1/2}) \leq L^{\mathrm{cl}}_{0,2}\lambda - c_1 b\, \mathcal{H}^1(\partial R(\lambda))\lambda^{1/2} + c_2 b^2 a(\lambda)^2.$$

Rearranging, assuming that $\lambda > 2\pi^2$, using the bound from Step 1, and choosing b sufficiently small one concludes that

$$a(\lambda) \leq C + o(1), \quad \text{as } \lambda \to \infty.$$

As a result it suffices to consider the optimization in a bounded family of rectangles.

Step 3: (Uniform two-term spectral asymptotics) The third step is to prove that the two-term asymptotic expansion in (2.6) is valid with a *uniform* remainder term on any compact subset of $\mathcal{A}$. Specifically, our aim is to prove uniform two-term asymptotics on the compact subset found in Step 2.

For the class $\mathcal{A}$ in (3.4) this is not a very difficult step in the proof. Indeed, since the eigenvalues of the Dirichlet Laplacian on R_a are given by

$$\pi^2(j^2a^2 + k^2a^{-2}), \quad \text{for } j, k \in \mathbb{N},$$

the problem is equivalent to counting integer lattice points in a quarter ellipse. For this problem precise asymptotic formulae are well-known in the number theory literature (see e.g. [50]).

Step 4: (Identifying the limiting geometry) The final step is to combine what has been proved in Steps 2 and 3 to conclude that any optimal rectangle converges to the square. The idea is to repeat the argument of Step 2, but with the bound (3.5) replaced by the two-term expansion obtained in Step 3. By the maximality of $R(\lambda)$ it holds that

$$N(\lambda; -\Delta^{\mathcal{D}}_{R_1}) \leq N(\lambda; -\Delta^{\mathcal{D}}_{R(\lambda)}).$$

By Step 2 the optimal rectangle $R(\lambda)$ remains in a compact subset of $\mathcal{A}$. Consequently, Step 3 implies that, as $\lambda \to \infty$,

$$L^{\mathrm{cl}}_{0,2}\lambda - \frac{L^{\mathrm{cl}}_{0,1}}{4}\mathcal{H}^1(\partial R_1)\lambda^{1/2} + o(\lambda^{1/2}) \leq L^{\mathrm{cl}}_{0,2}\lambda - \frac{L^{\mathrm{cl}}_{0,1}}{4}\mathcal{H}^1(\partial R(\lambda))\lambda^{1/2} + o(\lambda^{1/2}).$$

After rearranging one finds that

$$\mathcal{H}^1(\partial R(\lambda)) \leq \mathcal{H}^1(\partial R_1) + o(1), \quad \text{as } \lambda \to \infty.$$

Since the square is the unique minimizer of the perimeter among domains in $\mathcal{A}$ the claimed convergence follows.

Although the outline given here is for the specific problem studied in [2] the strategy can be adapted to many similar problems. However, there are several parts of the proof which for more general problems present substantial challenges.

Firstly, the argument in Step 2 proving that optimal rectangles are uniformly bounded is highly non-trivial. Indeed, the two-term bound (3.5) is a stronger bound than that conjectured by Pólya mentioned in Section 2.3. As the Pólya conjecture remains a challenging open problem it is not reasonable to believe that a similar argument can be applied for general classes of admissible domains. Secondly, even if such a bound holds true the argument of Step 2 implies that the perimeter of the optimal domains remain uniformly bounded. Without additional constraints on $\mathcal{A}$ or stronger a priori information about optimizers this is not sufficient to imply the desired compactness. Furthermore, proving uniform two-term asymptotic expansions on compact subsets of more general classes $\mathcal{A}$ poses a considerable challenge.

A recurring idea in the results of the thesis will be that spectral shape optimization problems with cost functions depending nicely on several eigenvalues can be expected to be more well behaved than the shape optimization problem for the individual eigenvalues. For instance, one can replace the counting function by Riesz means (2.8), see Papers C and F. The main ingredients of the strategy outlined above remain the same: spectral bounds capturing the correct asymptotic behaviour and uniform two-term asymptotic expansions.

As outlined the strategy is adapted to asymptotic shape optimization problems for which the leading term in the asymptotic expansion of the cost function is fixed by a constraint. Here this role was played by the area (or volume) constraint $|\Omega| = 1$. The general strategy can without much trouble be adapted to problems of similar nature but where this constraint is replaced by different ones.

For problems where the first term is not fixed by the constraint a similar, yet somewhat simpler, strategy can be employed (see e.g. [13, 33]). Indeed, the important term in the asymptotics is now that of leading order which in general is easier to analyse. The goal of Step 2 is achieved by proving that, up to a multiplicative constant, the leading term in the asymptotic expansion provides a uniform inequality (cf. Theorems 2.11, 2.12 and 2.13). Similarly, Step 3 is reduced to proving leading order asymptotics with uniform remainder.

Relation to universal spectral inequalities

Naturally, there is a close relationship between spectral shape optimization and geometric spectral inequalities. For instance, the sharp constant in the inequality

$$N(\lambda; -\Delta_\Omega^{\mathcal{D}}) \leq L_d^{\mathcal{D}} |\Omega| \lambda^{d/2}$$

is by definition given by

$$L_d^{\mathcal{D}} = \sup\Big\{ \frac{N(\lambda; -\Delta_\Omega^{\mathcal{D}})}{|\Omega|\lambda^{d/2}} : \lambda \geq 0,\ \Omega \subset \mathbb{R}^d \text{ open} \Big\}.$$

By scaling of Laplacian eigenvalues this is for a fixed λ equivalent to (3.2) with $\mathcal{A}$ chosen as the class of open subsets of $\mathbb{R}^d$. By (2.5) and (2.14) it follows that $L_{0,d}^{\mathrm{cl}} \leq L_d^{\mathcal{D}} \leq (\frac{d+2}{d})^{d/2} L_{0,d}^{\mathrm{cl}}$. Analogously, the sharp constant in the inequality $N(\lambda; -\Delta_\Omega^{\mathcal{N}}) \geq L_d^{\mathcal{N}} |\Omega| \lambda^{d/2}$ can be written in terms of the shape optimization problem (3.3). As mentioned in Section 2.3 Pólya conjectured that $L_d^{\mathcal{D}} = L_d^{\mathcal{N}} = L_{0,d}^{\mathrm{cl}}$.

In a recent paper Colbois and El Soufi [18] showed that

$$\lim_{\lambda \to \infty} \frac{\sup\{N(\lambda; -\Delta_\Omega^{\mathcal{D}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = 1\}}{\lambda^{d/2}} = L_d^{\mathcal{D}}, \tag{3.6}$$

and

$$\lim_{\lambda \to \infty} \frac{\inf\{N(\lambda; -\Delta_\Omega^{\mathcal{N}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = 1\}}{\lambda^{d/2}} = L_d^{\mathcal{N}}. \tag{3.7}$$

Therefore, to prove (or disprove) Pólya's conjecture it suffices to compute the leading order behaviour of (3.2) and (3.3) as λ tends to infinity. In an upcoming paper Freitas, Lagacé, and Payette [34] revisit and refine this argument.

Colbois and El Soufi [18] formulate their results and proofs in terms of the eigenvalues, but they can equivalently be formulated in terms of the counting functions. The key insight in [18] is that the functions, defined for $\eta \geq 0$,

$$\begin{aligned} f_{\mathcal{D}}(\eta) &= \sup\{N(\eta^{2/d}; -\Delta_\Omega^{\mathcal{D}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = 1\}, \\ f_{\mathcal{N}}(\eta) &= \inf\{N(\eta^{2/d}; -\Delta_\Omega^{\mathcal{N}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = 1\} \end{aligned}$$

are superadditive and subadditive, respectively. Indeed, since the class of admissible domains is closed under taking disjoint unions

$$\begin{aligned} f_{\mathcal{D}}(\eta) \geq\ & \sup\{N(\eta^{2/d}; -\Delta_\Omega^{\mathcal{D}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = \mu\} \\ &+ \sup\{N(\eta^{2/d}; -\Delta_\Omega^{\mathcal{D}}) : \Omega \subset \mathbb{R}^d \text{ open},\ |\Omega| = 1 - \mu\}, \end{aligned} \tag{3.8}$$

for any $\mu \in [0,1]$. By scaling behaviour of Laplacian eigenvalues (3.8) implies that

$$f_{\mathcal{D}}(\eta) \geq f_{\mathcal{D}}(\mu\eta) + f_{\mathcal{D}}((1-\mu)\eta) \quad \text{for all } \mu \in [0,1],$$

which is the claimed superadditivity. The analogous argument proves the subadditivity of $f_{\mathcal{N}}$. The equalities in (3.6) and (3.7) thus follow from Fekete's lemma: if $f\colon \mathbb{R}_+ \to \mathbb{R}$ is measurable and subadditive (superadditive), then $\lim_{x\to\infty} f(x)/x$ exists and is equal to $\inf_{x>0} f(x)/x$ ($\sup_{x>0} f(x)/x$) [47].

There is a technical point that should be addressed. Namely, in the proof of sub- resp. superadditivity we used that the disjoint union of two optimal domains in the shape optimization problem with smaller measure could be used as a competitor in the original problem. A priori optimal domains need not exist and, moreover, if they do their disjoint union might not be well defined as a domain of $\mathbb{R}^d$. Indeed, it could be the case that one of the optimizers is dense in $\mathbb{R}^d$. For the Dirichlet case it follows from results in the existing literature [12, 15, 81] that these problems do not occur. For the Neumann problem the corresponding result is to the author's knowledge not known. However, the problem can be circumvented by considering almost minimizers for the shape optimization problem [18].

If one ignores issues concerning the existence of bounded extremal domains, the ideas of Colbois and El Soufi lift almost directly to shape optimization problems for Riesz means. Set

$$\begin{aligned}
f_{\mathcal{D},\gamma}(\eta) &= \eta^{-2\gamma/d} \sup\{\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \eta^{2/d})_-^\gamma : \Omega \subset \mathbb{R}^d \text{ open}, |\Omega| = 1\},\\
f_{\mathcal{N},\gamma}(\eta) &= \eta^{-2\gamma/d} \inf\{\mathrm{Tr}(-\Delta_\Omega^{\mathcal{N}} - \eta^{2/d})_-^\gamma : \Omega \subset \mathbb{R}^d \text{ open}, |\Omega| = 1\}.
\end{aligned}$$

Arguing as in (3.8) and using the behaviour of Laplacian eigenvalues under scaling one concludes that $f_{\mathcal{D},\gamma}$ is superadditive while $f_{\mathcal{N},\gamma}$ is subadditive. Applying Fekete's lemma yields that the limits

$$L_{d,\gamma}^{\mathcal{D}} = \lim_{\eta\to\infty} \frac{f_{\mathcal{D},\gamma}(\eta)}{\eta} \quad \text{and} \quad L_{d,\gamma}^{\mathcal{N}} = \lim_{\eta\to\infty} \frac{f_{\mathcal{N},\gamma}(\eta)}{\eta}$$

are well defined. Moreover, the limits are the sharp constants in the inequalities

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma \leq L_{d,\gamma}^{\mathcal{D}} |\Omega| \lambda^{\gamma+d/2} \quad \text{and} \quad \mathrm{Tr}(-\Delta_\Omega^{\mathcal{N}} - \lambda)_-^\gamma \geq L_{d,\gamma}^{\mathcal{N}} |\Omega| \lambda^{\gamma+d/2}.$$

Recall that for $\gamma \geq 1$ the sharp constant in both these inequalities is $L_{\gamma,d}^{\mathrm{cl}}$ (see Theorems 2.11 and 2.12). However, for $\gamma \in [0,1)$ the best constants are to the author's knowledge unknown.

Also for the inequalities of Theorem 2.13 one can formulate the corresponding result. For γ, d as in Theorem 2.13 and $V_1, V_2 \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\lim_{|x|\to\infty} \mathrm{Tr}(-\Delta_{\mathbb{R}^d} - V_1(\,\cdot\,+x) - V_2(\,\cdot\,-x))_-^\gamma = \mathrm{Tr}(-\Delta_{\mathbb{R}^d} - V_1)_-^\gamma + \mathrm{Tr}(-\Delta_{\mathbb{R}^d} - V_2)_-^\gamma,$$

see e.g. [23, 24, 39]. As a consequence one can prove the superadditivity of the function

$$g_\gamma(\eta) = \sup\bigl\{\mathrm{Tr}(-\Delta_{\mathbb{R}^d} - \eta^{1/(\gamma+d/2)} V)_-^\gamma : \|V\|_{L^{\gamma+d/2}(\mathbb{R}^d)}^{\gamma+d/2} = 1\bigr\}.$$

Therefore, Fekete's lemma implies that the limit

$$\lim_{\eta\to\infty}\frac{g_\gamma(\eta)}{\eta}=L_{\gamma,d}$$

exists and coincides with the sharp constant in the Lieb–Thirring inequality

$$\operatorname{Tr}(-\Delta+V)_-^\gamma\le L_{\gamma,d}\int_{\mathbb{R}^d}V(x)_-^{\gamma+d/2}\,dx.$$

It should however be noted that if the sharp constant in any of the inequalities discussed in this section is *not* the semiclassical one, then the solutions of the corresponding shape optimization problem must in a sense degenerate in the semiclassical limit. For definiteness we sketch the argument only for the case of the Dirichlet Laplacian and the shape optimization problem for the counting function (3.2).

Assume that the conjecture of Pólya is false, then there exists a λ_0 and a domain Ω_0, with $|\Omega_0|=1$, such that

$$N(\lambda_0;-\Delta^{\mathcal{D}}_{\Omega_0})\ge(L^{\mathrm{cl}}_{0,d}+\delta)\lambda_0^{d/2}$$

for some $\delta>0$. We claim that this contradicts the existence of a maximizing domain of the shape optimization problem with any component of measure much larger than $\lambda^{-d/2}$ for λ large enough. Note that a maximizing domain cannot have connected components of measure smaller than $\omega_d\lambda_1(B_1)^{d/2}\lambda^{-d/2}$. Indeed, by the Faber–Krahn inequality, $|\Omega|^{2/d}\lambda_1(-\Delta^{\mathcal{D}}_\Omega)\ge\omega_d^{d/2}\lambda_1(-\Delta^{\mathcal{D}}_{B_1})$, the first eigenvalue of a component of this size is greater than λ, and hence one can construct a better candidate by removing the component in question and rescaling the remaining domain.

Assume for contradiction that Ω^* is an isolated component of a maximizer for $\lambda\gg\lambda_0$. By (2.5) and the assumption $|\Omega^*|\gg\lambda^{-d/2}$,

$$N(\lambda;-\Delta^{\mathcal{D}}_{\Omega^*})=L^{\mathrm{cl}}_{0,d}|\Omega^*|\lambda^{d/2}+o(|\Omega^*|\lambda^{d/2}).$$

Let $\Omega(\lambda)$ be the disjoint union of $M=\lfloor|\Omega^*|(\lambda/\lambda_0)^{d/2}\rfloor$ copies of $(\lambda_0/\lambda)^{1/2}\Omega_0$ and a ball of radius r chosen so that $|\Omega(\lambda)|=|\Omega^*|$. Then

$$\begin{aligned}N(\lambda;-\Delta^{\mathcal{D}}_{\Omega(\lambda)})&=\sum_{j=1}^M N(\lambda_0;-\Delta^{\mathcal{D}}_{\Omega_0})+N(\lambda;-\Delta^{\mathcal{D}}_{B_r})\\&\ge(L^{\mathrm{cl}}_{0,d}+\delta)\lambda_0^{d/2}M=(L^{\mathrm{cl}}_{0,d}+\delta)|\Omega^*|\lambda^{d/2}+O(1).\end{aligned}$$

For large enough λ it would thus be better to replace Ω^* with a large number of small copies of Ω_0. This contradicts that Ω^* was a component of a maximizing domain.

4 Mathematical aspects of many-body quantum mechanics

In the final chapter of this introduction we recall some elements of the mathematical framework of quantum mechanics. Again the discussion is kept as brief as possible and focuses only on what is necessary to understand the results of the thesis and how it connects to what has been discussed above. For more extensive introductions to the mathematics of quantum mechanics the reader is referred to, for instance, [21, 66, 68, 97]. Since the results obtained do not concern quantum dynamics we keep our discussion to the stationary case, that is, we ignore all dependence on time. Our discussion is also restricted to the setting where the number of particles is given, but typically in the limit as this number tends to infinity.

Wave functions and particle statistics

In many-body quantum mechanics a state of N quantum particles in $\mathbb{R}^d$ is described by a normalized *wave function* $\Psi \in L^2(\mathbb{R}^{dN};\mathbb{C})$.(1) The modulus squared of the wave function $|\Psi(\mathrm{x})|^2$ is interpreted as the probability density of finding the particles at $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Here and in what follows boldface symbols denote individual particle positions $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^d$, while upright symbols denote the corresponding vector in the N-particle configuration space $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN}$. This notational convention is followed also in Papers G and H.

If the N particles are indistinguishable then a permutation of two particle positions should not affect the density. That is, for any indices $j \neq k$, the wave function Ψ should satisfy

$$|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N)|^2 = |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)|^2.$$

For Ψ this leaves the possibility of the exchange resulting in a phase change

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = e^{i\alpha\pi}\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N),$$

(1) Here we consider only $\mathbb{C}$-valued wave functions but in general one should consider functions with values in $\mathbb{C}^q$. For instance, this is the case if one takes particle spin into account.

where $\alpha \in [0, 2)$ is called the statistics parameter. It can be shown that the exchange phase is necessarily independent of which particle pair is exchanged.

For particles propagating in three or higher dimensions the only logically consistent choices of the statistics parameter are $\alpha = 0$ and $\alpha = 1$ corresponding to the subspace of symmetric and antisymmetric wave functions, respectively. In what follows these two subspaces will be denoted by $L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ and $L^2_{\mathrm{asym}}(\mathbb{R}^{dN})$. Particles whose wave functions are in $L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ are called *bosons* while those described by wave functions in $L^2_{\mathrm{asym}}(\mathbb{R}^{dN})$ are called *fermions*. The properties of these two species of quantum particles are fundamentally different and exhibit highly diverse behaviour. The symmetry of the bosonic wave functions leads to phenomena such as the coherent propagation of light and Bose–Einstein condensation. The antisymmetry of fermionic wave functions,

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = -\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N), \tag{4.1}$$

is the mathematical embodiment of the Pauli exclusion principle which is crucial for the structure of atoms, conduction bands, the Fermi sea, etc.

In two dimensions, however, there is the logical possibility of quantum particles (or quasiparticles) with any statistics parameter. Such particles are called *anyons* and are the topic of Paper G. For a thorough discussion of the mathematics behind the logical possibility for anyons in low dimensions we refer to the lecture notes by Lundholm [72] (see also [61, 84]).

The energy of a quantum state

In appropriately chosen units, the total energy of a quantum state Ψ is given by the quadratic form of a self-adjoint operator. For non-interacting particles the operator in question is the direct sum of N copies of the same operator acting in each particle separately. For non-relativistic particles subject to the external potential V, the non-interacting energy of a state Ψ is given by

$$\Big\langle \Psi, \sum_{j=1}^{N} (-\Delta_{\mathbf{x}_j} + V(\mathbf{x}_j))\Psi \Big\rangle. \tag{4.2}$$

Here $-\Delta_{\mathbf{x}_j}$ is the operator acting as the Laplacian in the j-th copy of $\mathbb{R}^d$ and as the identity in the rest. The two terms of the operator represent the kinetic and the potential part of the energy, respectively. If the particles are further subject to an external magnetic field the Laplacian term for each particle should be replaced by the corresponding magnetic operator.

For relativistic particles the sum of Laplacians in (4.2) should be replaced by $\sum_j (-\Delta_{\mathbf{x}_j})^{1/2}$. Although our main interest in this thesis concerns the non-relativistic case, the results of Paper H concern kinetic energy bounds in the many-body setting for the fractional Laplace operator.

For particles with pair-interaction the operator in (4.2) is replaced by

$$\sum_{j=1}^{N}(-\Delta_{\mathbf{x}_j} + V(\mathbf{x}_j)) + \sum_{1\leq j<k\leq N} W(\mathbf{x}_j, \mathbf{x}_k).$$

Here the extra term corresponds to the pairwise interactions of different particles via the scalar pair-potential W. Similarly, one can consider models with n-particle interactions by adding terms of the form $W(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$ and summing over all choices of n distinct particle indices.

It is also possible to define operators with *magnetic interactions* by replacing the kinetic energy by corresponding magnetic versions where the magnetic potential, and thus also the corresponding magnetic field, depends on the positions of the particles. A particular case of magnetically interacting particles arises naturally when considering anyons.

The magnetic gauge picture of anyons

For non-interacting bosonic or fermionic particles the energy can be completely understood in terms of the one-body operator. This is due to the fact that both the subspaces of symmetric, resp. antisymmetric, wave functions in $L^2(\mathbb{R}^{dN})$ are tensor products of the corresponding one-particle spaces. However, for anyons such a correspondence is not available due to the more complicated nature of the Hilbert space of wave functions satisfying the anyonic exchange relation. Technically, one should not consider functions but rather elements of the Hilbert space of square integrable sections of a certain complex line bundle, for details see [72].

However, through a singular gauge transformation it is possible to model anyons as magnetically interacting bosons [72, 77, 83]. The corresponding model for anyons is called the *magnetic gauge picture*. The model described by a many-body operator in the Hilbert space of sections of the appropriate line bundle is referred to as the *anyonic gauge picture*. More precisely, the non-relativistic kinetic energy of anyons with statistics parameter α is unitarily equivalent to the magnetically interacting kinetic energy

$$\sum_{j=1}^{N}(-i\nabla_{\mathbf{x}_j} + \alpha\mathbf{A}_j(\mathbf{x}_j))^2,$$

acting on bosonic wave functions. Here $\mathbf{A}_j\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the singular magnetic potential

$$\mathbf{A}_j(\mathbf{x}) = \sum_{k\neq j} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|^2},$$

where $\mathbf{x}^\perp$ denotes the vector $\mathbf{x}$ rotated by $\pi/2$ counterclockwise around the origin, that is $(x, y)^\perp = (-y, x)$. The corresponding magnetic field $\mathbf{B}_j = \operatorname{curl}\mathbf{A}_j$ is identically zero away from the $\mathbf{x}_k$, for $k \neq j$, while at each particle position $\mathbf{x}_k$ there is an Aharonov–Bohm type magnetic field (a non-trivial magnetic field concentrated at a single point).

It was proved by Lundholm and Solovej [78] that the natural form domain of this operator is obtained by taking the closure of corresponding quadratic form initially defined on $C_0^\infty(\mathbb{R}^{2N} \setminus \triangle) \cap L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ where

$$\triangle = \{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{2N} : \mathbf{x}_i = \mathbf{x}_j \text{ for some } j \neq k\}.$$

Many-body Lieb–Thirring inequalities

In this thesis our concern is mostly directed towards one of the most fundamental properties of a many-body quantum mechanical system, namely, its *ground-state energy* defined as the bottom of the spectrum of the corresponding many-body operator. In fact, our focus is lower bounds for this energy which display the correct asymptotic behaviour with respect to N in the limit as the particle number tends to infinity.

Since there are approximately $N \sim 10^{23}$ particles in a single gram of matter the many-body limit is highly relevant in applications. Moreover, as the many-body operators become increasingly difficult to analyse when the particle number grows large it is important to develop rigorous approximations which capture the relevant asymptotic behaviour.

Define the *one-body density* $\varrho_\Psi \colon \mathbb{R}^d \to [0, \infty)$ associated to a quantum state $\Psi \in L^2(\mathbb{R}^{dN})$, with $\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$, by

$$\varrho_\Psi(\mathbf{x}) = \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{k \neq j} d\mathbf{x}_k.$$

Note that $\varrho_\Psi \in L^1(\mathbb{R}^d)$ with normalization chosen so that $\|\varrho_\Psi\|_{L^1(\mathbb{R}^d)} = N$, and $\int_\Omega \varrho_\Psi(\mathbf{x})\, d\mathbf{x}$ is the expected number of particles in $\Omega \subset \mathbb{R}^d$.

Naturally, it is not possible to recover Ψ from its one-body density. However, the potential energy of a state Ψ can be computed knowing only ϱ_Ψ,

$$\Big\langle \Psi, \sum_{j=1}^N V(\mathbf{x}_j)\Psi \Big\rangle = \int_{\mathbb{R}^d} V(\mathbf{x})\varrho_\Psi(\mathbf{x})\, d\mathbf{x}.$$

To give an approximation of the kinetic energy of Ψ in terms of ϱ_Ψ is unfortunately far from simple. Similarly, it is difficult to precisely understand the contribution to the energy from particle interactions in terms of ϱ_Ψ.

The main topic of this subsection is a many-body version of the Lieb–Thirring inequality of Theorem 2.13 with $\gamma = 1$. The content of the inequality is a lower bound for the non-relativistic kinetic energy of a fermionic wave function Ψ in terms of its one-body density.

Theorem 4.1 (The many-body Lieb–Thirring inequality; see [69, 70]). *For all $N \geq 1$ and any $\Psi \in H^1(\mathbb{R}^{dN})$, with $\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$, satisfying* (4.1) *it holds that*

$$\left\langle \Psi, \sum_{j=1}^{N} (-\Delta_{\mathbf{x}_j})\Psi \right\rangle \geq C \int_{\mathbb{R}^d} \varrho_\Psi(\mathbf{x})^{1+2/d}\, d\mathbf{x}. \tag{4.3}$$

Here $C > 0$ is a universal constant independent of N and Ψ.

Theorem 4.1 was first obtained by Lieb and Thirring through Theorem 2.13 with $\gamma = 1$ and a duality argument. As mentioned in Section 2.3 the inequality (4.3) was a crucial ingredient in their proof of the stability of quantum mechanical matter [69, 70].

The integral of the one-body density on the right-hand side of (4.3) is, up to a multiplicative constant, the so-called Thomas–Fermi approximation of the kinetic energy. For background and details on Thomas–Fermi theory, see e.g. [64, 66] and references therein. However, it is in the context of this thesis worth noting that the core of the theory lies in a semiclassical approximation.

We emphasize that without the antisymmetry condition (4.1) an inequality of the form (4.3) *cannot* hold with a constant independent of N. In fact, the best one can obtain is

$$\left\langle \Psi, \sum_{j=1}^{N} (-\Delta_{\mathbf{x}_j})\Psi \right\rangle \geq \frac{C}{N^{2/d}} \int_{\mathbb{R}^d} \varrho_\Psi(\mathbf{x})^{1+2/d}\, d\mathbf{x}.$$

That this is the best possible behaviour can be seen by considering the bosonic wave function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{j=1}^{N} u(\mathbf{x}_j)$, for any $u \in H^1(\mathbb{R}^d)$ which is normalized in $L^2(\mathbb{R}^d)$ (see e.g. [67]).

It is an interesting question whether one can replace the antisymmetry condition in Theorem 4.1 by different assumptions. For instance, for a system of interacting bosons with a sufficiently strong repulsion between the particles one might expect similar inequalities to hold (see e.g. [74]).

In a similar direction Lundholm and Solovej [77] proved Lieb–Thirring inequalities for anyons by utilizing that the magnetic interaction is strong enough to imply exclusion principles effectively replacing that for fermions (see also [76]). The strategy developed by Lundholm and Solovej provides a rather simple yet powerful machinery for proving many-body kinetic energy or Lieb–Thirring inequalities by combining local uncertainty and exclusion principles with a clever geometric covering argument. This strategy will be the core of the analysis in Papers G and H. For a description of the main steps of the strategy we refer to Section 3 of Paper H.

5 Summary of results

With the exception of Paper A, the results of this thesis can be split into two separate categories:

I. Papers B, C, D, and E focus on the relation between the shape of a domain $\Omega \subset \mathbb{R}^d$ and the spectrum of the Dirichlet or Neumann Laplace operator defined in $L^2(\Omega)$. More precisely, the focus of the results revolves around sharp estimates and shape optimization in the semiclassical regime, i.e. in the limit of large eigenvalues. Paper F concerns a related problem but where the optimization over the shape of a domain is replaced by an optimization problem within a one-parameter family of Schrödinger operators.

II. Papers G and H concern problems linked more directly to many-body quantum mechanics. Specifically, the results concern Lieb–Thirring inequalities and related energy estimates for wave functions describing indistinguishable particles satisfying exclusion principles weaker than (4.1).

Summary of Paper A

A bound for the perimeter of inner parallel bodies
Journal of Functional Analysis (2016).

In Paper A we study the size of the perimeter of the inner parallel sets of a convex set $\Omega \subset \mathbb{R}^d$. That is, we are interested in the set of points in Ω which are a certain distance $t > 0$ from the boundary of Ω. Specifically, we are concerned with how the size of this set varies with t, and our goal is to find a sharp lower bound for its $(d-1)$-dimensional measure.

Although the topic of the paper lies outside the main theme of this thesis, our interest in the problem stems from applications within spectral theory. In particular, the results obtained are crucial ingredients in Papers B and C. However, the problem has connections and applications also in other parts of mathematics. For instance, it is closely connected to the Eikonal abrasion model [26].

Let Ω_t denote the inner parallel set of $\Omega \subset \mathbb{R}^d$ at distance $t \geq 0$, which is defined by

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq t\}.$$

If Ω is bounded then $\Omega_t = \emptyset$ for t large enough. Define $r_{in}(\Omega)$ the inradius of Ω by $r_{in}(\Omega) = \sup\{t > 0 : \Omega_t \neq \emptyset\}$. The inradius can equivalently be defined as the radius of the largest ball contained in Ω or the supremum of the distance from $x \in \Omega$ to $\partial\Omega$.

The main result of Paper A is the following theorem

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with inradius r. Then, for any inner parallel set Ω_t, $t \geq 0$, it holds that*

$$\mathcal{H}^{d-1}(\partial\Omega_t) \geq \left(1 - \frac{t}{r}\right)_+^{d-1} \mathcal{H}^{d-1}(\partial\Omega). \tag{5.1}$$

Moreover, this bound is proved to be sharp and exact conditions for equality are established.

The question of whether a bound similar to (5.1) was valid for arbitrary two-dimensional convex domains was posed by Geisinger, Laptev, and Weidl in [35] which motivated the analysis in Paper A. In fact, the results of Paper B are almost entirely based on combining Theorem 5.1 with the techniques and results of [35]. It is natural to ask if one can find corresponding lower bounds for the perimeter of inner parallel sets when Ω is not assumed to be convex. However, such bounds would necessarily need to involve further geometric quantities or additional assumptions on either Ω or the smallness of t.

When working with inner parallel sets one encounters two main difficulties. Firstly the regularity of $\partial\Omega_t$ can be, and most often is, worse than that of $\partial\Omega$. Secondly given a set Ω and a positive number t there is only rarely (even in the restricted convex case) a unique set $\widetilde{\Omega}$ such that $\widetilde{\Omega}_t = \Omega$, i.e. the inverse problem does not have a unique solution.

The key idea that goes into the proof of Theorem 5.1 is nonetheless to turn the problem around and look at an inverse problem. Namely, given a convex set Ω and a positive number t we construct a 'maximal' solution to the inverse problem. Specifically, we find a maximizing set of the optimization problem

$$\sup\bigl\{\mathcal{H}^{d-1}(\partial\widehat{\Omega}) : \widehat{\Omega} \text{ convex and } \widehat{\Omega}_t = \Omega\bigr\}. \tag{5.2}$$

That is, given Ω and $t \geq 0$ we construct a convex set $\widetilde{\Omega}$ such that $\widetilde{\Omega}_t = \Omega$ and for any other convex set satisfying $\widehat{\Omega}_t = \Omega$ it holds that $\mathcal{H}^{d-1}(\partial\widehat{\Omega}) \leq \mathcal{H}^{d-1}(\partial\widetilde{\Omega})$. By proving that the inequality (5.1) holds for the inner parallel body of $\widetilde{\Omega}$ at distance t the statement of Theorem 5.1 follows. The techniques used to solve the optimization problem (5.2) and the proof of the inequality for the extremal set $\widetilde{\Omega}$ are based on elementary convex geometry. In particular, the proofs make use of the theory of mixed volumes and properties of the support function of a convex body.

Summary of Paper B

On the remainder term of the Berezin inequality on a convex domain
Proceedings of the American Mathematical Society (2017).

Paper B uses the main result of Paper A to improve upon certain results obtained by Geisinger, Laptev, and Weidl in [35]. In that paper the authors derive improved versions of the Berezin–Li–Yau inequality,

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma \leq L_{d,\gamma}^{\mathrm{cl}} |\Omega| \lambda^{\gamma + d/2},$$

by subtracting positive terms of lower order in λ from the right-hand side of the inequality. Similar improvements of this inequality have been the topic of several recent papers. The main result of Paper B is also a result in this direction under an additional convexity assumption and for $\gamma \geq 3/2$.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded convex domain. For $\gamma \geq 3/2$ there exists a constant $c(\gamma, d) > 0$ such that*

(i) *if $\lambda \leq \frac{\pi^2}{4r_{in}(\Omega)^2}$, then*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma = 0;$$

(ii) *if $\lambda > \frac{\pi^2}{4r_{in}(\Omega)^2}$, then*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma \leq L_{\gamma,d}^{\mathrm{cl}} |\Omega| \lambda^{\gamma + d/2} - c(\gamma, d) L_{\gamma,d-1}^{\mathrm{cl}} \mathcal{H}^{d-1}(\partial\Omega) \lambda^{\gamma + (d-1)/2}.$$

For planar convex domains this result was proved under an additional geometric assumption in [35]. Namely, they assumed that

$$\mathcal{H}^1(\partial\Omega_t) \geq \Big(1 - \frac{3t}{w(\Omega)}\Big)_+ \mathcal{H}^1(\partial\Omega), \quad \text{for all } t \geq 0.$$

Here $w(\Omega)$ is the width of Ω defined as the smallest number $w \geq 0$ such that Ω is contained between two parallel hyperplanes separated by a distance w. Theorem 5.1 implies that this assumptions is always true. That the assumption was valid for all convex planar domains was conjectured in [35] which, as mentioned above, motivated the analysis in Paper A. Moreover, using Theorem 5.1 the methods of [35] used in the case of planar convex domains can be extended to convex domains of arbitrary dimension resulting in Theorem 5.2.

We remark that Theorem 5.2 remains valid also for $\gamma \in [1, 3/2)$. In fact, for all $\gamma \geq 1$ the corresponding result can be obtained as a consequence of an inequality proved in [40] combined with a result of Paper D (see Theorem 2.4 in Paper C).

Summary of Paper C

Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains

Journal of Spectral Theory, published online.

In Paper C we consider the shape optimization problem

$$\sup\{\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma : \Omega \subset \mathbb{R}^d \text{ convex open}, |\Omega| = 1\},$$

for $\lambda > 0$ and $\gamma \geq 1$. In particular, we are interested in what happens to optimal domains as λ tends to infinity.

Let $\mathcal{K}^d$ denote the set of non-empty, bounded convex domains in $\mathbb{R}^d$. This is a metric space when equipped with the Hausdorff metric. The main result in the paper is the following

Theorem 5.3. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^d$. Fix $\gamma \geq 1$ and let $\Omega_{\lambda,\gamma}(\mathcal{A})$ denote any extremal domain for the shape optimization problem*

$$\sup\{\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Then the following statements hold:

(i) *For any sequence $\{\lambda_j\}_{j\geq 1}\uparrow\infty$ the corresponding sequence $\{\Omega_{\lambda_j,\gamma}(\mathcal{A})\}_{j\geq 1}$ has a subsequence which, up to translation, converges in $\mathcal{A}$. Moreover, the limit Ω_∞ of such a subsequence has unit measure.*

(ii) *Under the additional assumption that*

$$\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma = L_{\gamma,d}^{\mathrm{cl}}|\Omega|\lambda^{\gamma+d/2} - \frac{L_{\gamma,d-1}^{\mathrm{cl}}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{\gamma+(d-1)/2} + o(\lambda^{\gamma+(d-1)/2}),$$

as $\lambda \to \infty$, uniformly for Ω in compact subsets of $\mathcal{A}$, then the limit Ω_∞ also minimizes the perimeter in $\mathcal{A}$:

$$\mathcal{H}^{d-1}(\partial\Omega_\infty) = \inf\{\mathcal{H}^{d-1}(\partial\Omega) : \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

For two natural classes of convex domains it is also proved that the assumption in (ii) is valid. Namely, if $\mathcal{A}$ is the set of convex polytopes of no more than m faces, or if the elements of $\mathcal{A}$ are assumed to have boundaries which are uniformly C^1-regular. However, by the results obtained in Paper E the assumption in (ii) is valid for *any* choice of $\mathcal{A} \subseteq \mathcal{K}^d$. In particular, we can take $\mathcal{A} = \mathcal{K}^d$ and obtain that in the limit $\lambda \to \infty$ any optimizer will, up to translation, converge to a ball of unit measure (see Corollary 5.9 below).

The main tools in the proof is an extension of Theorem 5.2 to all $\gamma \geq 1$, the Blaschke selection principle, and two-term asymptotic expansions of $\operatorname{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma$ as $\lambda \to \infty$ with uniform control of the error term. Combining these tools with the strategy described in Chapter 3 one obtains the main result.

It is also shown that the ideas can be extended to understand the asymptotic behaviour of the shape optimization problem if one also allows for disjoint unions of convex domains.

Corollary 5.4. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^d$ which is invariant under dilations and satisfies the assumption in* (ii) *of Theorem 5.3. Fix $\gamma \geq 1$ and let $\Omega_{\lambda,\gamma}(\mathcal{A}^\infty)$ denote any extremal domain of the shape optimization problem*

$$\sup\bigl\{\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma : |\Omega| = 1,\ \Omega = \textstyle\bigcup_{k\geq 1}\Omega_k,\ \Omega_k \in \mathcal{A},\ \Omega_k \cap \Omega_{k'} = \emptyset \textit{ if } k \neq k'\bigr\}.$$

Let also Ω_λ^1 denote the largest of the components of $\Omega_{\lambda,\gamma}(\mathcal{A}^\infty)$.

For any sequence $\{\lambda_j\}_{j\geq 1} \uparrow \infty$ the corresponding sequence $\{\Omega_{\lambda_j}^1\}_{j\geq 1}$ has a subsequence which, up to rigid transformations, converges in $\mathcal{A}$. Moreover, the limit Ω_∞ of such a subsequence has unit measure and minimizes the perimeter in $\mathcal{A}$:

$$\mathcal{H}^{d-1}(\partial\Omega_\infty) = \inf\{\mathcal{H}^{d-1}(\partial\Omega) : \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Note that Corollary 5.4 can be interesting even for extremely simple choices of $\mathcal{A}$. For instance, it implies that among unions of disjoint balls the maximizers of the Riesz means will in the semiclassical limit converge to a *single* ball of unit measure.

Summary of Paper D

Asymptotic behaviour of cuboids optimising Laplacian eigenvalues
(joint with K. Gittins)
Integral Equations and Operator Theory (2017).

In the same spirit as Paper C, the topic of Paper D is an asymptotic problem in spectral shape optimization. In contrast to Paper C the class of admissible domains is much more restrictive, but the main focus is towards optimizing individual eigenvalues and not the more regular Riesz means studied in Paper C. Before stating the main results we require some additional notation.

For a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary denote by $\{\lambda_k(\Omega)\}_{k\geq 1}$ and $\{\mu_k(\Omega)\}_{k\geq 0}$ the increasingly ordered eigenvalues counted with multiplicity of $-\Delta_\Omega^{\mathcal{D}}$ and $-\Delta_\Omega^{\mathcal{N}}$, respectively.

For $\bar{a} = (a_1, \ldots, a_d)$, with $0 < a_1 \leq a_2 \leq \ldots \leq a_d$, define the cuboid $R_{\bar{a}} = \prod_{j=1}^d (0, a_j) \subset \mathbb{R}^d$ and let Q denote the unit cube $(0,1)^d$.

Finally, for $k \geq 1$ define

$$\lambda_k^* = \inf\{\lambda_k(R_{\bar{a}}) : |R_{\bar{a}}| = 1\}, \quad \text{and} \quad \mu_k^* = \sup\{\mu_k(R_{\bar{a}}) : |R_{\bar{a}}| = 1\}.$$

The aim of Paper D is to analyse the behaviour of cuboids realizing the infimum and supremum in the limit as k tends to infinity.

The main result of Paper D is

Theorem 5.5. *Let $d \geq 2$. For $k \in \mathbb{N}$ let $R_k^{\mathcal{D}}$ and $R_k^{\mathcal{N}}$ denote two d-dimensional unit measure cuboids such that*

$$\lambda_k(R_k^{\mathcal{D}}) = \lambda_k^*, \quad \text{and} \quad \mu_k(R_k^{\mathcal{N}}) = \mu_k^*.$$

Then, for some $\theta_d < d-1$ and as $k \to \infty$, it holds that

$$\begin{aligned} a_{1,k}^{\mathcal{D}} &= 1 + O(k^{(\theta_d-(d-1))/(2d)}) \\ a_{1,k}^{\mathcal{N}} &= 1 + O(k^{(\theta_d-(d-1))/(2d)}), \end{aligned}$$

where $a_{1,k}^{\mathcal{D}/\mathcal{N}}$ is the shortest side-length of the cuboid $R_k^{\mathcal{D}/\mathcal{N}}$.

That is, in the limit $k \to \infty$ the sequences $\{R_k^{\mathcal{D}}\}_{k\geq 1}$ and $\{R_k^{\mathcal{N}}\}_{k\geq 1}$ converge to the unit cube Q.

For $d = 2$ the result of Theorem 5.5 was first obtained by Antunes and Freitas for the Dirchlet problem, and van den Berg, Bucur, and Gittins for the Neumann problem [2, 5]. Later van den Berg and Gittins [6] also proved the result for the Dirichlet problem when $d = 3$.

The quantity θ_d in Theorem 5.5 is given through a closely related lattice point counting problem. For $d \geq 2$, θ_d can be taken as any number such that for all $a_1, \ldots, a_d > 0$,

$$\#\{z \in \mathbb{Z}^d : a_1^{-2} z_1^2 + \ldots + a_d^{-2} z_d^2 \leq r^2\} - \omega_d r^d \prod_{j=1}^d a_j = O(r^{\theta_d}), \tag{5.3}$$

as $r \to \infty$, uniformly for a_j on compact subsets of $(0, \infty)$.

Finding the sharp remainder in (5.3) when $d = 2$ and $a_1 = a_2 = 1$ is the well-known, and still open, Gauss circle problem. If the dimension is greater than 5, then (5.3) is valid with $\theta_d = d - 2$ which is known to be sharp [38]. For $d = 3, 4$ Herz [46] proved (5.3) with $\theta_d = \frac{d(d-1)}{d+1}$. For $d = 2$ the expansion (5.3) is valid for $\theta_2 = \frac{46}{73} + \varepsilon$, for any $\varepsilon > 0$ [50]. In particular, for any $d \geq 2$ (5.3) is valid for some $\theta_d < d - 1$.

In addition to the geometric convergence of the extremal domains, results concerning the distance between λ_k^* and μ_k^* and the eigenvalues of the cube are obtained.

Theorem 5.6. *Let $d \geq 2$. Then, as $k \to \infty$,*

$$\begin{aligned} |\lambda_k(Q) - \lambda_k^*| &= O(k^{(\theta_d-(d-2))/(2d)}) \\ |\mu_k(Q) - \mu_k^*| &= O(k^{(\theta_d-(d-2))/(2d)}). \end{aligned}$$

For $d \geq 5$ Theorem 5.6 implies that the difference between the extremal eigenvalues and those of the unit cube are bounded uniformly with respect to k.

The strategy of the proofs of Theorems 5.5 and 5.6 relies on the fact that the eigenvalues of both the Dirichlet and Neumann Laplacians on a cuboid are explicitly known. Namely for the cuboid $R_{\bar{a}}$ they are given by

$$\frac{\pi^2 j_1^2}{a_1^2} + \ldots + \frac{\pi^2 j_d^2}{a_d^2},$$

where $j_1, \ldots, j_d$ are positive integers for the Dirichlet eigenvalues and non-negative integers for the eigenvalues of the Neumann Laplacian. Through this observation one can reformulate the problem in terms of finding ellipsoids containing the largest, or smallest, number of positive, respectively non-negative, integer lattice points.

With this in hand one can follow the strategy outlined in Chapter 3 to reduce the proof to that of minimizing the perimeter among cuboids of unit measure. In particular, the proof of uniform asymptotic expansions is reduced to well-studied problems in number theory and the proof of universal spectral inequalities can be reduced to proving a one-dimensional inequality by applying the Aizenman–Lieb identity (2.10).

Summary of Paper E

Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain
(joint with R. L. Frank)
Preprint 2019.

Paper E deals with semiclassical asymptotics for the Dirichlet Laplacian in domains with rough boundary. Specifically we prove a two-term asymptotic expansion of $\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-$ in the limit $\lambda \to \infty$ under a Lipschitz assumption on $\partial\Omega$. As discussed in Section 2.2 this corresponds to an averaged version of Weyl's conjecture.

The main result of the paper is

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with Lipschitz boundary. Then, as $\lambda \to \infty$,*

$$\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_- = L_{1,d}^{\mathrm{cl}} |\Omega| \lambda^{1+d/2} - \frac{L_{1,d-1}^{\mathrm{cl}}}{4} \mathcal{H}^{d-1}(\partial\Omega) \lambda^{1+(d-1)/2} + o(\lambda^{1+(d-1)/2}).$$

The method employed to prove the result is based on combining techniques of local trace asymptotics developed by Solovej–Spitzer [94] and Frank–Geisinger [30, 31], with ideas from geometric measure theory developed by Brown [11] in order to prove (2.7) for Lipschitz domains.

The main result of Frank and Geisinger [30] is a version of Theorem 5.7 valid for $\Omega \subset \mathbb{R}^d$ with $C^{1,\alpha}$-regular boundary. In the subsequent paper [31], where the same authors consider the Laplace operator with Robin boundary conditions, the regularity assumption is lowered to C^1.

However, the step from C^1-regular boundary to Lipschitz requires fundamentally different tools. The treatment of the boundary in [30, 31] relies on locally changing coordinates in such a way that the boundary of the domain is mapped to a hyperplane while maintaining control of how the Laplace operator is changed by this mapping. If the boundary is merely Lipschitz, such an approach is not possible. Indeed, straightening a Lipschitz boundary requires a Lipschitz change of coordinates which can lead to arbitrarily large perturbations of the Laplacian.

The approach employed to overcome this issue is to utilize a geometric construction of Brown [11]. The main ingredient is a quantitative way of saying that the boundary of a Lipschitz domain is around most points well approximated by a hyperplane, at least at a sufficiently small scale.

The techniques developed in the proof of Theorem 5.7 are not only applicable to prove asymptotic estimates but can also be used to prove universal spectral inequalities. For instance, if Ω is assumed to be convex we prove

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a convex bounded open set. Then, for all $\lambda > 0$,*

$$\Big|\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_- - L_d|\Omega|\lambda^{1+d/2} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2}\Big| \leq C\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2}\big(r_{in}(\Omega)\sqrt{\lambda}\big)^{-1/11},$$

where the constant C depends only on the dimension.

As a corollary of Theorem 5.8 the a priori regularity assumptions in the main theorem of Paper C can be removed.

Corollary 5.9. *Let $\gamma \geq 1$. For $\lambda > 0$ let $\Omega_{\lambda,\gamma}$ denote any extremal domain of the shape optimization problem*

$$\sup\{\mathrm{Tr}(-\Delta_\Omega^{\mathcal{D}} - \lambda)_-^\gamma : \Omega \subset \mathbb{R}^d \text{ convex open}, \ |\Omega| = 1\}.$$

Then, up to translation, $\Omega_{\lambda,\gamma}$ converges in the Hausdorff metric to a ball of unit measure as $\lambda \to \infty$.

Summary of Paper F

Maximizing Riesz means of anisotropic harmonic oscillators
Arkiv för Matematik, to appear.

Paper F concerns an asymptotic optimization problem for anisotropic harmonic oscillators. The problem arose in connection to spectral shape optimization and provides an interesting toy model for which the behaviour differs from that observed in, for instance, [2, 5, 6] and Papers C and D. In the same manner as the results of Paper D can be formulated in terms of finding ellipsoids containing the most points of $\mathbb{N}^d$ the problem at hand has a similar interpretation, but where the ellipsoids are replaced by triangles.

For $\beta > 0$, let L_β denote the self-adjoint operator in $L^2(\mathbb{R}^2)$ acting as

$$L_\beta = -\Delta + \beta x^2 + \beta^{-1} y^2.$$

We refer to this operator as the *anisotropic harmonic oscillator*. The spectrum of L_β is discrete and consists of an infinite number of positive eigenvalues which we denote by, repeating each eigenvalue according to its multiplicity,

$$0 < \lambda_1(\beta) \leq \lambda_2(\beta) \leq \ldots \leq \lambda_k(\beta) \leq \ldots.$$

The eigenvalues of L_β are in one-to-one correspondence with $\mathbb{N}^2$, explicitly the correspondence is given by the map

$$\mathbb{N}^2 \ni (k_1, k_2) \mapsto 2(k_1 - 1/2)\sqrt{\beta} + 2(k_2 - 1/2)/\sqrt{\beta}.$$

In Paper F we are interested in the behaviour of β_k realizing the infimum

$$\inf\{\lambda_k(\beta) : \beta > 0\}$$

as k tends to infinity. Although we are not able to completely understand the behaviour of such minimizing β we prove a number of results concerning a regularized version of the problem. Namely, the corresponding problem for Riesz means.

From the viewpoint of counting lattice points a natural family of generalizations of this problem is the following: for $\sigma, \tau > -1$ what $\beta = \beta(\lambda)$ maximizes the quantity

$$\#\bigl\{(k_1, k_2) \in \mathbb{N}^2 : (k_1 + \sigma)\sqrt{\beta} + (k_2 + \tau)/\sqrt{\beta} \leq \lambda\bigr\}? \tag{5.4}$$

For $\sigma = \tau = -1/2$ this is exactly the problem of minimizing eigenvalues of anisotropic harmonic oscillators but formulated in terms of the eigenvalue counting function. These problems were introduced by Laugesen and Liu in [59, 60]. In the first of these articles it was conjectured that when $\sigma = \tau = 0$ the set of maximizing values of (5.4) converges to a highly non-trivial set as $\lambda \to \infty$. Specifically they conjectured that the limiting set consists of an infinite set of rational numbers. A partial positive answer to this conjecture was subsequently given by Marshall and Steinerberger [79]. Their main result states that the limiting set contains an infinite subset of $\mathbb{Q}$.

The main goal of Paper F is to study the following version of the problem in a setting corresponding to Riesz means of harmonic oscillators: for $\gamma > 0$ and $\sigma, \tau > -1$ find β realizing the supremum

$$\sup\bigl\{R^\gamma_{\sigma,\tau}(\beta, \lambda) : \beta > 0\bigr\}, \tag{5.5}$$

where

$$R^\gamma_{\sigma,\tau}(\beta, \lambda) = \sum_{(k_1,k_2)\in\mathbb{N}^2} \bigl(\lambda - (k_1 + \sigma)\sqrt{\beta} + (k_2 + \tau)/\sqrt{\beta}\bigr)^\gamma_+.$$

Putting $\gamma = 0$ and interpreting the sum appropriately this is precisely the problem of maximizing (5.4).

Our main results state that for a certain range of the parameters σ, τ, and γ the maximizing β has a well-defined limit as $\lambda \to \infty$.

Theorem 5.10. *For $\lambda > 0$ let $\beta^{\gamma}_{\sigma,\tau}(\lambda)$ denote any β realizing the supremum in* (5.5)*. Then, for all $\gamma > 0$ and $\sigma, \tau > -1/2$ it holds that*

$$\lim_{\lambda\to\infty} \beta^{\gamma}_{\sigma,\tau}(\lambda) = \frac{1+2\tau}{1+2\sigma}.$$

Moreover, for all $\gamma > 1$ it holds that

$$\lim_{\lambda\to\infty} \beta^{\gamma}_{-1/2,-1/2}(\lambda) = 1.$$

In the case $(\sigma, \tau) \in (-1, \infty)^2 \setminus ((-1/2, \infty)^2 \cup \{(-1/2, -1/2)\})$ any sequence of maximizing β must degenerate in the semiclassical limit. In the sense that for any fixed compact set $I \subset \mathbb{R}_+$ all maximizers $\beta(\lambda)$ are in I^c if λ is large enough.

For $(\sigma, \tau) \in (-1/2, \infty)^2 \cup \{(-1/2, -1/2)\}$ and $\gamma \geq 0$ not covered by Theorem 5.10 it is conjectured that the corresponding result is false and that the problem exhibits behaviour resembling that studied in [79], i.e. the case $\sigma = \tau = \gamma = 0$.

The main strategy is that discussed in Chapter 3 and applied in Papers C and D. The main tools in following this strategy are combinations of precise estimates for what corresponds to a one-dimensional versions of the considered Riesz means, the Aizenman–Lieb identity (2.10), and precise asymptotic expansions of the sum in (5.5) as $\lambda \to \infty$.

The asymptotic expansion is derived by generalizing a calculation for $\beta = 1$ and $\sigma = \tau = -1/2$ made in [41, 42, 43] which is based on the Laplace transform and precise application of the residue theorem. In order to state the result precisely let $\zeta\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ denote the Hurwitz ζ-function [25, Chapter 25]. Let also $\{x\}$ denote the fractional part of $x \in \mathbb{R}$, i.e. $\{x\} = x - \lfloor x \rfloor$.

Theorem 5.11. *For any $\gamma > 0$, $M \in \mathbb{N}$, $\delta > 0$, $\beta \in \mathbb{R}_+$ and $\sigma, \tau > -1$, there are constants $\alpha_k = \alpha_k(\beta, \sigma, \tau, \gamma)$ such that*

$$R^{\gamma}_{\sigma,\tau}(\beta, \lambda) = \sum_{k=0}^{M+1} \alpha_k \lambda^{2-k+\gamma} + \operatorname{osc}(\beta, \lambda) + o(\lambda^{-M+\gamma+\delta}), \qquad \text{as } \lambda \to \infty.$$

The coefficients α_k are continuous in β and $|\operatorname{osc}(\beta, \lambda)| \leq C_\beta(\lambda + 1)$. Moreover, C_β and the implicit constant of the remainder term are uniformly bounded for β in compact subsets of $\mathbb{R}_+$.

Furthermore,

(*i*) *if $\beta = \frac{\mu}{\nu} \in \mathbb{Q}_+$, $\gcd(\mu, \nu) = 1$, then, with $x = \sqrt{\mu\nu}\lambda - \mu\sigma - \nu\tau$,*

$$\operatorname{osc}(\beta, \lambda) = \frac{\zeta(-\gamma, \{x\})}{(\mu\nu)^{\frac{1+\gamma}{2}}}\lambda + O(1), \qquad \text{as } \lambda \to \infty;$$

(ii) if $\beta \in \mathbb{R}_+ \setminus \mathbb{Q}$, it holds that

$$\begin{aligned}
\operatorname{osc}(\beta, \lambda) &= \frac{\beta^{-\gamma/2}\Gamma(1+\gamma)}{(2\pi)^{1+\gamma}} \sum_{k=1}^{\Lambda(\lambda)/\sqrt{\beta}} \frac{\sin(\pi k(2\lambda\sqrt{\beta} - (1+2\sigma)\beta - 2\tau) - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\beta)} \\
&\quad + \frac{\beta^{\gamma/2}\Gamma(1+\gamma)}{(2\pi)^{1+\gamma}} \sum_{k=1}^{\Lambda(\lambda)\sqrt{\beta}} \frac{\sin(\pi k(2\lambda/\sqrt{\beta} - 2\sigma - (1+2\tau)/\beta) - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k/\beta)} \\
&\quad + o(\lambda^{-M+\gamma+\delta}),
\end{aligned}$$

as $\lambda \to \infty$ and where $\Lambda(\lambda) = O(\lambda^{\frac{M+2-\gamma}{\gamma}})$.

For an explicit formula for the coefficients α_k, see Paper F. For the maximization problems that we consider it is only the first few coefficients that are important

$$\begin{aligned}
\alpha_0 &= \frac{1}{(1+\gamma)(2+\gamma)}, \qquad \alpha_1 = -\frac{(1+2\sigma)\sqrt{\beta} + (1+2\tau)/\sqrt{\beta}}{2(1+\gamma)}, \\
\alpha_2 &= \frac{(1+2\sigma)(1+2\tau)}{4} + \frac{(1+6\sigma(1+\sigma))\beta + (1+6\tau(1+\tau))/\beta}{12}.
\end{aligned}$$

The behavioural transition observed in the optimization problem at $\sigma = -1/2$ and $\tau = -1/2$ is a consequence of the function $\beta \mapsto \alpha_1$ changing its character. If $\sigma, \tau > -1/2$, then the function achieves its unique maximum at $\beta = \frac{1+2\tau}{1+2\sigma}$. If instead $\min\{\sigma, \tau\} \leq -1/2$, then the function is maximized in one, or both, of the limits $\beta \to 0$ or $\beta \to \infty$. Heuristically, $\beta^{\gamma}_{\sigma,\tau}(\lambda)$ should approach the maximum of α_1 as $\lambda \to \infty$ which leads to the degenerate behaviour discussed above. The criticality of the harmonic oscillators $\sigma = \tau = -1/2$ is a result of $\alpha_1(\beta, -1/2, -1/2, \gamma) \equiv 0$, and hence the optimization takes place at the level of the lower order term $\alpha_2\lambda^{\gamma}$. However, for this term to dictate the asymptotic behaviour it needs to be asymptotically much larger than $\operatorname{osc}(\beta, \lambda)$. In other words we need $\gamma > 1$. Otherwise, the oscillating behaviour will dominate and one expects to find a more complicated limiting set of maximizing β.

Summary of Paper G

Exclusion bounds for extended anyons (joint with D. Lundholm)
Archive for Rational Mechanics and Analysis (2018).

As discussed in Chapter 4 there is the mathematical possibility of two-dimensional quantum particles called *anyons* which escape the classical boson/fermion dichotomy. Through the magnetic gauge picture such particles can be modelled as bosons interacting through a self-generated magnetic field. It has been argued that this is actually one of the more probable manners in which one might expect anyons to arise in practice; that is, as bosons to which a quantity of magnetic flux effectively has been attached (see [75] and references therein).

However, in such a model the magnetic flux attached has some positive extent. In contrast, in the magnetic gauge picture which we described in Chapter 4 the magnetic field is completely concentrated at a point. Therefore, it is interesting to try to understand the properties of a two-dimensional gas of bosons interacting via a magnetic field of positive extent attached to each particle. This is precisely the topic of Paper G.

Technically the model considered is that described in Chapter 4 but the singular magnetic potential

$$\mathbf{A}_j(\mathbf{x}) = \sum_{k\neq j} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|^2}$$

is replaced by

$$\mathbf{A}_j^R(\mathbf{x}) = \mathbf{A}_j * \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}) = \sum_{k\neq j} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|_R^2},$$

where $R > 0$ describes the size of the attached flux and $|\mathbf{x}|_R = \max\{|\mathbf{x}|, R\}$. Note that while $\mathbf{A}_j$ corresponds to a magnetic field with point fluxes at each $\mathbf{x}_k$, $k \neq j$, the vector potential $\mathbf{A}_j^R$ corresponds to the magnetic field

$$\mathbf{B}_j^R(\mathbf{x}) = \operatorname{curl} \mathbf{A}_j^R(\mathbf{x}) = 2\pi \sum_{k\neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}(\mathbf{x})}{\pi R^2}$$

(a disk with radius R of constant magnetic field around each $\mathbf{x}_k$). Note that the magnetic flux attached to each $\mathbf{x}_k$ (the integral of the magnetic field) is independent of R.

For $R > 0, \alpha \in \mathbb{R}$ we are interested in understanding the magnetic kinetic energy operator

$$T_\alpha^R = \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j} + \alpha \mathbf{A}_j^R(\mathbf{x}_j))^2$$

in the bosonic Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$. Let also T_α^0 denote the corresponding operator with $\mathbf{A}_j^R$ replaced by $\mathbf{A}_j$, i.e. the non-extended case.

Paper G focuses on one of the most fundamental properties of the extended anyon gas, namely, its ground-state energy. In particular, we consider the energy per particle of the homogeneous anyon gas confined to a square Q in the limit as the particle number N and the size of the square $|Q|$ tend to infinity, while keeping the average particle density $\bar{\varrho} = N/|Q|$ fixed.

In the case of ideal anyons, i.e. when $R = 0$, this problem was studied by Lundholm and Solovej [77]. In their results the dependence on the statistics parameter α enters through the rather peculiar quantity

$$\alpha_* = \begin{cases} \frac{1}{\nu} & \text{if } \alpha = \frac{\mu}{\nu} \in \mathbb{Q} \text{ with } \gcd(\mu,\nu) = 1 \text{ and } \mu \text{ odd}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

In particular, their results imply a positive energy per particle in the thermodynamic limit for α such that $\alpha_* \neq 0$. The quantity (5.6) appears also in the results of Paper G. In particular, it determines the behaviour in the limit as R tends to zero. For $R = 0$ our results improve upon the results of Lundholm and Solovej. However, again the method is in this case able to yield positive energy per particle in the thermodynamic limit only if $\alpha_* \neq 0$. The dependence on α_* might appear rather unnatural, and indeed Lundholm and Seiringer [76] recently proved that the thermodynamic energy per particle is zero only for $\alpha \in 2\mathbb{Z}$, that is, only if the particles are bosons. However, it should be noted that there are arguments for why number-theoretic properties of the statistics parameter α might be relevant for the precise thermodynamic energy [71].

The main result of Paper G is

Theorem 5.12. *Let $e(\alpha, \bar{\gamma})$, where $\bar{\gamma} = R\bar{\varrho}^{1/2}$, denote the ground-state energy per particle and unit density of the extended anyon gas in the thermodynamic limit at fixed $\alpha \in \mathbb{R}$, $R \geq 0$ and density $\bar{\varrho} > 0$ where Dirichlet boundary conditions have been imposed, that is*

$$e(\alpha, \bar{\gamma}) = \liminf_{\substack{N, |Q| \to \infty \\ N/|Q| = \bar{\varrho}}} \left(\frac{1}{\bar{\varrho} N} \inf_{\substack{\Psi \in D(T_\alpha^R) \cap C_0^\infty(Q^N) \\ \|\Psi\|_2 = 1}} \langle \Psi, T_\alpha^R \Psi \rangle \right).$$

Then

$$e(\alpha, \bar{\gamma}) \geq C \biggl(2\pi \frac{|\alpha| \min\{2(1 - \bar{\gamma}^2/4)^{-1}, K_\alpha\}}{K_\alpha + 2|\alpha| \ln(2/\bar{\gamma})} \mathbb{1}_{\bar{\gamma} < 2} + 2\pi|\alpha| \mathbb{1}_{\bar{\gamma} \geq 2} + \pi g(c\alpha_*, 12\bar{\gamma}/\sqrt{2})^2 (1 - 12\bar{\gamma}/\sqrt{2})_+^3 \biggr),$$

for some universal constants $C, c > 0$. Here K_α is defined by

$$K_\alpha = \sqrt{2|\alpha|} \frac{I_0(\sqrt{2|\alpha|})}{I_1(\sqrt{2|\alpha|})},$$

where I_ν is the modified Bessel function of order ν, and $g(\nu, \gamma)$ for $\nu \in \mathbb{R}_+$ and $0 \leq \gamma < 1$ is the square root of the smallest positive solution λ associated with the Bessel equation $-u'' - u'/r + \nu^2 u/r^2 = \lambda u$ on the interval $[\gamma, 1]$ with Neumann boundary conditions, while $g(\nu, \gamma) = \nu$ for $\gamma \geq 1$.

Furthermore, for any $\alpha \in \mathbb{R}$ we have for the ideal anyon gas that

$$e(\alpha, 0) \geq \pi\alpha_* \bigl(1 - O(\alpha_*^{1/3})\bigr).$$

The result looks rather daunting, but the important message is that for all values of $\alpha, \bar{\gamma} > 0$ the energy per particle in the thermodynamic limit is positive.

The method of the main proof follows the strategy developed in [77] to understand the gas of ideal anyons. The main idea of this strategy is to combine local exclusion principles with a partitioning of the configuration space. In order to extend the methods to the case $R > 0$ we prove new magnetic Hardy inequalities based on ideas in [57, 77]. Moreover, the analysis requires proving lower bounds for certain two-dimensional Schrödinger operators with oscillating radial scalar potentials. Although this part of the argument does not require anything deep, it is nonetheless fairly involved at a technical level.

Summary of Paper H

Lieb–Thirring inequalities for wave functions vanishing on the diagonal set
(joint with D. Lundholm and P. T. Nam)
Preprint 2019.

Paper H is devoted to a generalization of the Lieb–Thirring inequality for many-body wave functions discussed in Chapter 4. Specifically we consider the kinetic energy operator acting as the fractional Laplacian in each particle. For $s > 0$, the fractional Laplace operator $(-\Delta)^s$ is defined using Theorem 2.1 through the quadratic form

$$q(u) = \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2\, d\xi, \quad u \in H^s(\mathbb{R}^d), \tag{5.7}$$

where $\hat{u}$ denotes the Fourier transform of u. Specifically we consider inequalities of the form

$$\left\langle \Psi, \sum_{j=1}^N (-\Delta_{\mathbf{x}_j})^s \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \varrho_\Psi(\mathbf{x})^{1+2s/d}\, d\mathbf{x}, \tag{5.8}$$

for $\Psi \in H^s(\mathbb{R}^{dN})$, with $\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$, and where ϱ_Ψ is the one-body density

$$\varrho_\Psi(\mathbf{x}) = \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{i \neq j} d\mathbf{x}_i.$$

Importantly, the constant C should be independent of Ψ and N.

Under the assumption that Ψ satisfies the Pauli exclusion principle,

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \operatorname{sgn}(\sigma)\Psi(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(N)})$$

for all permutations $\sigma \in S_N$ (or equivalently (4.1)), the inequality (5.8) is known to be valid for all $s > 0$ and $d \geq 1$ [22, 69, 70]. In particular, when $s = 1$ this is precisely Theorem 4.1.

Noting that the Pauli exclusion principle implies that Ψ must vanish on the diagonal set

$$\triangle = \left\{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN} : \mathbf{x}_i = \mathbf{x}_j \text{ for some } i \neq j\right\}$$

it is natural to ask if (5.8) remains valid under this weaker assumption. The main result of Paper H is that the answer to this question is *yes, if and only if* $2s > d$.

In fact, we prove a more general result. Define for $d \geq 1, s > 0$ and $k \geq 2$ the function space

$$\mathcal{H}_k^{s,N}(\mathbb{R}^d) = \overline{\left\{\Psi \in C_0^\infty(\mathbb{R}^{dN}) : \Psi|_{\triangle_k} = 0\right\}}^{H^s(\mathbb{R}^{dN})},$$

where

$$\triangle_k = \left\{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN} : \mathbf{x}_{j_1} = \ldots = \mathbf{x}_{j_k} \text{ for some } j_1 < \ldots < j_k\right\}$$

is the k-particle diagonal set in $\mathbb{R}^{dN}$. Note that by definition $\triangle = \triangle_2$.

The main result of Paper H is

Theorem 5.13. *Let $d \geq 1$, $k \geq 2$ and $2s > d(k-1)$. Then for every $N \geq 1$ and every $\Psi \in \mathcal{H}_k^{s,N}(\mathbb{R}^d)$, with $\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$, we have*

$$\left\langle \Psi, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \varrho_\Psi(\mathbf{x})^{1+2s/d} \, d\mathbf{x}.$$

Here $C = C(d, s, k) > 0$ is a universal constant independent of N and Ψ.

Theorem 5.13 provides a positive answer to a question posed by Lundholm, Nam, and Portmann in [73] which served as the original motivation for our work.

The proof of Theorem 5.13 is based on the local strategy of deriving Lieb–Thirring inequalities mentioned in Chapter 4. The main new ingredient in Paper H is a reduction of the proof of a local exclusion principle to simply the positivity of a local energy. This refines a bootstrap argument developed in [76]. After this reduction the final ingredient in the proof of Theorem 5.13 is a new many-body Poincaré inequality for wave functions vanishing on $\triangle_k$, which could be of independent interest (Theorem 5.1 in Paper H).

Contributions of the author

The results of Papers D, E, G, and H were obtained through collaborations. The role of the author in each of these is indicated below.

Paper D (joint with K. Gittins)
The paper is a result of a collaboration in which both authors contributed approximately equal amounts to every aspect of the work. The problem studied is a generalization of one studied by Gittins and co-authors in [5, 6] (see also [2]). The main contribution of the author to the paper is how to use the product structure of the domains together with the Aizenman–Lieb identity (2.10) to reduce the d-dimensional case to one one-dimensional problem. This is key in constructing a unified proof removing complications appearing when increasing the dimension.

Paper E (joint with R. L. Frank)
The paper is a result of a collaboration in which both authors contributed approximately equal amounts to every aspect of the work. The idea of combining the techniques of [30] with those in [11] was proposed by Frank. The argument to control the geometric approximation error by proving two-term semiclassical asymptotics for the localized Laplacian on cones was proposed by the author. The geometric construction in the proof of uniformity in the setting of convex domains is based on ideas of the author appearing also in Papers A and C.

Paper G (joint with D. Lundholm)
The paper is a result of a collaboration in which both authors contributed approximately equal amounts to every aspect of the work. The study of the problem was proposed by Lundholm as a continuation on earlier work on ideal anyons [77].

Paper H (joint with D. Lundholm and P. T. Nam)
The paper is a result of a collaboration in which all authors contributed approximately equal amounts to every aspect of the work. The idea to generalize the reduction of a local exclusion bound by Lundholm and Seiringer [76] was proposed by Nam (Lemma 4.1 in Paper H). The main contributions of the author relate to the application for wave functions vanishing on diagonal sets.

References

[1] M. Aizenman and E. H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429.

[2] P. R. S. Antunes and P. Freitas, *Optimal spectral rectangles and lattice ellipses*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **469** (2013), no. 2150.

[3] P. Bérard, *Transplantation et isospectralité. I*, Math. Ann. **292** (1992), no. 3, 547–559.

[4] F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167.

[5] M. van den Berg, D. Bucur, and K. Gittins, *Maximising Neumann eigenvalues on rectangles*, Bull. Lond. Math. Soc. **48** (2016), no. 5, 877–894.

[6] M. van den Berg and K. Gittins, *Minimising Dirichlet eigenvalues on cuboids of unit measure*, Mathematika **63** (2017), 468–482.

[7] M. Sh. Birman, *On the spectrum of singular boundary-value problems*, Mat. Sb. (N.S.) **55 (97)** (1961), 125–174.

[8] M. Sh. Birman and A. Laptev, *The negative discrete spectrum of a two-dimensional Schrödinger operator*, Comm. Pure Appl. Math. **49** (1996), no. 9, 967–997.

[9] M. Sh. Birman and M. Z. Solomjak, *The principal term of the spectral asymptotics for "non-smooth" elliptic problems*, Funkcional. Anal. i Priložen. **4** (1970), no. 4, 1–13.

[10] M. Sh. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht, 1987.

[11] R. M. Brown, *The trace of the heat kernel in Lipschitz domains*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 889–900.

[12] D. Bucur, *Minimization of the k-th eigenvalue of the Dirichlet Laplacian*, Arch. Ration. Mech. Anal. **206** (2012), no. 3, 1073–1083.

[13] D. Bucur and P. Freitas, *Asymptotic behaviour of optimal spectral planar domains with fixed perimeter*, J. Math. Phys. **54** (2013), no. 5, 053504.

[14] D. Bucur and A. Henrot, *Maximization of the second non-trivial Neumann eigenvalue*, Acta. Math. (to appear).

[15] D. Bucur and D. Mazzoleni, *A surgery result for the spectrum of the Dirichlet Laplacian*, SIAM J. Math. Anal. **47** (2015), no. 6, 4451–4466.

[16] G. Buttazzo and G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Ration. Mech. Anal. **122** (1993), no. 2, 183–195.

[17] C. Clark, *The asymptotic distribution of eigenvalues and eigenfunctions for elliptic boundary value problems*, SIAM Rev. **9** (1967), 627–646.

[18] B. Colbois and A. El Soufi, *Extremal eigenvalues of the Laplacian on Euclidean domains and closed surfaces*, Math. Z. **278** (2014), no. 1-2, 529–546.

[19] R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. I*, Interscience Publishers, Inc., New York, 1953.

[20] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. (2) **106** (1977), no. 1, 93–100.

[21] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, study ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.

[22] I. Daubechies, *An uncertainty principle for fermions with generalized kinetic energy*, Comm. Math. Phys. **90** (1983), no. 4, 511–520.

[23] E. B. Davies, *The twisting trick for double well Hamiltonians*, Comm. Math. Phys. **85** (1982), no. 3, 471–479.

[24] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge Univ. Press, Cambridge, 1995.

[25] *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/, Release 1.0.21, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders (eds.).

[26] G. Domokos and Z. Lángi, *The isoperimetric quotient of a convex body decreases monotonically under the eikonal abrasion model*, Mathematika **65** (2019), no. 1, 119–129.

[27] G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitz. Ber. Bayer. Akad. Wiss. (1923), 169–172.

[28] S. Fournais and B. Helffer, *Spectral methods in surface superconductivity*, Progress in Nonlinear Differential Equations and their Applications, vol. 77, Birkhäuser Boston, Inc., Boston, MA, 2010.

[29] R. L. Frank, *Remarks on eigenvalue estimates and semigroup domination*, Spectral and scattering theory for quantum magnetic systems, Contemp. Math., vol. 500, Amer. Math. Soc., Providence, RI, 2009, pp. 63–86.

[30] R. L. Frank and L. Geisinger, *Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain*, Mathematical results in quantum physics, World Sci. Publ., Hackensack, NJ, 2011, pp. 138–147.

[31] R. L. Frank and L. Geisinger, *Semi-classical analysis of the Laplace operator with Robin boundary conditions*, Bull. Math. Sci. **2** (2012), no. 2, 281–319.

[32] R. L. Frank, D. Hundertmark, M. Jex, and P. T. Nam, *The Lieb–Thirring inequality revisited*, `arXiv:1808.09017 [math-ph]` (2018).

[33] P. Freitas, *Asymptotic behaviour of extremal averages of Laplacian eigenvalues*, J. Stat. Phys. **167** (2017), no. 6, 1511–1518.

[34] P. Freitas, J. Lagagé, and J. Payette, *Optimal unions of scaled copies of domains and Pólya's conjecture*, (2018), in preparation.

[35] L. Geisinger, A. Laptev, and T. Weidl, *Geometrical versions of improved Berezin–Li–Yau inequalities*, J. Spectr. Theory **1** (2011), 87–109.

[36] K. Gittins and S. Larson, *Asymptotic behaviour of cuboids optimising Laplacian eigenvalues*, Integral equations and operator theory **89** (2017), no. 5, 607–629.

[37] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), no. 1, 1–22.

[38] F. Götze, *Lattice point problems and values of quadratic forms*, Invent. Math. **157** (2004), no. 1, 195–226.

[39] E. M. Harrell, *Double wells*, Comm. Math. Phys. **75** (1980), no. 3, 239–261.

[40] E. Harrell II and J. Stubbe, *Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian*, J. Spectr. Theory **8** (2018), no. 4, 1529–1550.

[41] B. Helffer and D. Robert, *Riesz means of bound states and semiclassical limit connected with a Lieb–Thirring's conjecture*, Asymptotic Anal. **3** (1990), no. 2, 91–103.

[42] B. Helffer and D. Robert, *Riesz means of bounded states and semi-classical limit connected with a Lieb–Thirring conjecture. II*, Ann. Inst. H. Poincaré Phys. Théor. **53** (1990), no. 2, 139–147.

[43] B. Helffer and J. Sjöstrand, *On diamagnetism and de Haas-van Alphen effect*, Ann. Inst. H. Poincaré Phys. Théor. **52** (1990), no. 4, 303–375.

[44] R. Hempel, L. A. Seco, and B. Simon, *The essential spectrum of Neumann Laplacians on some bounded singular domains*, J. Funct. Anal. **102** (1991), no. 2, 448–483.

[45] A. Henrot (ed.), *Shape optimization and spectral theory*, de Gruyter Open, Warsaw, 2017.

[46] C. S. Herz, *On the number of lattice points in a convex set*, Amer. J. Math. **84** (1962), 126–133.

[47] E. Hille, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, vol. 31, Amer. Math. Soc., New York, 1948.

[48] D. Hundertmark, P. Kunstmann, T. Ried, and S. Vugalter, *Cwikel's bound reloaded*, `arXiv:1809.05069 [math-ph]` (2018).

[49] D. Hundertmark, E. H. Lieb, and L. E. Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator*, Adv. Theor. Math. Phys. **2** (1998), no. 4, 719–731.

[50] M. N. Huxley, *Area, lattice points, and exponential sums*, London Mathematical Society Monographs. New Series, vol. 13, Oxford University Press, New York, 1996.

[51] V. Ivrii, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34.

[52] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), no. 4, part II, 1–23.

[53] E. Krahn, *Über eine von Rayleigh Formulierte Minimaleigenschaft des Krieses*, Math. Ann. **94** (1925), 97–100.

[54] E. Krahn, *Über Minimaleigenshaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Dorpat. **A9** (1926), 1–44.

[55] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. **106** (1992), no. 2, 353–357.

[56] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151** (1997), no. 2, 531–545.

[57] A. Laptev and T. Weidl, *Hardy inequalities for magnetic Dirichlet forms*, Mathematical results in quantum mechanics (Prague, 1998), Oper. Theory Adv. Appl., vol. 108, Birkhäuser, Basel, 1999, pp. 299–305.

[58] A. Laptev and T. Weidl, *Sharp Lieb–Thirring inequalities in high dimensions*, Acta Math. **184** (2000), no. 1, 87–111.

[59] R. S. Laugesen and S. Liu, *Optimal stretching for lattice points and eigenvalues*, Ark. Mat. **56** (2018), no. 1, 111–145.

[60] R. S. Laugesen and S. Liu, *Shifted lattices and asymptotically optimal ellipses*, J. Anal. **26** (2018), no. 1, 71–102.

[61] J. M. Leinaas and J. Myrheim, *On the theory of identical particles*, Il Nuovo Cimento B (1971-1996) **37** (1977), no. 1, 1–23.

[62] P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, 309–318.

[63] E. H. Lieb, *The classical limit of quantum spin systems*, Comm. Math. Phys. **31** (1973), 327–340.

[64] E. H. Lieb, *The stability of matter*, Rev. Modern Phys. **48** (1976), no. 4, 553–569.

[65] E. H. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, RI, 1980, pp. 241–252.

[66] E. H. Lieb, *The stability of matter: from atoms to stars*, fourth ed., Springer, Berlin, 2005, Selecta of Elliott H. Lieb, Edited by W. Thirring, and with a preface by F. Dyson.

[67] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc., Providence, RI, 2001.

[68] E. H. Lieb and R. Seiringer, *The stability of matter in quantum mechanics*, Cambridge Univ. Press, Cambridge, 2010.

[69] E. H. Lieb and W. E. Thirring, *Bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett. **35** (1975).

[70] E. H. Lieb and W. E. Thirring, *Inequalities for the moments of the eigenvalues of the schrödinger hamiltonian and their relation to sobolev inequalities*, Studies in Mathematical Physics, vol. 35, Princeton Univ. Press, 1976.

[71] D. Lundholm, *Many-anyon trial states*, Phys. Rev. A **96** (2017), 012116.

[72] D. Lundholm, *Methods of modern mathematical physics: Uncertainty and exclusion principles in quantum mechanics*, `arXiv:1805.03063 [math-ph]` (2018).

[73] D. Lundholm, P. T. Nam, and F. Portmann, *Fractional Hardy–Lieb–Thirring and related inequalities for interacting systems*, Arch. Ration. Mech. Anal. **219** (2016), no. 3, 1343–1382.

[74] D. Lundholm, F. Portmann, and J. P. Solovej, *Lieb–Thirring bounds for interacting Bose gases*, Comm. Math. Phys. **335** (2015), no. 2, 1019–1056.

[75] D. Lundholm and N. Rougerie, *Emergence of fractional statistics for tracer particles in a laughlin liquid*, Phys. Rev. Lett. **116** (2016), 170401.

[76] D. Lundholm and R. Seiringer, *Fermionic behaviour of ideal anyons*, Lett. Math. Phys. **108** (2018), no. 11, 2523–2541.

[77] D. Lundholm and J. P. Solovej, *Hardy and Lieb–Thirring inequalities for anyons*, Comm. Math. Phys. **322** (2013), no. 3, 883–908.

[78] D. Lundholm and J. P. Solovej, *Local exclusion and Lieb–Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré **15** (2014), no. 6, 1061–1107.

[79] N. F. Marshall and S. Steinerberger, *Triangles capturing many lattice points*, Mathematika **64** (2018), no. 2, 551–582.

[80] V. G. Maz′ya, *Sobolev spaces with applications to elliptic partial differential equations*, Grundlehren der Mathematischen Wissenschaften, Springer, Heidelberg, 2011.

[81] D. Mazzoleni and A. Pratelli, *Existence of minimizers for spectral problems*, J. Math. Pures Appl. (9) **100** (2013), no. 3, 433–453.

[82] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Natl. Acad. Sci. USA **51** (1964), 542.

[83] J. Mund and R. Schrader, *Hilbert spaces for nonrelativistic and relativistic "free" plektons (particles with braid group statistics)*, Advances in dynamical systems and quantum physics (Capri, 1993), World Sci. Publ., River Edge, NJ, 1995, pp. 235–259.

[84] J. Myrheim, *Anyons*, Topological aspects of low dimensional systems (A. Comtet, T. Jolicœur, S. Ouvry, and F. David, eds.), Les Houches - Ecole d'Ete de Physique Theorique, vol. 69, Springer-Verlag, Berlin, 1999, pp. 265–413.

[85] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. Lond. Math. Soc. **11** (1961), 419–433.

[86] J. W. S. Rayleigh, *The Theory of Sound*, 1st ed., Macmillan, London, 1877.

[87] J. W. S. Rayleigh, *Remarks upon the law of complete radiation.*, Phil. Mag. (5) **49** (1900), 539–540.

[88] M. Reed and B. Simon, *Methods of modern mathematical physics. I–IV*, Academic Press, New York-London, 1972-1979.

[89] G. V. Rozenblum, *The distribution of the discrete spectrum for singular differential operators.*, Sov. Math., Dokl. **13** (1972), 245–249.

[90] G. V. Rozenblum, *On the eigenvalues of the first boundary value problem in unbounded domains.*, Math. USSR, Sb. **18** (1973), 235–248.

[91] G. V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Soviet Math. (Iz. VUZ) **20** (1976), no. 1, 63–71.

[92] Y. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Transl. of Math. Monog., vol. 155, Amer. Math. Soc., Providence, RI, 1997.

[93] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97** (1976), no. 2, 279–288.

[94] J. P. Solovej and W. L. Spitzer, *A new coherent states approach to semiclassics which gives Scott's correction*, Comm. Math. Phys. **241** (2003), no. 2-3, 383–420.

[95] T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), no. 1, 169–186.

[96] G. Szegő, *Inequalities for certain eigenvalues of a membrane of given area*, J. Ration. Mech. Anal. **3** (1954), 343–356.

[97] G. Teschl, *Mathematical methods in quantum mechanics*, Graduate Studies in Mathematics, vol. 99, Amer. Math. Soc., Providence, RI, 2009.

[98] D. Vassiliev, *Two-term asymptotic behavior of the spectrum of a boundary value problem in the case of a piecewise smooth boundary*, Dokl. Akad. Nauk SSSR **286** (1986), no. 5, 1043–1046.

[99] T. Weidl, *On the Lieb–Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$*, Comm. Math. Phys. **178** (1996), no. 1, 135–146.

[100] H. F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Ration. Mech. Anal. **5** (1956), 633–636.

[101] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479.

[102] H. Weyl, *Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze*, J. Reine Angew. Math. **143** (1913), 177–202.

Part II

Scientific papers

Paper A

A bound for the perimeter of inner parallel bodies

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A bound for the perimeter of inner parallel bodies

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ABSTRACT

We provide a sharp lower bound for the perimeter of the inner parallel sets of a convex body Ω. The bound depends only on the perimeter and inradius r of the original body and states that

$$|\partial\Omega_t| \geq \left(1 - \frac{t}{r}\right)_+^{n-1} |\partial\Omega|.$$

In particular the bound is independent of any regularity properties of $\partial\Omega$. As a by-product of the proof we establish precise conditions for equality. The proof, which is straightforward, is based on the construction of an extremal set for a certain optimization problem and the use of basic properties of mixed volumes.

1. Introduction

Given a convex domain $\Omega \subset \mathbb{R}^n$ we consider the family of its *inner parallel sets*. We denote by Ω_t the inner parallel set at distance $t \geq 0$, which is defined by

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \geq t\} = \Omega \sim tB.$$

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0022-1236/

Here B is the unit ball in $\mathbb{R}^n$ and $\sim$ denotes the *Minkowski difference*; a precise definition is given in Section 1.1. Correspondingly, the *outer parallel set* at distance $t \geq 0$ is the set

$$\{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) \leq t\} = \Omega + tB,$$

where $+$ denotes the *Minkowski sum*. In this paper we provide a lower bound for the perimeter of Ω_t in terms of the perimeter of Ω.

An important result in the theory of outer parallel sets is the so-called *Steiner formula*

$$|\Omega + tB| = \sum_{i=0}^{n} \binom{n}{i} t^i W_i(\Omega), \tag{1}$$

where coefficients W_i of the polynomial are the *quermassintegrals* of Ω, which are a special case of mixed volumes (see Section 1.1). The set of quermassintegrals contains several important geometric quantities: for instance we have that $W_0(\Omega) = |\Omega|$ and $nW_1(\Omega) = |\partial\Omega|$. There are analogous formulae to (1), called the *Steiner formulae* [14], that express the value of the i-th quermassintegral of $\Omega + tB$ in terms of $W_j(\Omega)$, for $i \leq j \leq n$. The Steiner formula appears not only in convex geometry, and important applications may be found in Federer's work on curvature measures in geometric measure theory (see [5]) and Weyl's tube formula in differential geometry (see [17]).

For inner parallel sets there is, in general, no counterpart to the Steiner formula. Matheron conjectured in [11] that the volume of a Minkowski difference is bounded from below by the *alternating Steiner polynomial*. If we restrict our attention to inner parallel sets he conjectured that

$$|\Omega \sim tB| \geq \sum_{i=0}^{n} \binom{n}{i} (-t)^i W_i(\Omega).$$

The precise conjecture was a more general statement where B is replaced by a general convex body and the quermassintegrals are replaced by mixed volumes. However, the conjecture was proved to be false by Hernández Cifre and Saorín in [7].

In addition to the lack of a Steiner-type formula, the Minkowski difference is far from being as well behaved as the Minkowski sum. In contrast to the Minkowski sum the difference is not a vectorial operation. Moreover, the regularity properties of $\Omega \sim tB$ may be very different from those of Ω. Both of these properties are demonstrated in Fig. 1. Nonetheless, the theory of inner parallel sets is rich and has several beautiful applications in both convex geometry and analysis (see for instance [2,4,10,12,13]).

In [8] the authors prove bounds for the quermassintegrals of inner parallel sets in a more general setting than that described above. Instead of considering the sets $\Omega \sim tB$, $t \geq 0$, they consider $\Omega \sim tE$ for some convex set E. The inequalities obtained in this paper are closely related to those in [8], and using similar techniques the results here could, at least in some sense, be generalized to the same setting. However, in such

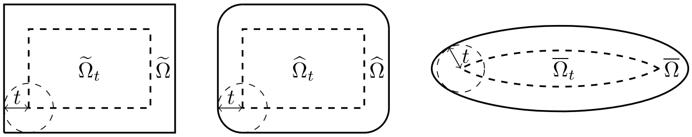

Fig. 1. The inner parallel sets of some convex bodies in $\mathbb{R}^2$. Note that $\widetilde{\Omega}_t$ is equal to $\widehat{\Omega}_t$ even though $\widetilde{\Omega} \neq \widehat{\Omega}$.

generalizations the connection to the spectral theoretic applications that motivated the work in this paper is lost.

The main result of this paper is an improvement of the following theorem which is obtained in [16] using the Steiner formula to bound the perimeter of outer parallel sets.

Theorem 1.1 *(Modified Steiner inequality [16]). Let Ω be a convex domain in $\mathbb{R}^n$ with volume $|\Omega|$ and surface area $|\partial\Omega|$ such that at each point the principal curvatures of $\partial\Omega$ are bounded from above by $1/K$ for some $K > 0$. Then for any $t \geq 0$ we have the bound*

$$|\partial\Omega_t| \geq |\partial\Omega|\Big(1 - \frac{n-1}{K}t\Big)_+.$$

Further, for any $0 \leq t < K$ the principal curvatures of $\partial\Omega_t$ are bounded from above by $(K-t)^{-1}$.

Our study of this problem is motivated by work of Geisinger, Laptev and Weidl in [6] where they use Theorem 1.1 to obtain bounds on the Riesz eigenvalue means for the Dirichlet Laplacian on a convex domain $\Omega \subset \mathbb{R}^n$. For convex domains in the plane satisfying the inequality

$$|\partial\Omega_t| \geq \Big(1 - \frac{3t}{\omega}\Big)_+ |\partial\Omega|, \tag{2}$$

where ω denotes the width of Ω, the authors further improve these bounds. Moreover, the authors conjecture that (2) holds for any planar convex domain. In this paper we prove that the bound (2) holds for any convex set in $\mathbb{R}^2$ and that similar bounds hold in arbitrary dimension.

We now turn to our main result which is contained in the next theorem. In [9] the techniques of [6] are combined with this result to obtain further geometrical improvements of Berezin-type bounds for the Dirichlet eigenvalues of the Laplacian on convex domains.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with inradius r. Then, for any inner parallel set Ω_t, $t \geq 0$, it holds that*

$$|\partial\Omega_t| \geq \Big(1 - \frac{t}{r}\Big)_+^{n-1} |\partial\Omega|.$$

Further, equality holds for some $t \in (0, r)$ if and only if Ω is homothetic to its form body.[1] *If this is the case equality holds for all $t \geq 0$.*

Using the above theorem and known bounds for the inradius and width of a convex body we are able to conclude that the conjectured inequality (2) holds and provide the following generalization to higher dimensions.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with width ω. Then, for the inner parallel sets of Ω we have that*

$$\begin{aligned} |\partial\Omega_t| &\geq \Big(1 - \frac{2\sqrt{n}}{\omega}\,t\Big)_+^{n-1} |\partial\Omega| && \text{if } n \text{ is odd,} \\ |\partial\Omega_t| &\geq \Big(1 - \frac{2(n+1)}{\omega\sqrt{n+2}}\,t\Big)_+^{n-1} |\partial\Omega| && \text{if } n \text{ is even.} \end{aligned}$$

In both cases equality holds if Ω is a regular $(n+1)$-simplex.

The result developed here is in several aspects an improvement of Theorem 1.1. Firstly, the assumptions on Ω are less restrictive. We require only convexity whilst the earlier result requires the principal curvatures of $\partial\Omega$ to be bounded. Further, by noting that

$$\Big(1 - \frac{t}{K}\Big)_+^{n-1} \geq \Big(1 - \frac{(n-1)}{K}t\Big)_+$$

and that the maximum of the principal curvatures of the boundary of a convex set is always larger than the reciprocal of its inradius one can conclude that Theorem 1.2 implies Theorem 1.1. We also note that if t is less than the reciprocal of the maximal principal curvature then $\Omega = \Omega_t + tB$. In general the set Ω cannot be determined from Ω_t and t.

1.1. Notation and preliminaries

Let $\mathcal{K}_0^n$ denote the set of all convex bodies in $\mathbb{R}^n$ that have nonempty interior. Throughout the paper Ω will belong to $\mathcal{K}_0^n$. Let B denote the closed unit ball in $\mathbb{R}^n$ and let $\mathbb{S}^{n-1}$ denote the corresponding sphere. A closed ball of radius r centered at $x \in \mathbb{R}^n$ is denoted by $B_r(x)$. For notational simplicity we denote both volume and surface measure by $|\cdot|$. This will appear in two forms, the volume of a set $|\Omega|$ and the surface measure of its boundary $|\partial\Omega|$. Further, we will make use of the notation $x_\pm = (|x| \pm x)/2$.

For two sets $K, L \in \mathcal{K}_0^n$ the *Minkowski sum* $(+)$ and *difference* $(\sim)$ are defined by

$$\begin{aligned} K + L &:= \{x + y : x \in K, y \in L\}, \\ K \sim L &:= \{x \in \mathbb{R}^n : x + L \subseteq K\}. \end{aligned}$$

[1] The precise definition of the form body of a convex set Ω will be given in Section 1.1 (see also [8,14]).

It is a direct consequence of the definitions that we, as claimed in the introduction, equivalently can define the inner parallel body Ω_t, $t \geq 0$, as $\Omega \sim tB$ [14]. Similarly the outer parallel body can be written as $\Omega + tB$.

The *inradius* r of a set $\Omega \in \mathcal{K}_0^n$ is defined as the radius of the largest ball contained in Ω, or equivalently (see for instance [14]) as

$$r = \sup\{\lambda \geq 0 : \Omega \sim \lambda B \neq \emptyset\}.$$

The observation contained in the next lemma is intuitively clear but of central importance in what follows.

Lemma 1.4. *Let $\Omega \in \mathcal{K}_0^n$ have inradius r_0. Then, for any $t \in [0, r_0]$ the inradius r_t of Ω_t satisfies*

$$r_t = r_0 - t.$$

Proof. Let $x_0 \in \mathbb{R}^n$ be such that $B_{r_0}(x_0) \subseteq \Omega$. For each $x \in B_{r_0}(x_0)$ we have that $\operatorname{dist}(x, \partial\Omega) \geq \operatorname{dist}(x, \partial B_{r_0}(x_0))$ and hence

$$\Omega_t \supseteq \big(B_{r_0}(x_0)\big)_t = B_{(r_0-t)}(x_0).$$

We conclude that $r_t \geq r_0 - t$. To prove the reverse inequality we observe that for any $x_t \in \mathbb{R}^n$ such that $B_{r_t}(x_t) \subseteq \Omega_t$ we have that $\operatorname{dist}(B_{r_t}(x_t), \partial\Omega) \geq t$. Which implies that

$$B_{(r_t+t)}(x_t) = B_{r_t}(x_t) + tB \subseteq \Omega,$$

and consequently $r_0 \geq r_t + t$. $\square$

A classic result in convex geometry is that the volume of a Minkowski sum $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is, for $\lambda_1, \ldots, \lambda_m \geq 0$ and $K_1, \ldots, K_m \in \mathcal{K}_0^n$, a homogeneous n-th degree polynomial in the λ_i with positive coefficients (see [3,14]). That is, we can write

$$|\lambda_1 K_1 + \cdots + \lambda_m K_m| = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} W(K_{i_1}, \ldots, K_{i_n}),$$

where W is symmetric with respect to its arguments. The $W(K_{i_1}, \ldots, K_{i_n})$ are called the *mixed volumes* of $K_1, \ldots, K_m$. In what follows we will use several properties of W. We list the properties here and for proofs refer to [3,14]:

- W is a symmetric functional on n-tuples of sets in $\mathcal{K}_0^n$.
- W is multilinear with respect to Minkowski addition:

$$W(\lambda K + \lambda' K', K_2, \ldots, K_n) = \lambda W(K, K_2, \ldots, K_n) + \lambda' W(K', K_2, \ldots, K_n).$$

- W is monotonically increasing with respect to inclusions.
- W is invariant under translations in each argument.
- The perimeter of $K \in \mathcal{K}_0^n$ is, up to a constant, equal to a mixed volume:

$$|\partial K| = nW(B, K, \dots, K).$$

We will by $h(K, u)$ denote the *support function* of $K \in \mathcal{K}_0^n$ which is defined for any $u \in \mathbb{R}^n$ as

$$h(K, u) = \sup_{x \in K} \langle x, u \rangle.$$

The restriction of $h(K, u)$ to $u \in \mathbb{S}^{n-1}$ reduces to the function describing the distance from the origin to the supporting hyperplane of K with normal u. In what follows we denote such a supporting hyperplane by $H(K, u)$. We then have the following characterization of the supporting hyperplanes of K:

$$H(K, u) = \{x \in \mathbb{R}^n : \langle x, u \rangle = h(K, u)\}.$$

The following properties of $h(K, u)$ will be needed later:

- For any $K, L \in \mathcal{K}_0^n$ and $\alpha, \beta > 0$ it holds that

$$h(\alpha K + \beta L, u) = \alpha h(K, u) + \beta h(L, u).$$

- For any $u \in \mathbb{S}^{n-1}$ and $K, L \in K_0^n$ it holds that

$$h(K \sim L, u) \leq h(K, u) - h(L, u).$$

- For $x \in \partial(K \sim L)$ there exists a normal vector u of $\partial(K \sim L)$ at x such that

$$h(K \sim L, u) = h(K, u) - h(L, u).$$

Proofs of the above properties can be found in [14].

The *width* ω of $\Omega \in \mathcal{K}_0^n$ is defined as

$$\omega = \inf\{h(\Omega, u) + h(\Omega, -u) : u \in \mathbb{S}^{n-1}\}.$$

A point $x \in \partial\Omega$ is called *regular* if the supporting hyperplane at x is uniquely defined, that is if there is a unique $u \in \mathbb{S}^{n-1}$ such that

$$x \in H(\Omega, u) \cap \Omega.$$

The set of all regular points of $\partial\Omega$ is denoted by $\mathrm{reg}(\Omega)$. We also let $\mathcal{U}(\Omega)$ denote the set of all outward pointing unit normals to $\partial\Omega$ at points of $\mathrm{reg}(\Omega)$.

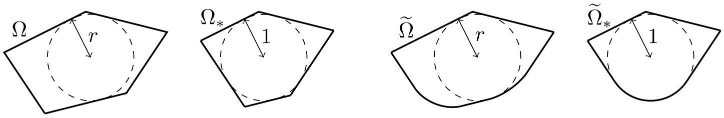

Fig. 2. The form body of two convex sets in $\mathbb{R}^2$.

We are now ready to define the *form body* Ω_* of a set $\Omega \in \mathcal{K}_0^n$, which, following [14], is defined by

$$\Omega_* = \bigcap_{u \in \mathcal{U}(\Omega)} \{x \in \mathbb{R}^n : \langle x, u\rangle \le 1\}.$$

Two planar convex bodies and their form bodies are shown in Fig. 2. If Ω is a polytope then Ω_* is the polytope that has the same set of normals as Ω, but with each face translated so that it is tangent to the unit ball. If instead the boundary of Ω is smooth (in which case every point is regular) then $\Omega_* = B$.

The following lemma will be needed in our main proof and is an almost direct consequence of the definitions of Ω_* and r combined with the fact that almost every point of $\partial\Omega$ is regular.

Lemma 1.5. *Let $\Omega \in \mathcal{K}_0^n$ have inradius r. Then there exists $x \in \mathbb{R}^n$ such that $x + r\Omega_* \subseteq \Omega$.*

2. Proof of the main result

The idea of the proof is as follows: Given a set $\Omega \in \mathcal{K}_0^n$ and $t \ge 0$ we construct a convex set $\widetilde{\Omega}$ such that $\widetilde{\Omega}_t = \Omega$ and $|\partial\widetilde{\Omega}| \ge |\partial\widehat{\Omega}|$ for any other set $\widehat{\Omega} \in \mathcal{K}_0^n$ satisfying $\widehat{\Omega}_t = \Omega$. If we can prove Theorem 1.2 for such $\widetilde{\Omega}$ it clearly holds also for any other convex set satisfying $\widehat{\Omega}_t = \Omega$. Since the choice of Ω and t was arbitrary this completes the proof.

We begin by constructing the set $\widetilde{\Omega}$. In the case where Ω is a polygon this problem has the fairly intuitive solution that $\widetilde{\Omega}$ is the polygon with the same faces as Ω, only moved a distance t along their outward pointing normals. The following lemma tells us that a generalization of this intuitive solution actually works for any possible Ω.

Lemma 2.1. *Let $\Omega \in \mathcal{K}_0^n$ and let Ω_* denote its form body. Then, for any $t \ge 0$ the maximization problem*

$$\max\{|\partial\widehat{\Omega}| : \widehat{\Omega} \in \mathcal{K}_0^n,\ \widehat{\Omega}_t = \Omega\}$$

is solved by $\widetilde{\Omega} = \Omega + t\Omega_*$.

Proof. Recalling the properties of the support function, we have that for any $x \in \partial\Omega = \partial(\widehat{\Omega} \sim tB)$ there exists a $u \in \mathbb{S}^{n-1}$ normal to $\partial\Omega$ at x such that

$$h(\widehat{\Omega} \sim tB, u) = h(\widehat{\Omega}, u) - h(tB, u) = h(\widehat{\Omega}, u) - t.$$

Rearranging this we find for all $u \in \mathcal{U}(\Omega)$ that $h(\widehat{\Omega}, u) = h(\Omega, u) + t$. Therefore, it follows that

$$\begin{aligned}\widehat{\Omega} &\subseteq \bigcap_{u\in\mathcal{U}(\Omega)} \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(\Omega, u) + t\} \\ &= \bigcap_{u\in\mathcal{U}(\Omega)} \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(\Omega, u) + h(t\Omega_*, u)\} \\ &= \bigcap_{u\in\mathcal{U}(\Omega)} \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(\Omega + t\Omega_*, u)\} \\ &= \Omega + t\Omega_*,\end{aligned}$$

where we used that $h(\Omega_*, u) = 1$ for $u \in \mathcal{U}(\Omega)$ and the last equality follows from [14, Theorem 2.2.6]. Since the perimeter is increasing under inclusion of convex sets, we conclude that $|\partial\widehat{\Omega}| \leq |\partial(\Omega + t\Omega_*)|$.

What remains to complete the proof is to show that $\Omega + t\Omega_*$ is an admissible set in the above maximization problem. That $\Omega \subseteq (\Omega + t\Omega_*)_t$ follows from the argument above so we only need to establish the opposite inclusion. Let $x \in \operatorname{reg}(\Omega)$. Then, with u being the unique normal to $\partial\Omega$ at x, we have that

$$h(\Omega + t\Omega_*, u) = h(\Omega, u) + t.$$

Since a convex body can be written as the intersection of its supporting half-spaces we conclude that $x + tu \in \overline{(\Omega + t\Omega_*)^c}$ implying that $\operatorname{dist}(x, \partial(\Omega + t\Omega_*)) \leq t$. Combining this with the inclusion of Ω in $(\Omega + t\Omega_*)_t$ we find that $\operatorname{reg}(\Omega) \in \partial(\Omega + t\Omega_*)_t$. Since almost every point of $\partial\Omega$ is regular the statement follows. $\square$

We are now ready to prove Theorem 1.2. Let $t \geq 0$ and let $\Omega \in \mathcal{K}_0^n$ have inradius r. By the above lemma we have, for any convex body $\widehat{\Omega}$ such that $\widehat{\Omega}_t = \Omega$, the bound $|\partial\widehat{\Omega}| \leq |\partial(\Omega + t\Omega_*)|$, and by Lemma 1.4 any such $\widehat{\Omega}$ has the inradius $\hat{r} = r + t$. Thus it is sufficient to prove that

$$|\partial\Omega| \geq \left(1 - \frac{t}{\hat{r}}\right)_+^{n-1} |\partial(\Omega + t\Omega_*)|.$$

Using the multilinearity of W and fact that the perimeter of a convex set can be expressed as a mixed volume we find that

$$\begin{aligned}|\partial(\Omega + t\Omega_*)| &= nW(B, \underbrace{\Omega + t\Omega_*, \ldots, \Omega + t\Omega_*}_{n-1}) \\ &= n\sum_{m=0}^{n-1} \binom{n-1}{m} t^m W(B, \underbrace{\Omega, \ldots, \Omega}_{n-1-m}, \underbrace{\Omega_*, \ldots, \Omega_*}_{m}).\end{aligned}$$

By Lemma 1.5 there exists $x \in \mathbb{R}^n$ such that $x + r\Omega_* \subseteq \Omega$. Therefore by the translation invariance and the monotonicity of mixed volumes we find that

$$\begin{aligned}
|\partial(\Omega + t\Omega_*)| &= n \sum_{m=0}^{n-1} \binom{n-1}{m} t^m W(B, \underbrace{\Omega, \ldots, \Omega}_{n-1-m}, \underbrace{\Omega_*, \ldots, \Omega_*}_{m}) \\
&= n \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{t^m}{r^m} W(B, \underbrace{\Omega, \ldots, \Omega}_{n-1-m}, \underbrace{r\Omega_*, \ldots, r\Omega_*}_{m}) \\
&\le n \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{t^m}{r^m} W(B, \underbrace{\Omega, \ldots, \Omega}_{n-1}) \\
&= |\partial\Omega| \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{t^m}{r^m} \\
&= |\partial\Omega| \Big(1 + \frac{t}{r}\Big)^{n-1}.
\end{aligned}$$

Rearranging the terms and using that $\hat{r} = r + t$ one obtains the desired inequality. It is clear from the argument above that equality holds if and only if $\Omega = x + r\Omega_*$, that is when Ω is homothetic to Ω_*. This completes the proof of Theorem 1.2.

Deducing Corollary 1.3 is simply a matter of applying the following theorem due to Steinhagen [15]. We note that this theorem appeared in the case of planar convex bodies in earlier work by Blaschke [1], and this simpler case is sufficient for proving the inequality conjectured in [6].

Theorem 2.2 *(Steinhagen's inequality [15]). Let $\Omega \in \mathcal{K}_0^n$ have inradius r and width ω. Then the following two-sided inequality holds:*

$$\begin{aligned}
2r \le \omega \le 2\sqrt{n}\, r \qquad & \textit{if } n \textit{ is odd}, \\
2r \le \omega \le \frac{2(n+1)}{\sqrt{n+2}} r \qquad & \textit{if } n \textit{ is even}.
\end{aligned}$$

The lower bound is attained if Ω is a ball, and the upper bound is attained if Ω is a regular $(n+1)$-simplex.

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References

[1] W. Blaschke, Über den größten Kreis in einer konvexen Punktmenge, Jahresber. Dtsch. Math.-Ver. 23 (1914) 369–374.
[2] G. Bol, Beweis einer vermutung von H. Minkowski, Abh. Math. Semin. Univ. Hambg. 15 (1) (1943) 37–56.
[3] T. Bonnesen, W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, English translation: L. Boron, C. Christenson, B. Smith (eds.) Theory of Convex Bodies, BCS Associates, Moscow, 1987.
[4] N.S. Brannen, The Wills conjecture, Trans. Amer. Math. Soc. 349 (10) (1997) 3977–3987.
[5] H. Federer, Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–491.
[6] L. Geisinger, A. Laptev, T. Weidl, Geometrical versions of improved Berezin–Li–Yau Inequalities, J. Spectr. Theory 1 (2011) 87–109.
[7] M.A. Hernández Cifre, E. Saorín, On the volume of inner parallel bodies, Adv. Geom. 10 (2) (2010) 275–286.
[8] M.A. Hernández Cifre, E. Saorín, On inner parallel bodies and quermassintegrals, Israel J. Math. 177 (2010) 29–47.
[9] S. Larson, On the remainder term of the Berezin inequality on a convex domain, ArXiv e-prints, arXiv:1509.06705, 2015.
[10] E. Linke, E. Saorín Gómez, Decomposition of polytopes using inner parallel bodies, Monatsh. Math. 176 (4) (2015) 575–588.
[11] G. Matheron, La formule de Steiner pour les érosions, J. Appl. Probab. 15 (1) (1978) 126–135.
[12] J.R. Sangwine-Yager, Inner parallel bodies and geometric inequalities, Ph.D. thesis, University of California, Davis, ProQuest LLC, Ann Arbor, MI, 1978.
[13] J.R. Sangwine-Yager, A Bonnesen-style inradius inequality in 3-space, Pacific J. Math. 134 (1) (1988) 173–178.
[14] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, second expanded edn., Encyclopedia of Mathematics and Its Applications, vol. 151, Cambridge Univ. Press, Cambridge, 2014.
[15] P. Steinhagen, Über die größte Kugel in einer konvexen Punktmenge, Abh. Math. Semin. Univ. Hambg. 1 (1) (1922) 15–26.
[16] M. van den Berg, A uniform bound on trace $(e^{t\Delta})$ for convex regions in $\mathbb{R}^n$ with smooth boundaries, Comm. Math. Phys. 92 (4) (1984) 525–530.
[17] H. Weyl, On the volume of tubes, Amer. J. Math. 61 (2) (1939) 461–472.

Paper B

On the remainder term of the Berezin inequality on a convex domain

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ON THE REMAINDER TERM OF THE BEREZIN INEQUALITY ON A CONVEX DOMAIN

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ABSTRACT. We study the Dirichlet eigenvalues of the Laplacian on a convex domain in $\mathbb{R}^n$, with $n \geq 2$. In particular, we generalize and improve upper bounds for the Riesz means of order $\sigma \geq 3/2$ established in an article by Geisinger, Laptev and Weidl. This is achieved by refining estimates for a negative second term in the Berezin inequality. The obtained remainder term reflects the correct order of growth in the semi-classical limit and depends only on the measure of the boundary of the domain. We emphasize that such an improvement is for general $\Omega \subset \mathbb{R}^n$ not possible and was previously known to hold only for planar convex domains satisfying certain geometric conditions.

As a corollary we obtain lower bounds for the individual eigenvalues λ_k, which for a certain range of k improves the Li–Yau inequality for convex domains. However, for convex domains one can by using different methods obtain even stronger lower bounds for λ_k.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$ and let $-\Delta_\Omega$ be the Dirichlet Laplace operator on $L^2(\Omega)$, defined in the quadratic form sense with form domain $H_0^1(\Omega)$. If the volume of Ω, which we denote by $|\Omega|$, is finite, then the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and the spectrum of $-\Delta_\Omega$ is discrete. Further, the spectrum is positive and accumulates only at infinity. Thus we can write it as an increasing sequence of eigenvalues:

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots,$$

where an eigenvalue is repeated according to its multiplicity.

Letting $x_\pm = (|x| \pm x)/2$, the Riesz means of these eigenvalues are defined, for $\Lambda > 0$, by

$$\sum_{k=1}^{\infty} (\Lambda - \lambda_k)_+^\sigma = \operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma, \quad \sigma \geq 0.$$

In what follows we will be interested in establishing upper bounds for these means. In particular, we will study the case $\sigma \geq 3/2$ when Ω is convex.

The classical Weyl asymptotic formula (see [23]) states that for $\Omega \subset \mathbb{R}^n$ and $\sigma \geq 0$ the identity

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma = L_{\sigma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\sigma+n/2} + o(\Lambda^{\sigma+n/2}) \tag{1}$$

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holds as $\Lambda \to \infty$. Here, and in what follows, $L^{\mathrm{cl}}_{\sigma,n}$ denotes the Lieb–Thirring constant:

$$L^{\mathrm{cl}}_{\sigma,n} = \frac{\Gamma(\sigma+1)}{(4\pi)^{n/2}\Gamma(\sigma+1+n/2)}.$$

Following the work of Weyl the second term of the asymptotics has been further studied (see, for instance, [3,4,7–9,18]). Under certain conditions on the set Ω and its boundary $\partial\Omega$ it was proved by Ivriĭ in [8] that

$$\text{(2)}\quad \operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- = L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2} - \frac{1}{4}L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|\Lambda^{\sigma+(n-1)/2} + o(\Lambda^{\sigma+(n-1)/2})$$

holds as $\Lambda \to \infty$, for $\sigma \geq 1$ this was later generalized to a larger class of domains by Frank and Geisinger [4]. To simplify notation we write $|\Omega|$ for the n-dimensional volume of Ω, and $|\partial\Omega|$ for the $(n-1)$-dimensional surface area of its boundary.

In [2] Berezin proved that for $\Omega \subset \mathbb{R}^n$ and $\Lambda > 0$ the convex Riesz eigenvalue means, that is, when $\sigma \geq 1$, satisfy the bound

$$\text{(3)}\qquad \operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \leq \frac{1}{(2\pi)^n}\iint_{\Omega\times\mathbb{R}^n}(|p|^2-\Lambda)^\sigma_-\,dp\,dx = L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2}.$$

From the Weyl asymptotics (1) it follows that the constant $L^{\mathrm{cl}}_{\sigma,n}$ in this bound is sharp. That (3) remains true also for $\sigma = 0$ coincides with the Pólya conjecture on the number of eigenvalues of $-\Delta_\Omega$ less than Λ (see [16]). In view of (2) this raises the question of whether one, in a similar manner as for the semi-classical limit, can improve Berezin's inequality by a negative remainder term.

Given an open set $\Omega \subset \mathbb{R}^n$ one can increase $|\partial\Omega|$ without significantly increasing $\lambda_k(\Omega)$. Thus, it is in general, for any $C > 0$, not possible to subtract a term $C|\partial\Omega|\Lambda^{\sigma+(n-1)/2}$ from the right-hand side of (3). However, the main result of this paper is that if we restrict our attention to convex sets and $\sigma \geq 3/2$ such an improvement is possible. This result is contained in the following theorem which generalizes a result obtained by Geisinger, Laptev and Weidl [5, Theorem 5.1] for convex sets in $\mathbb{R}^2$ satisfying certain geometric assumptions (see Theorem 1.4).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain with inradius r and let $\sigma \geq 3/2$. Then there exists a constant $C(\sigma,n) > 0$ such that*

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- = 0 \qquad \text{if } \Lambda \leq \frac{\pi^2}{4r^2},$$

and

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \leq L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2} - C(\sigma,n)L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|\Lambda^{\sigma+(n-1)/2} \qquad \text{if } \Lambda > \frac{\pi^2}{4r^2}.$$

Further, we provide upper and lower bounds for the constants $C(\sigma,n)$.

Using techniques from [11] this result can be applied to find improved bounds for Riesz means on product domains, $\Omega = \Omega_1 \times \Omega_2$, where one of the factors is a convex domain. These considerations lead to the following corollary:

Corollary 1.2. *Let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^{n_1}$ is a bounded, convex domain with inradius r and $\Omega_2 \subset \mathbb{R}^{n_2}$ is bounded and open. Assume that $\sigma + n_2/2 \geq 3/2$ and that for all $\Lambda > 0$*

$$\operatorname{Tr}(-\Delta_{\Omega_2} - \Lambda)^\sigma_- \leq L^{\mathrm{cl}}_{\sigma,n_2}|\Omega_2|\Lambda^{\sigma+n_2/2}.$$

With $n = n_1 + n_2$ we have that for $\Lambda \leq \frac{\pi^2}{4r^2}$

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma = 0,$$

and if $\Lambda > \frac{\pi^2}{4r^2}$, then

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L_{\sigma,n}^{\mathrm{cl}}|\Omega|\Lambda^{\sigma+n/2} - C(n_1, \sigma + n_2/2)L_{\sigma,n-1}^{\mathrm{cl}}|\Omega_2|\,|\partial\Omega_1|\Lambda^{\sigma+(n-1)/2},$$

where $C(\sigma, n)$ is the constant appearing in Theorem 1.1.

In particular, if $n_2 \geq 3$ and Ω_2 satisfies the Pólya conjecture, for instance if Ω_2 is a tiling domain, we may apply this with $\sigma = 0$. Thus we obtain examples of domains for which the Pólya conjecture is true even if we subtract, from the right-hand side of (3), a term of order $\Lambda^{(n-1)/2}$.

Proof of Corollary 1.2. Since $\Omega = \Omega_1 \times \Omega_2$ we have that the eigenvalues of $-\Delta_\Omega$ are given by

$$\lambda_{kl} = \eta_k + \nu_l,$$

where η_k and ν_l are the eigenvalues of $-\Delta_{\Omega_1}$ and $-\Delta_{\Omega_2}$, respectively. Thus we find that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma = \sum_{\lambda_{kl} \leq \Lambda} (\Lambda - \lambda_{kl})^\sigma = \sum_{\eta_k \leq \Lambda} \Big(\sum_{\nu_l \leq \Lambda - \eta_k} ((\Lambda - \eta_k) - \nu_l)^\sigma \Big).$$

By the assumptions on Ω_2, one obtains that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L_{\sigma,n_2}^{\mathrm{cl}}|\Omega_2| \sum_{\eta_k \leq \Lambda} (\Lambda - \eta_k)^{\sigma+n_2/2} = L_{\sigma,n_2}^{\mathrm{cl}}|\Omega_2| \operatorname{Tr}(-\Delta_{\Omega_1} - \Lambda)_-^{\sigma+n_2/2}.$$

Applying Theorem 1.1 and using that $L_{\sigma,n_2}^{\mathrm{cl}}L_{\sigma+n_2/2,n_1}^{\mathrm{cl}} = L_{\sigma,n_1+n_2}^{\mathrm{cl}}$ yields the result. □

As it stands in [5] the theorem corresponding to Theorem 1.1 above contains an error. This error appears when the Aizenman–Lieb argument (see [1]) is used together with a bound for the case $\sigma = 3/2$ to obtain bounds for larger values of σ. However, the proof that is used in [5] for the case of $\sigma = 3/2$ generalizes without any difficulty to arbitrary $\sigma \geq 3/2$ (this is the method we use here). The only difference is that instead of a constant depending only on the dimension we obtain one depending also on the parameter σ, namely $C(\sigma, n)$. In fact, it is not very difficult to prove that this constant must depend on both σ and n.

The first result in the direction of improving Berezin's inequality (3) is due to Melas, who in [15] obtains an improvement for all $\sigma \geq 1$. However, the negative correction term that was established in [15] is not of the same order in Λ as the correction term in the semi-classical asymptotics (2). In the two-dimensional case it was proved in [10] that the order of the remainder term can be chosen arbitrarily close to the asymptotically correct one, namely $\sigma + 1/2$.

In the case of $\sigma \geq 3/2$, which is the case studied here, it was established in [22] that the Berezin inequality, for open sets $\Omega \subset \mathbb{R}^n$, can be strengthened by a negative term of the same order in Λ as the second term in (2). However, as remarked earlier any uniform improvement of (3) must depend on other geometric quantities. For instance, the remainder term found in [22] depends on projections onto hyperplanes and in [6] the authors derive a remainder term, of the correct order, depending only on $|\Omega|$.

The approach of [22] relies on using Lieb–Thirring inequalities for Schrödinger operators with operator-valued potentials, see [12], and reducing the problem to trace estimates for the one-dimensional Laplacian on open intervals. In [5] the authors employ the same approach but with different estimates for the one-dimensional problem. Moreover, the authors of [5] are able to refine these estimates if Ω is convex. We summarize these refinements in the following theorems.

Theorem 1.3 ([5], Corollary 3.5). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain with smooth boundary and assume that at each point the principal curvatures of $\partial\Omega$ are bounded from above by $1/K$. Then, for $\sigma \geq 3/2$ and all $\Lambda > 0$ we have that*

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2} - L^{\mathrm{cl}}_{\sigma,n}2^{-n-2}|\partial\Omega|\Lambda^{\sigma+(n-1)/2}\int_0^1\Big(1 - \frac{n-1}{4K\sqrt{\Lambda}}\,t\Big)_+\,dt.$$

Theorem 1.4 ([5], Theorem 5.1). *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex domain with width w and let $\Omega_t = \{x \in \Omega : \mathrm{dist}(x, \Omega^c) \geq t\}$ denote its inner parallel set at distance $t \geq 0$. Further, assume that each Ω_t satisfies the estimate*

$$|\partial\Omega_t| \geq \Big(1 - \frac{3t}{w}\Big)_+|\partial\Omega|.$$

Then, for $\sigma \geq 3/2$ we have that

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma = 0 \qquad \text{if } \Lambda \leq \frac{\pi^2}{w^2},$$

and

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L^{\mathrm{cl}}_{\sigma,2}\,|\Omega|\,\Lambda^{\sigma+1} - C(\sigma)\,L^{\mathrm{cl}}_{\sigma,1}\,|\partial\Omega|\,\Lambda^{\sigma+1/2} \qquad \text{if } \Lambda > \frac{\pi^2}{w^2},$$

for some $C(\sigma) > 0$. In particular

$$C(3/2) \geq \frac{11}{9\pi^2} - \frac{3}{20\pi^4} - \frac{2}{5\pi^2}\log\Big(\frac{4\pi}{3}\Big) > 0.0642.$$

As pointed out earlier the last theorem is stated in [5] with a constant not depending on σ (in place of $C(\sigma)$), as we shall see such a statement cannot hold. However, the proof provided in [5] for the case $\sigma = 3/2$ holds and through a straightforward generalization this can be used to prove the statement for all $\sigma \geq 3/2$.

Note that in Theorem 1.4 the remainder term reflects the correct order of growth in the semi-classical limit and depends only on $|\partial\Omega|$. As remarked above this is not possible in general. In this paper we use bounds for the perimeter of inner parallel sets, obtained in [13], to refine and generalize both Theorem 1.3 and Theorem 1.4 to arbitrary convex domains and any dimension.

We begin Section 2 with a short introduction to the theory and notation that we will need from [5]. We then proceed by applying the results of [13] to refine the arguments leading to the improved Berezin bounds. The generalization of Theorem 1.3 is proved in the same manner as in [5], the only difference being the application of results from [13] instead of a version of Steiner's inequality (see [21]). Also the argument leading to Theorem 1.1, the generalized version of Theorem 1.4, is an almost step-by-step generalization of the proof given in [5]. However, for general dimension the computations become slightly more complicated.

In Section 3 we use the obtained improvements of (3) to prove (implicit) lower bounds for individual eigenvalues $\lambda_k(\Omega)$, where Ω is convex. We are able to show that for a rather surprising number of the lower eigenvalues these bounds are an improvement of the Li–Yau inequality [14]:

$$\lambda_k(\Omega) \geq \Gamma\Big(\frac{n}{2}+1\Big)^{2/n} \frac{4\pi n}{n+2}\Big(\frac{k}{|\Omega|}\Big)^{2/n}. \tag{4}$$

We note that for convex Ω one can, through different methods, improve the bounds given by (4); see [20]. Even though our results in a certain range of k provide better bounds than (4), they fail to improve the results of [20] in general.

2. An improved Berezin inequality for convex domains

We begin with a short introduction of the relevant notation used in [5] and [13]. For an open set $\Omega \subset \mathbb{R}^n$ (which in our case will be a convex set) we let, for $x \in \Omega$ and $u \in \mathbb{S}^{n-1}$,

$$\theta(x,u) = \inf\{t > 0 : x + tu \notin \Omega\},$$
$$d(x,u) = \inf\{\theta(x,u), \theta(x,-u)\}$$

and

$$l(x,u) = \theta(x,u) + \theta(x,-u).$$

For a convex non-empty set $\Omega \subset \mathbb{R}^n$ we let $h(\Omega, \cdot\,)$ denote the support function of Ω, which is defined by

$$h(\Omega, x) = \sup_{y\in\Omega}\langle y, x\rangle, \quad x \in \mathbb{R}^n.$$

For a detailed account on properties of the support function and convex geometry in general we refer to Schneider's excellent book [19].

Letting $\delta(x)$ denote the distance from x to the boundary of Ω we have that

$$\delta(x) = \inf_{u\in\mathbb{S}^{n-1}} \theta(x,u).$$

We define the inradius r and width w of a convex set Ω by

$$r = \sup_{x\in\Omega}\delta(x), \qquad w = \inf_{u\in\mathbb{S}^{n-1}} h(\Omega,u) + h(\Omega,-u).$$

For a convex set $\Omega \subset \mathbb{R}^n$ with width w it holds (see for instance [19]) that

$$w = \inf_{u\in\mathbb{S}^{n-1}} \sup_{x\in\Omega} l(x,u).$$

The quantity on the right-hand side is in [5], for a general domain Ω, denoted by l_0.

As in Theorem 1.4 we let Ω_t denote the inner parallel body of a convex set Ω at distance $t \geq 0$, which is defined by

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) \geq t\}.$$

The inradius of Ω can now alternatively be written as $r = \sup\{t \geq 0 : \Omega_t \neq \emptyset\}$.

In [13] the main result is a lower bound for the $(n-1)$-dimensional surface area of the perimeter of an inner parallel set, the result is stated below and will be of central importance in what follows.

Theorem 2.1 ([13], Theorem 1.2). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with inradius r. Then, for any inner parallel set Ω_t, $t \geq 0$, we have that*

$$|\partial\Omega_t| \geq \Big(1 - \frac{t}{r}\Big)_+^{n-1} |\partial\Omega|.$$

Further, equality holds for some $t \in (0, r)$ if and only if Ω is homothetic to its form body. If this is the case equality holds for all $t \geq 0$.

For the precise definition of the form body of Ω we refer to [19]. Since the exact conditions for equality will be of little importance, we will not include the precise definition.

For a fixed $\varepsilon > 0$ let

$$A_\varepsilon(x) = \{a \in \mathbb{R}^n \setminus \overline{\Omega} : |x - a| < \delta(x) + \varepsilon\}$$

and for any $x \in \Omega$ let

$$\rho(x) = \inf_{\varepsilon>0} \sup_{a \in A_\varepsilon(x)} \frac{|B_{\delta(x)}(a) \setminus \overline{\Omega}|}{|B_1(0)|\,|x - a|^n},$$

where $B_\delta(x)$ denotes a ball of radius δ centred at $x \in \mathbb{R}^n$. For a convex domain Ω we have that $\rho(x) > 1/2$ for all $x \in \Omega$ (see [5]).

As in [5] we set for $\Lambda > 0$

$$M_\Omega(\Lambda) = \int_{R_\Omega(\Lambda)} \rho(x)\, dx,$$

where $R_\Omega(\Lambda) = \{x \in \Omega : \delta(x) < 1/(4\sqrt{\Lambda})\} = \Omega \setminus \Omega_{1/(4\sqrt{\Lambda})}$.

The following theorem and its proof in [5] form the starting point for most of the remaining arguments of this paper.

Theorem 2.2 ([5], Theorem 3.3). *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite volume and $\sigma \geq 3/2$. Then for all $\Lambda > 0$ we have that*

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L_{\sigma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\sigma+n/2} - L_{\sigma,n}^{\mathrm{cl}} 2^{-n+1} \Lambda^{\sigma+n/2} M_\Omega(\Lambda).$$

Using Theorem 2.1 and the same argument that leads to Corollary 3.5 in [5], we deduce the following bound.

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain with inradius r. Then for all $\sigma \geq 3/2$ and all $\Lambda > 0$ we have that*

$$\begin{aligned}\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma &\leq L_{\sigma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\sigma+n/2} \\ &\quad - L_{\sigma,n}^{\mathrm{cl}} 2^{-n-2} |\partial\Omega| \Lambda^{\sigma+(n-1)/2} \int_0^1 \Big(1 - \frac{s}{4r\sqrt{\Lambda}}\Big)_+^{n-1} ds.\end{aligned}$$

Proof. Consider the remainder term in Theorem 2.2. Inserting into the definition of $M_\Omega(\Lambda)$ that $\rho(x) > 1/2$ when Ω is convex we find that

$$M_\Omega(\Lambda) = \int_{R_\Omega(\Lambda)} \rho(x)\, dx > \int_{R_\Omega(\Lambda)} \frac{1}{2}\, dx = \frac{1}{2} \int_0^{1/(4\sqrt{\Lambda})} |\partial\Omega_t|\, dt.$$

Applying Theorem 2.1 yields

$$M_\Omega(\Lambda) > \frac{|\partial\Omega|}{2} \int_0^{1/(4\sqrt{\Lambda})} \Big(1 - \frac{t}{r}\Big)_+^{n-1} dt = \frac{|\partial\Omega|}{8\sqrt{\Lambda}} \int_0^1 \Big(1 - \frac{s}{4r\sqrt{\Lambda}}\Big)_+^{n-1} ds,$$

which proves the claim. ∎

Using the inequality $\lambda_1(\Omega) \geq \frac{\pi^2}{4r^2}$ (see [17]) Corollary 2.3 actually implies that Theorem 1.1 holds for some positive constant $C(\sigma, n)$. However, by applying more refined techniques we can prove Theorem 1.1 with substantially larger values for $C(\sigma, n)$.

We now turn our attention to the main results of this paper, namely Theorem 1.1. As noted earlier this generalizes a result obtained in [5], in particular we are able to relax certain geometric constraints and generalize the result to dimensions $n \geq 2$. We emphasize that the remainder term in Theorem 1.1 reflects the behaviour of the second term of the semi-classical limit $\Lambda \to \infty$; see (2). It has the correct order in Λ and depends only on the size of $\partial\Omega$. Since it is not possible to obtain a uniform remainder term of this form for a general domain $\Omega \subset \mathbb{R}^n$, it would be of interest to know under what geometric conditions such a bound holds.

For $n = 2$ and $\sigma = 3/2$ the constant can be estimated in a similar manner as in [5] with the slightly improved result

$$C(3/2, 2) > 0.0846 > \frac{11}{9\pi^2} - \frac{3}{20\pi^4} - \frac{2}{5\pi^2} \ln\Big(\frac{4\pi}{3}\Big) \approx 0.0642,$$

where the constant on the right-hand side is the one found by Geisinger, Laptev and Weidl. The lower bound obtained for $C(\sigma, n)$ takes the form of an integral. This integral can, for fixed dimension and given σ, be expressed in terms of certain hypergeometric functions. However, these expressions quickly become rather complicated. For the first few dimensions and some different values of σ numerical values of the obtained upper and lower bounds for $C(\sigma, n)$ are displayed in Table 1.

TABLE 1. The obtained upper / lower bounds for $C(\sigma, n)$ for dimensions two through six and some different values of σ.

U/L	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$\sigma = 3/2$	0.1334 / 0.0846	0.0819 / 0.0538	0.0572 / 0.0391	0.0430 / 0.0305	0.0339 / 0.0247
$\sigma = 2$	0.1228 / 0.0808	0.0762 / 0.0515	0.0537 / 0.0375	0.0407 / 0.0293	0.0323 / 0.0239
$\sigma = 5/2$	0.1143 / 0.0775	0.0716 / 0.0495	0.0508 / 0.0361	0.0387 / 0.0283	0.0308 / 0.0231
$\sigma = 3$	0.1074 / 0.0747	0.0678 / 0.0477	0.0484 / 0.0349	0.0370 / 0.0274	0.0296 / 0.0224

We proceed by giving the proof of Theorem 1.1, which largely follows along the same lines as the corresponding proof in [5].

Proof of Theorem 1.1. The first part of the theorem follows directly from $\lambda_1(\Omega) \geq \frac{\pi^2}{4r^2}$; see [17]. Therefore we may focus on the second case.

Equation (13) in [5] states that for an open bounded set $\Omega \subset \mathbb{R}^n$, $\sigma \geq 3/2$ and $\Lambda > 0$ we have that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L^{\mathrm{cl}}_{\sigma,n} \Lambda^{\sigma+n/2} \int_\Omega \int_{\mathbb{S}^{n-1}} \Big(1 - \frac{1}{4\Lambda d(x,u)^2}\Big)_+^{\sigma+n/2} d\nu(u)\, dx, \tag{5}$$

where $d\nu(u)$ is the normalized measure on the sphere. This inequality will be the starting point for the second part of the proof.

Fix $x \in \Omega$ and choose $u_0 \in \mathbb{S}^{n-1}$ such that $\delta(x) = d(x, u_0)$. Since everything is coordinate invariant we may assume that $u_0 = (1, 0, \dots, 0)$ and let $\mathbb{S}^{n-1}_+ = \{u \in \mathbb{S}^{n-1} : \langle u, u_0 \rangle > 0\}$. Denote by a the intersection point of the ray $\{x + tu_0, t > 0\}$ with $\partial\Omega$. Similarly, for $u \in \mathbb{S}^{n-1}_+$ let b_u be the intersection point of the ray $\{x + tu, t > 0\}$ with the hyperplane through a orthogonal to u_0; we note that this is nothing but the supporting hyperplane of Ω with normal u_0.

We have that $d(x,u) \le |x-b_u|$, and with θ_u denoting the angle between u and u_0 we find that

$$d(x,u) \le |x-b_u| = \frac{|x-a|}{\cos\theta_u} = \frac{\delta(x)}{\cos\theta_u}.$$

Using the the antipodal symmetry of $d(x,u)$ and inserting the above estimate into (5) one obtains that

$$\begin{aligned}\text{(6)}\qquad \operatorname{Tr}(-\Delta_\Omega-\Lambda)_-^\sigma &\le 2L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}\int_\Omega\int_{\mathbb{S}^{n-1}_+}\Big(1-\frac{1}{4\Lambda d(x,u)^2}\Big)_+^{\sigma+n/2}d\nu(u)\,dx\\ &\le 2L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}\int_\Omega\int_{\mathbb{S}^{n-1}_+}\Big(1-\frac{\cos^2\theta_u}{4\Lambda\delta(x)^2}\Big)_+^{\sigma+n/2}d\nu(u)\,dx.\end{aligned}$$

We now switch to n-dimensional spherical coordinates such that u_0 is given by setting all angular coordinates to zero. Together with the rotational symmetry around u_0, this yields that

$$\operatorname{Tr}(-\Delta_\Omega-\Lambda)_-^\sigma \le L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}C_n\int_\Omega\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda d(x)^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\,dx,$$

where the normalization constant C_n is given by

$$C_n = \Big(\int_0^{\pi/2}(\sin\theta)^{n-2}\,d\theta\Big)^{-1} = \frac{2\,\Gamma\big(\frac{n}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n-1}{2}\big)}.$$

We begin by rewriting the integral in (6) to more easily obtain an expression of the desired form,

$$\begin{aligned}&\operatorname{Tr}(-\Delta_\Omega-\Lambda)_-^\sigma\\ &\quad\le L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}C_n\int_\Omega\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda\delta(x)^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\,dx\\ &\quad= L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2}\\ &\qquad - L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}\int_\Omega\Big(1-C_n\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda\delta(x)^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)dx\\ &\quad= L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2}\\ &\qquad - L^{\mathrm{cl}}_{\sigma,n}\Lambda^{\sigma+n/2}\int_{\mathbb{R}_+}|\partial\Omega_t|\Big(1-C_n\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda t^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)dt.\end{aligned}$$

In the last step we make use of the coarea formula and that the distance function $\delta(x)$ satisfies the Eikonal equation $|\nabla\delta| = 1$ almost everywhere.

By the definition of C_n the expression in the outer integral is non-negative, that is,

$$1-C_n\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda t^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta \ge 0.$$

Therefore, using Theorem 2.1 one obtains that

$$\begin{aligned}&\operatorname{Tr}(-\Delta_\Omega-\Lambda)_-^\sigma \le L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2}\\ &\quad - L^{\mathrm{cl}}_{\sigma,n}|\partial\Omega|\Lambda^{\sigma+n/2}\int_{\mathbb{R}_+}\Big(1-\frac{t}{r}\Big)_+^{n-1}\Big(1-C_n\int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda t^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)dt.\end{aligned}$$

Letting $s = 2\sqrt{\Lambda}\, t$ and using that $\Lambda \geq \frac{\pi^2}{4r^2}$ we find

$$\begin{aligned}\int_{\mathbb{R}_+} &\Big(1-\frac{t}{r}\Big)_+^{n-1}\Big(1 - C_n \int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{4\Lambda t^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)dt\\ &\geq \frac{1}{2\sqrt{\Lambda}}\int_{\mathbb{R}_+}\Big(1-\frac{s}{\pi}\Big)_+^{n-1}\Big(1 - C_n \int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{s^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)ds.\end{aligned}$$

Since the integral above depends only on n and σ the claim follows with a lower bound on $C(\sigma, n)$ given by

$$C(\sigma,n) \geq \frac{L^{\mathrm{cl}}_{\sigma,n}}{2\,L^{\mathrm{cl}}_{\sigma,n-1}} I(\sigma,n),$$

where

$$\begin{aligned} I(\sigma,n) &= \int_{\mathbb{R}_+}\Big(1-\frac{s}{\pi}\Big)_+^{n-1}\Big(1 - C_n \int_0^{\pi/2}\Big(1-\frac{\cos^2\theta}{s^2}\Big)_+^{\sigma+n/2}(\sin\theta)^{n-2}\,d\theta\Big)ds\\ &= \int_0^{\pi}\Big(1-\frac{s}{\pi}\Big)^{n-1}\Big(1 - C_n \int_0^{1}\Big(1-\frac{\varphi^2}{s^2}\Big)_+^{\sigma+n/2}(1-\varphi^2)^{(n-3)/2}\,d\varphi\Big)ds.\end{aligned}$$

To find upper estimates for the constants $C(\sigma, n)$ we argue as follows. For $\Lambda > \frac{\pi^2}{4r^2}$ our theorem says that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\sigma \leq L^{\mathrm{cl}}_{\sigma,n}|\Omega|\Lambda^{\sigma+n/2} - C(\sigma,n)L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|\Lambda^{\sigma+(n-1)/2}.$$

But we know that the left-hand side is positive, and thus any positive zero of the polynomial on the right must be contained in the interval $(0, \frac{\pi^2}{4r^2}]$. Clearly the polynomial has exactly one positive zero Λ_0, given by

$$\Lambda_0 = \left(\frac{C(\sigma,n)L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|}{L^{\mathrm{cl}}_{\sigma,n}|\Omega|}\right)^2.$$

Therefore we must have that

$$\frac{\pi^2}{4r^2} \geq \left(\frac{C(\sigma,n)L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|}{L^{\mathrm{cl}}_{\sigma,n}|\Omega|}\right)^2.$$

Rearranging the terms we find that for any convex domain Ω it should hold that

$$C(\sigma,n) \leq \frac{\pi}{2r}\frac{L^{\mathrm{cl}}_{\sigma,n}|\Omega|}{L^{\mathrm{cl}}_{\sigma,n-1}|\partial\Omega|}. \tag{7}$$

By using the coarea formula and Theorem 2.1 we find that

$$\frac{|\Omega|}{r|\partial\Omega|} = \frac{1}{r|\partial\Omega|}\int_0^r |\partial\Omega_t|\,dt \geq \frac{1}{r}\int_0^r\Big(1-\frac{t}{r}\Big)^{n-1}dt = \frac{1}{n},$$

where equality holds for a certain class of sets (see [13]). Inserting this into (7) we find that

$$C(\sigma,n) \leq \frac{\sqrt{\pi}\,\Gamma(\sigma+\frac{n+1}{2})}{4n\,\Gamma(\sigma+1+\frac{n}{2})},$$

as this tends to zero when σ or n tends to infinity it is clear that the constant in Theorem 1.1 must depend on both quantities.

Comparing the obtained upper and lower bounds we find that our proof provides a rather good estimate for $C(\sigma, n)$. This is also indicated by the numerical values in Table 1. □

3. Bounds on individual eigenvalues

Using the same methods as in [5] we would like to obtain bounds for individual eigenvalues. However to analytically solve the equation that one obtains for Λ is no simple task, since it involves solving an n-th order polynomial equation. It is, however, not difficult to numerically compute lower bounds. Nonetheless, we are able to conclude that the bounds implicitly given by our improved trace bounds in fact improve those given by the Li–Yau inequality for a certain range of k (which, in a rather complicated way, depends on n). As an introduction to what is to come, we state and prove the following result for the two-dimensional case. The proof is precisely the same as that given in [5].

Corollary 3.1 ([5], Corollary 5.2)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. Then with $C = C(3/2, 2)$ given by Theorem 1.1 we for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$ have that*

$$\frac{\lambda_k(\Omega)}{1-\alpha} \geq 10\pi\alpha^{3/2}\frac{k}{|\Omega|} + \frac{15\pi C}{8}\frac{|\partial\Omega|}{|\Omega|}\sqrt{10\pi\alpha^{3/2}\frac{k}{|\Omega|} + \frac{225\pi^2C^2}{256}\frac{|\partial\Omega|^2}{|\Omega|^2}} + \frac{225\pi^2C^2}{128}\frac{|\partial\Omega|^2}{|\Omega|^2}.$$

Proof. We let $N(\Lambda) = \operatorname{Tr}(-\Delta_\Omega - \Lambda)^0_-$ be the counting function of eigenvalues less than Λ. For $\sigma > 0$ and all $\Lambda > 0, \tau > 0$ it is shown in [11] that

$$N(\Lambda) \leq (\tau\Lambda)^{-\sigma}\operatorname{Tr}(-\Delta_\Omega - (1+\tau)\Lambda)^\sigma_-. \tag{8}$$

Applying this with $\sigma = 3/2$, we can use Theorem 1.1 and for $\Lambda \geq \frac{\pi^2}{4r^2}$ estimate

$$\begin{aligned} N(\Lambda) &\leq (\tau\Lambda)^{-3/2}\big(L^{\mathrm{cl}}_{3/2,2}|\Omega|((1+\tau)\Lambda)^{5/2} - C(3/2,2)L^{\mathrm{cl}}_{3/2,1}|\partial\Omega|((1+\tau)\Lambda)^2\big) \\ &= L^{\mathrm{cl}}_{3/2,2}|\Omega|\frac{(1+\tau)^{5/2}}{\tau^{3/2}}\Lambda - C(3/2,2)L^{\mathrm{cl}}_{3/2,1}|\partial\Omega|\frac{(1+\tau)^2}{\tau^{3/2}}\sqrt{\Lambda}. \end{aligned}$$

Substituting $\tau = \alpha/(1-\alpha)$ for $\alpha \in (0,1)$ and using that $N(\lambda_k) \geq k$ we find that

$$k \leq L^{\mathrm{cl}}_{3/2,2}\alpha^{-3/2}|\Omega|\frac{\lambda_k(\Omega)}{1-\alpha} - C(3/2,2)L^{\mathrm{cl}}_{3/2,1}|\partial\Omega|\alpha^{-3/2}\sqrt{\frac{\lambda_k(\Omega)}{1-\alpha}}.$$

Since the right-hand side is a convex quadratic polynomial in $\sqrt{\frac{\lambda_k(\Omega)}{1-\alpha}}$ which vanishes at zero, there is exactly one positive solution to where this polynomial is equal to k. By monotonicity this solution provides a lower bound for $\sqrt{\frac{\lambda_k(\Omega)}{1-\alpha}}$. Through some algebraic manipulations this yields that

$$\begin{aligned} \frac{\lambda_k(\Omega)}{1-\alpha} &\geq \left(\frac{C(3/2,2)L^{\mathrm{cl}}_{3/2,1}|\partial\Omega| + \big((C(3/2,2)L^{\mathrm{cl}}_{3/2,1}|\partial\Omega|)^2 + 4kL^{\mathrm{cl}}_{3/2,2}|\Omega|\alpha^{3/2}\big)^{1/2}}{2L^{\mathrm{cl}}_{3/2,2}|\Omega|}\right)^2 \\ &= \frac{\alpha^{3/2}k}{L^{\mathrm{cl}}_{3/2,2}|\Omega|} + C(3/2,2)\frac{L^{\mathrm{cl}}_{3/2,1}}{L^{\mathrm{cl}}_{3/2,2}}\frac{|\partial\Omega|}{|\Omega|}\sqrt{\frac{\alpha^{3/2}k}{L^{\mathrm{cl}}_{3/2,2}|\Omega|} + \frac{C(3/2,2)^2}{4}\left(\frac{L^{\mathrm{cl}}_{3/2,1}}{L^{\mathrm{cl}}_{3/2,2}}\right)^2\frac{|\partial\Omega|^2}{|\Omega|^2}} \\ &\quad + \frac{C(3/2,2)^2}{2}\left(\frac{L^{\mathrm{cl}}_{3/2,1}}{L^{\mathrm{cl}}_{3/2,2}}\right)^2\frac{|\partial\Omega|^2}{|\Omega|^2}. \end{aligned}$$

Inserting the values of the Lieb–Thirring constants we obtain the desired expression. □

For dimensions three and four the same method can be used to get explicit bounds for $\lambda_k(\Omega)$, but the expressions obtained become rather intractable as they involve the formula for a root of a third respectively fourth degree polynomial. However, since we already know that the bounds given by the Li–Yau inequality are better for large k a problem of interest is to find in what range of k our bound is an improvement of that given by Li–Yau (4). In this direction the estimates in Corollary 5.2 of [5] improve the Li–Yau inequality for $n = 2$ when $k \leq 23$. With the new improved estimates for $C(3/2, 2)$ obtained here this is increased to all $k < 40$.

Let $B(\Omega, k, n)$ be such that $\lambda_k(\Omega) \geq B(\Omega, k, n)$ is the bound implied by Theorem 1.1. In what follows we will provide, for general n, a lower bound for k^*, which is such that for any integer $k < k^*$ we have that

$$\lambda_k(\Omega) \geq B(\Omega, k, n) > \Gamma\Big(\frac{n}{2}+1\Big)^{2/n} \frac{4\pi n}{n+2}\Big(\frac{k}{|\Omega|}\Big)^{2/n},$$

where the right-hand side is the Li–Yau inequality. As in [5] we also consider the question for which λ_k, with $k > 2$, the bound is an improvement of that implied by the Krahn–Szegő inequality:

$$\lambda_k(\Omega) \geq \lambda_2(\Omega) \geq \pi\Gamma\Big(\frac{n}{2}+1\Big)^{-2/n}\Big(\frac{2}{|\Omega|}\Big)^{2/n} j^2_{n/2-1,1}, \tag{9}$$

where $j_{m,1}$ denotes the first positive zero of the Bessel function J_m.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. Then, there exist $k_*, k^* > 0$ depending only on the dimension such that for all k satisfying $k_* < k < k^*$ the lower bound*

$$\lambda_k(\Omega) \geq B(\Omega, k, n)$$

is an improvement of both the Li–Yau inequality and the bound in (9). *Moreover, for n sufficiently small the set of such k is non-empty and we have that*

$$k^* \geq \frac{3^n}{2^n}\frac{\pi^n n^n}{\Gamma\big(\frac{n}{2}+1\big)^2}\left(\frac{C(3/2,n)(n+2)^{1/2}(n+3)^{2+n/2}\Gamma(n+2)}{3\cdot 2^n n(n+3)^{(n+3)/2}\Gamma\big(\frac{n}{2}+2\big)\Gamma\big(\frac{n}{2}\big)-3^{3/2}(n+2)^{n/2}\Gamma(n+4)}\right)^n,$$

$$k_* \leq \Big(\frac{n+2}{n}\Big)^{n/2}\frac{2^{1-n}}{\Gamma\big(\frac{n}{2}+1\big)^2} j^n_{n/2-1,1}.$$

In particular, for the first few dimensions the obtained bounds are displayed in Table 2.

Table 2. The upper respectively lower bounds for k_*, k^*.

$n =$	2	3	4	5	6	7	8
$k^* \geq$	40	91	165	255	332	392	412
$k_* \leq$	6	10	16	25	38	59	91

As is indicated by Table 2 the gap between k^* and k_* has a maximum around dimension $n = 7$, after which the gap appears to close rather quickly. Using the obtained upper bounds for $C(3/2, n)$ it is not difficult to show that k^* will tend to zero as $n \to \infty$.

Proof of Theorem 3.2. By the Li–Yau inequality we know that for an open set $\Omega \subset \mathbb{R}^n$ we have that

$$\lambda_k(\Omega) \geq \Gamma\Big(\frac{n}{2}+1\Big)^{2/n} \frac{4\pi n}{n+2}\Big(\frac{k}{|\Omega|}\Big)^{2/n}.$$

Solving this for k we find that it is equivalent to the bound

$$k \leq \Big(\frac{n+2}{4\pi n}\Big)^{n/2} \frac{|\Omega|}{\Gamma\big(\frac{n}{2}+1\big)} \lambda_k^{n/2}.$$

By monotonicity this implies that

$$N(\Lambda) \leq \Big(\frac{n+2}{4\pi n}\Big)^{n/2} \frac{|\Omega|}{\Gamma\big(\frac{n}{2}+1\big)} \Lambda^{n/2} =: P_{LY}(\Lambda).$$

Using (8) we conclude from Theorem 1.1 that if $\Lambda > 0$ and $\tau > 0$, then

$$\begin{aligned} &N(\Lambda) \\ &\leq \Big(L^{\mathrm{cl}}_{3/2,n}|\Omega|\frac{(1+\tau)^{(n+3)/2}}{\tau^{3/2}}\Lambda^{n/2} - C(3/2,n)L^{\mathrm{cl}}_{3/2,n-1}|\partial\Omega|\frac{(1+\tau)^{1+n/2}}{\tau^{3/2}}\Lambda^{(n-1)/2}\Big)_+ \\ &=: P(\Lambda). \end{aligned}$$

It is clear that both $P(\Lambda)$ and $P_{LY}(\Lambda)$ are continuous and monotonically increasing for $\Lambda \geq 0$. By monotonicity the bound given by $\lambda_k \geq B(\Omega, k, n) = P^{-1}(k)$ is sharper than the Li–Yau inequality precisely when $P(\Lambda) < P_{LY}(\Lambda)$.

Further, we have that

$$L^{\mathrm{cl}}_{3/2,n}\frac{(1+\tau)^{(n+3)/2}}{\tau^{3/2}} \geq L^{\mathrm{cl}}_{3/2,n}\frac{(n+3)^{(n+3)/2}}{3^{3/2}n^{n/2}} > \Big(\frac{n+2}{4\pi n}\Big)^{n/2}\Gamma\Big(\frac{n}{2}+1\Big)^{-1},$$

where we used that the left-hand side is minimal (for $\tau > 0$) when $\tau = n/3$. And thus the polynomial P_{LY} is asymptotically larger than P. Hence it is clear that there exists a unique $\Lambda^* > 0$ such that $P(\Lambda^*) = P_{LY}(\Lambda^*)$. Correspondingly, if we let k^* be such that for all $k < k^*$ we have

$$P^{-1}(k) > P^{-1}_{LY}(k) = \Gamma\Big(\frac{n}{2}+1\Big)^{2/n}\frac{4\pi n}{n+2}\Big(\frac{k}{|\Omega|}\Big)^{2/n},$$

then k^* is the smallest integer larger than $P(\Lambda^*) = P_{LY}(\Lambda^*)$. By monotonicity finding a lower bound for Λ^* corresponds to finding a lower bound for k^*.

We proceed by calculating $\Lambda^* = \Lambda^*(n, \Omega, \tau)$. After equating the two polynomials, a simple calculation using that $\Lambda^* > 0$ gives us the solution

$$\begin{aligned} \Lambda^* &= \left(\frac{C(3/2,n)|\partial\Omega|\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n-1}\frac{(1+\tau)^{1+n/2}}{\tau^{3/2}}}{|\Omega|\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n}\frac{(1+\tau)^{(n+3)/2}}{\tau^{3/2}} - |\Omega|\big(\frac{n+2}{4\pi n}\big)^{n/2}}\right)^2 \\ &= \left(\frac{C(3/2,n)\,\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n-1}(1+\tau)^{1+n/2}}{\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n}(1+\tau)^{(n+3)/2} - \big(\frac{n+2}{4\pi n}\big)^{n/2}\tau^{3/2}}\right)^2\frac{|\partial\Omega|^2}{|\Omega|^2}. \end{aligned}$$

We can now insert this expression into either of our two polynomials to attempt to estimate k^*. Since P_{LY} is a monomial it makes our computations slightly simpler,

$$\begin{aligned} P_{LY}(\Lambda^*) &= \Big(\frac{n+2}{4\pi n}\Big)^{n/2}\frac{|\Omega|}{\Gamma\big(\frac{n}{2}+1\big)}(\Lambda^*)^{n/2} \\ &= \Big(\frac{n+2}{4\pi n}\Big)^{n/2}\frac{1}{\Gamma\big(\frac{n}{2}+1\big)}\Bigg(\frac{C(3/2,n)\,\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n-1}(1+\tau)^{1+n/2}}{\Gamma\big(\frac{n}{2}+1\big)L^{\mathrm{cl}}_{3/2,n}(1+\tau)^{(n+3)/2}-\big(\frac{n+2}{4\pi n}\big)^{n/2}\tau^{3/2}}\Bigg)^n\frac{|\partial\Omega|^n}{|\Omega|^{n-1}}. \end{aligned}$$

Note that $P_{LY}(\Lambda^*)$ behaves very nicely with respect to both the isoperimetric ratio of our domain and the constant $C(3/2,n)$.

Now as this expression is rather messy, especially in its dimensional dependence, it is not the easiest task to calculate its integer part. Even trying to optimize this in τ is a rather intricate problem. But considering where τ comes from in our argument, and that the bound holds for any $\tau > 0$, we simply choose the τ which minimizes the leading coefficient of $P(\Lambda)$. A simple calculation shows that this is attained at $\tau = 3/n$. Inserting this into the expression above we lose the dependence of τ and obtain that

$$k^* \geq \frac{3^n}{2^n}\frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)}\Bigg(\frac{C(3/2,n)(n+2)^{1/2}(n+3)^{2+n/2}\Gamma(n+2)}{3\cdot 2^n n(n+3)^{(n+3)/2}\Gamma\big(\frac{n}{2}+2\big)\Gamma\big(\frac{n}{2}\big)-3^{3/2}(n+2)^{n/2}\Gamma(n+4)}\Bigg)^n\frac{|\partial\Omega|^n}{|\Omega|^{n-1}}.$$

Using the isoperimetric inequality we may further bound this, and thus also lose the domain dependence. This gives the following bound which now depends only on the dimension:

$$k^* \geq \frac{3^n}{2^n}\frac{\pi^n n^n}{\Gamma\big(\frac{n}{2}+1\big)^2}\Bigg(\frac{C(3/2,n)(n+2)^{1/2}(n+3)^{2+n/2}\Gamma(n+2)}{3\cdot 2^n n(n+3)^{(n+3)/2}\Gamma\big(\frac{n}{2}+2\big)\Gamma\big(\frac{n}{2}\big)-3^{3/2}(n+2)^{n/2}\Gamma(n+4)}\Bigg)^n.$$

As in [5] we can supplement these bounds from below. Let Λ_{KZ} denote the bound for $\lambda_2(\Omega)$ given by (9), that is,

$$\Lambda_{KZ} = \pi\Gamma\Big(\frac{n}{2}+1\Big)^{-2/n}\Big(\frac{2}{|\Omega|}\Big)^{2/n} j^2_{n/2-1,1},$$

where again $j_{m,1}$ denotes the first positive zero of the Bessel function J_m. By the same reasoning as before we can conclude that $k_* \leq P(\Lambda_{KZ})$. If $\Lambda_{KZ} \leq \Lambda^*$ we have that $P(\Lambda_{KZ}) \leq P_{LY}(\Lambda_{KZ})$ thus if this is the case $P_{LY}(\Lambda_{KZ})$ is an upper bound for k_*. Moreover, if $\Lambda_{KZ} > \Lambda^*$, then $k_* \geq k^*$ and the range of k where our implicit bounds improve the Li–Yau bound and that implied by Krahn–Szegő is empty, and therefore we can restrict our interest to the first case. Calculating we find that

$$k_* \leq P_{LY}(\Lambda_{KZ}) = \Big(\frac{n+2}{n}\Big)^{n/2}\frac{2^{1-n}}{\Gamma\big(\frac{n}{2}+1\big)^2} j^n_{n/2-1,1},$$

which completes the proof. □

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References

[1] Michael Aizenman and Elliott H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429, DOI 10.1016/0375-9601(78)90385-7. MR598768

[2] F. A. Berezin, *Covariant and contravariant symbols of operators* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167. MR0350504

[3] R. Courant and D. Hilbert, *Methoden der mathematischen Physik. 1. Band. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 12.)*, Berlin: J. Springer, gr. 8°, XIII, 450 S (1924), 1924.

[4] Rupert L. Frank and Leander Geisinger, *Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain*, Mathematical results in quantum physics, World Sci. Publ., Hackensack, NJ, 2011, pp. 138–147, DOI 10.1142/9789814350365_0012. MR2885166

[5] Leander Geisinger, Ari Laptev, and Timo Weidl, *Geometrical versions of improved Berezin-Li-Yau inequalities*, J. Spectr. Theory **1** (2011), no. 1, 87–109, DOI 10.4171/JST/4. MR2820887

[6] L. Geisinger and T. Weidl, *Universal bounds for traces of the Dirichlet Laplace operator*, J. Lond. Math. Soc. (2) **82** (2010), no. 2, 395–419, DOI 10.1112/jlms/jdq033. MR2725046

[7] Lars Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218. MR0609014

[8] V. Ja. Ivriĭ, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary* (Russian), Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34. MR575202

[9] Victor Ivrii, *Microlocal analysis and precise spectral asymptotics*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1631419

[10] Hynek Kovařík, Semjon Vugalter, and Timo Weidl, *Two-dimensional Berezin-Li-Yau inequalities with a correction term*, Comm. Math. Phys. **287** (2009), no. 3, 959–981, DOI 10.1007/s00220-008-0692-1. MR2486669

[11] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151** (1997), no. 2, 531–545, DOI 10.1006/jfan.1997.3155. MR1491551

[12] Ari Laptev and Timo Weidl, *Sharp Lieb-Thirring inequalities in high dimensions*, Acta Math. **184** (2000), no. 1, 87–111, DOI 10.1007/BF02392782. MR1756570

[13] Simon Larson, *A bound for the perimeter of inner parallel bodies*, J. Funct. Anal. **271** (2016), no. 3, 610–619, DOI 10.1016/j.jfa.2016.02.022. MR3506959

[14] Peter Li and Shing Tung Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, 309–318. MR701919

[15] Antonios D. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 631–636 (electronic), DOI 10.1090/S0002-9939-02-06834-X. MR1933356

[16] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc. (3) **11** (1961), 419–433. MR0129219

[17] M. H. Protter, *A lower bound for the fundamental frequency of a convex region*, Proc. Amer. Math. Soc. **81** (1981), no. 1, 65–70, DOI 10.2307/2043987. MR589137

[18] Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Translations of Mathematical Monographs, vol. 155, American Mathematical Society, Providence, RI, 1997. Translated from the Russian manuscript by the authors. MR1414899

[19] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183

[20] Hajime Urakawa, *Lower bounds for the eigenvalues of the fixed vibrating membrane problems*, Tohoku Math. J. (2) **36** (1984), no. 2, 185–189, DOI 10.2748/tmj/1178228846. MR742593

[21] M. van den Berg, *A uniform bound on trace$(e^{t\Delta})$ for convex regions in $\mathbf{R}^n$ with smooth boundaries*, Comm. Math. Phys. **92** (1984), no. 4, 525–530. MR736409

[22] Timo Weidl, *Improved Berezin-Li-Yau inequalities with a remainder term*, Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 253–263, DOI 10.1090/trans2/225/17. MR2509788

[23] Hermann Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)* (German), Math. Ann. **71** (1912), no. 4, 441–479, DOI 10.1007/BF01456804. MR1511670

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Paper C

Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains

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Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains

Simon Larson

Abstract. For $\Omega \subset \mathbb{R}^n$, a convex and bounded domain, we study the spectrum of $-\Delta_\Omega$ the Dirichlet Laplacian on Ω. For $\Lambda \geq 0$ and $\gamma \geq 0$ let $\Omega_{\Lambda,\gamma}(\mathcal{A})$ denote any extremal set of the shape optimization problem

$$\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\},$$

where $\mathcal{A}$ is an admissible family of convex domains in $\mathbb{R}^n$. If $\gamma \geq 1$ and $\{\Lambda_j\}_{j\geq 1}$ is a positive sequence tending to infinity we prove that $\{\Omega_{\Lambda_j,\gamma}(\mathcal{A})\}_{j\geq 1}$ is a bounded sequence, and hence contains a convergent subsequence. Under an additional assumption on $\mathcal{A}$ we characterize the possible limits of such subsequences as minimizers of the perimeter among domains in $\mathcal{A}$ of unit measure. For instance if $\mathcal{A}$ is the set of all convex polygons with no more than m faces, then $\Omega_{\Lambda,\gamma}$ converges, up to rotation and translation, to the regular m-gon.

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Keywords. Shape optimization, Riesz eigenvalue means, eigenvalue sums, Dirichlet–Laplace operator, Weyl asymptotics, convexity.

Contents

1. Introduction and main results

1.1. Introduction. This paper deals with the existence of an asymptotically optimal shape in a certain family of shape optimization problems. By a shape optimization problem we mean a variational problem where given a cost functional $\mathcal{F}$ and an admissible class of domains $\mathcal{A}$ one wishes to solve the optimization problem

$$\inf\{\mathcal{F}(\Omega) : \Omega \in \mathcal{A}\}.$$

For an introduction to the general theory of shape optimization we refer the reader to the books [10, 28].

In recent years the study of shape optimization for spectral problems, where the cost functional $\mathcal{F}$ depends on the spectrum of an operator defined on Ω, has been of large interest, see for instance [27] and references therein. This type of problem has a long history which can be traced back to Lord Rayleigh [50] who conjectured that the disk minimizes the first eigenvalue of the Dirichlet Laplacian among all planar domains of fixed area. Rayleigh's conjecture was proved independently by Faber [15] and Krahn [32]; the latter of whom also generalized the result to higher dimensions [33]. From this result one can prove a similar statement concerning the second eigenvalue, namely that it is minimized by the union of two disjoint balls of equal measure [32, 33, 54]. For even higher eigenvalues the corresponding problems have only in recent years seen much progress. Using techniques coming from free boundary problems in partial differential equations it has been possible to prove the existence of extremal sets within the larger class of quasi-open sets[1] for the problem

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n \text{ quasi-open}, |\Omega| = 1\},$$

where $\lambda_k(\Omega)$ denotes the k-th eigenvalue of the Dirichlet Laplacian on Ω (see [9, 12, 13, 46]). Within the same framework one can treat more general functionals depending on the eigenvalues of some spectral problem (see [12, 44, 46, 55]).

Here we are interested in a two-parameter family of spectral shape optimization problems for the Dirichlet Laplacian, parametrized by $\gamma, \Lambda \geq 0$ in (2) below. In the case $\gamma = 0$ the problem essentially reduces to that of minimizing individual eigenvalues of the Dirichlet Laplacian but formulated in terms of the *eigenvalue counting function*:

$$N_\Omega(\Lambda) := \#\{k \in \mathbb{N} : \lambda_k(\Omega) < \Lambda\}. \tag{1}$$

Here we shall mainly consider the case $\gamma \geq 1$.

[1] A quasi-open set is a superlevel set of a function in $H^1(\mathbb{R}^n)$, for a precise definition see [13].

For $\gamma \geq 1$ and $\Lambda \geq 0$ the cost functionals we consider fit into the above mentioned framework for proving existence of extremal sets in the class of quasi-open sets of fixed measure. In the case $\gamma = 1$ the problem is equivalent to that of minimizing the sum of the first m eigenvalues for certain values of m, and thus it follows from [12, 47] that the optimal sets are open and their boundary is smooth up to exceptional sets of lower dimension. For $\gamma > 1$ the question of whether the extremal sets are open is to the author's knowledge not covered by existing theory. However, this will not be the question dealt with in this paper. Instead, we restrict ourselves to the much simpler case of considering the problem when restricting the admissible class $\mathcal{A}$ to certain families of convex domains. Before we are able to properly define the functional considered it is necessary to introduce some additional notation.

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, and let $-\Delta_\Omega$ denote the Dirichlet Laplace operator on $L^2(\Omega)$, which we define in the quadratic form sense with the Sobolev space $H_0^1(\Omega)$ as its form domain. If we assume that the measure of Ω is finite then the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and hence the spectrum of $-\Delta_\Omega$ is discrete. Moreover, the spectrum consists of an infinite sequence of positive eigenvalues accumulating at infinity only. We enumerate these eigenvalues in an increasing sequence where each eigenvalue is repeated according to its multiplicity,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots .$$

An open ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$ will be denoted by $B_r(x)$; if the centre of the ball is irrelevant we write simply B_r. For the ball of unit measure centred at the origin we write B.

We can now define the two-parameter family of functionals studied here. For $\gamma \geq 0$ and $\Lambda \geq 0$ the *Riesz eigenvalue means* of $-\Delta_\Omega$ are defined by

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = \sum_{k:\lambda_k(\Omega)<\Lambda} (\Lambda - \lambda_k(\Omega))^\gamma, \tag{2}$$

where $x_\pm := (|x| \pm x)/2$.

Given $\gamma \geq 0$, $\Lambda \geq 0$ and an admissible class of domains $\mathcal{A}$, we are interested in the shape optimization problem

$$\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\}. \tag{3}$$

Here and in what follows we denote the n-dimensional measure of a set $\Omega \subset \mathbb{R}^n$ by $|\Omega|$ and the $(n-1)$-dimensional measure of its boundary by $|\partial\Omega|$. For fixed γ, Λ, and $\mathcal{A}$ let $\Omega_{\Lambda,\gamma}(\mathcal{A})$ denote any extremal domain of (3). We emphasize that it is not a priori clear that any such domain exists. We shall here restrict our attention to $\gamma \geq 1$ and admissible classes $\mathcal{A}$ which are families of convex domains; without loss of generality we shall always assume that the admissible class $\mathcal{A}$ is closed under rigid transformations and contains at least one domain of unit measure. For such $\mathcal{A}$ the existence of extremal domains $\Omega_{\Lambda,\gamma}(\mathcal{A})$ will be proved in Lemma 3.1 below.

We note that for $\gamma = 0$ the Riesz mean is equal to the counting function of eigenvalues less than Λ. Thus, in this case (3) is in a sense dual to the problem of minimizing $\lambda_k(\Omega)$.

Moreover, the problem of maximizing the Riesz mean of order $\gamma = 1$ is equivalent to minimizing the sum of the m first eigenvalues for certain values of m. Indeed, since $\Omega_{\Lambda,1}(\mathcal{A})$ is extremal for (3) with $\gamma = 1$, we have for any $\Omega \in \mathcal{A}$ with $|\Omega| = 1$ that

$$\operatorname{Tr}(-\Delta_{\Omega_{\Lambda,1}(\mathcal{A})} - \Lambda)_- \geq \operatorname{Tr}(-\Delta_\Omega - \Lambda)_-. \tag{4}$$

By definition (4) is equivalent to

$$\sum_{k \leq N_{\Omega_{\Lambda,1}(\mathcal{A})}(\Lambda)} \lambda_k(\Omega_{\Lambda,1}(\mathcal{A})) \leq \sum_{k \leq N_\Omega(\Lambda)} \lambda_k(\Omega) - \Lambda(N_\Omega(\Lambda) - N_{\Omega_{\Lambda,1}(\mathcal{A})}(\Lambda)).$$

We claim that the right-hand side is no larger than the sum of the $N_{\Omega_{\Lambda,1}(\mathcal{A})}(\Lambda)$ first eigenvalues of $-\Delta_\Omega$. To this end let $m = N_{\Omega_{\Lambda,1}(\mathcal{A})}(\Lambda)$ and $\hat{m} = N_\Omega(\Lambda)$. If $m = \hat{m}$ the claim is clearly true. If $m < \hat{m}$ then

$$\begin{aligned}
\sum_{k \leq \hat{m}} \lambda_k(\Omega) - \Lambda(\hat{m} - m) &= \sum_{k \leq m} \lambda_k(\Omega) + \sum_{k=m+1}^{\hat{m}} \lambda_k(\Omega) - \Lambda(\hat{m} - m) \\
&< \sum_{k \leq m} \lambda_k(\Omega),
\end{aligned}$$

where we in the last step used that $\lambda_k(\Omega) < \Lambda$ for each $k \leq \hat{m}$. The remaining case follows almost identically. Hence $\Omega_{\Lambda,1}(\mathcal{A})$ is also extremal for the shape optimization problem

$$\inf\Big\{\sum_{k=1}^m \lambda_k(\Omega) \colon \Omega \in \mathcal{A},\ |\Omega| = 1\Big\},$$

with $m = N_{\Omega_{\Lambda,1}(\mathcal{A})}(\Lambda)$.

1.2. Main results. Let $\mathcal{K}^n$ denote the metric space defined as the set of all bounded convex domains $\Omega \subset \mathbb{R}^n$ with non-empty interior equipped with the Hausdorff distance [51]. We shall in this paper restrict our classes of admissible domains $\mathcal{A}$ to consisting of certain subsets of $\mathcal{K}^n$. In an upcoming paper it will be shown that these restrictions can be dropped [19]. We begin by defining two natural classes of convex domains.

(A) For an integer $m \geq n+1$ we let $\mathcal{P}_m \subset \mathcal{K}^n$ be the set of all bounded convex polytopes in $\mathbb{R}^n$ with no more than m faces. We note that $\mathcal{P}_m$ is a closed subset of $\mathcal{K}^n$: if a sequence $\{P_j\}_{j\geq 1} \subset \mathcal{P}_m$ converges to $P \in \mathcal{K}^n$ in the topology of $\mathcal{K}^n$, then $P \in \mathcal{P}_m$.

(B) Fix a continuous increasing function $\omega\colon [0, L) \to \mathbb{R}$, with $\omega(0) = 0$. Let $x \in \partial\Omega$, after rotation and translation we assume that $x = 0$ and the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$ is tangent to $\partial\Omega$ at x. Let D be the projection of $\partial\Omega \cap B_{L/2}$ onto this hyperplane. If $B_{L/2} \cap \partial\Omega$ can be represented as the graph of a function $f \in C^1(D)$ which satisfies

$$|\nabla f(x') - \nabla f(y')| \leq \omega(|x'-y'|), \quad \text{for all } x', y' \in D$$

we say that $\partial\Omega$ has C^1-modulus of continuity ω around x. We say that $\partial\Omega$ is ω-uniformly C^1 if this holds true with the same ω at every $x \in \partial\Omega$.

We let $\mathcal{K}^n_\omega$ denote the set of all $\Omega \in \mathcal{K}^n$ whose boundary is ω-uniformly C^1. By the uniform regularity assumption it follows that also $\mathcal{K}^n_\omega$ is a closed subset of $\mathcal{K}^n$: if a sequence $\{K_j\}_{j\geq 1} \subset \mathcal{K}^n_\omega$ converges to $K \in \mathcal{K}^n$ in the topology of $\mathcal{K}^n$, then $K \in \mathcal{K}^n_\omega$. We shall always assume that ω is such that $\mathcal{K}^n_\omega$ contains at least one domain of unit measure.

Our main results are contained in the following theorems.

Theorem 1.1. *Fix $\gamma \geq 1$ and $m \geq n+1$. Let $\{\Lambda_j\}_{j\geq 1} \subset \mathbb{R}_+$ be a sequence tending to infinity, and choose for each j a corresponding extremal domain $\Omega_j = \Omega_{\Lambda_j,\gamma}(\mathcal{P}_m)$.*

Then the sequence $\{\Omega_j\}_{j\geq 1}$ has a subsequence which, up to rigid transformations, converges to a domain $P_m \in \mathcal{P}_m$. Moreover, P_m is of unit measure and minimizes the measure of the perimeter among domains in $\mathcal{P}_m$ of the same measure:

$$|\partial P_m| = \inf\{|\partial\Omega| : \Omega \in \mathcal{P}_m,\ |\Omega| = 1\}.$$

We also prove the corresponding result in $\mathcal{K}^n_\omega$.

Theorem 1.2. *Fix $\gamma \geq 1$ and ω as in (B) above. Let $\{\Lambda_j\}_{j\geq 1} \subset \mathbb{R}_+$ be a sequence tending to infinity, and choose for each j a corresponding extremal domain $\Omega_j = \Omega_{\Lambda_j,\gamma}(\mathcal{K}^n_\omega)$.*

Then the sequence $\{\Omega_j\}_{j\geq 1}$ has a subsequence which, up to rigid transformations, converges to a domain $K_\omega \in \mathcal{K}^n_\omega$. Moreover, K_ω is of unit measure and minimizes the measure of the perimeter among domains in $\mathcal{K}^n_\omega$ of the same measure:

$$|\partial K_\omega| = \inf\{|\partial\Omega| \colon \Omega \in \mathcal{K}^n_\omega,\ |\Omega| = 1\}.$$

We note that if $\mathcal{A}$ is one of the admissible classes considered above, then the existence of a set Ω' realizing the infimum

$$\inf\{|\partial\Omega| : \Omega \in \mathcal{A},\ |\Omega| = 1\} \tag{5}$$

is an easy consequence of the strong compactness properties of $\mathcal{K}^n$ [51]. If the set Ω' is unique, up to rigid transformations, then for any choice of sequence $\{\Lambda_j\}_{j\geq 1}$ we find that the corresponding sequence of maximizers converges to Ω'. Since the choice of sequence was arbitrary we obtain that

$$\inf_{\substack{x_0\in\mathbb{R}^n\\ T\in \mathrm{O}(n)}} \mathrm{dist}_{\mathcal{K}^n}(\Omega_{\Lambda,\gamma}(\mathcal{A}), x_0 + T\Omega') = o(1) \quad \text{as } \Lambda \to \infty, \tag{6}$$

where $\mathrm{O}(n)$ is the orthogonal group in dimension n and $\mathrm{dist}_{\mathcal{K}^n}$ denotes the metric of $\mathcal{K}^n$. Since we do not know that the maximizers $\Omega_{\Lambda,\gamma}(\mathcal{A})$ are unique we emphasize that we mean that (6) is true when an arbitrary choice of maximizer is made for each Λ.

In particular if ω is such that the unit ball $B \in \mathcal{K}^n_\omega$ then it is up to translations the unique minimizer of (5) and hence $\Omega_{\Lambda,\gamma}(\mathcal{K}^n_\omega)$ converges to B, modulo translations. If the ball is not in $\mathcal{K}^n_\omega$ then minimizers of the perimeter need not be unique and different subsequences of $\Omega_{\Lambda_j,\gamma}(\mathcal{K}^n_\omega)$ may converge to different such minimizers.

The existence and characterization of minimizers of the perimeter in the class $\mathcal{P}_m$ is a classical problem. This problem is equivalent to that of finding which polytopes circumscribing a ball have minimal volume [23]. For $n = 2$ the regular m-gon is, up to rotations and translations, the unique minimizer. However, in higher dimensions this turns out to be a very difficult problem, and to the author's knowledge it is not known whether the minimizers are unique.

If Ω' realizing (5) is not unique then one can still conclude that all isolated minimizers of the perimeter are local asymptotic maximizers of our shape optimization problem in the following sense. Let $\Omega' \in \mathcal{A}$ realize the infimum (5) and assume that Ω' is isolated from any other such minimizer with respect to the Hausdorff topology (up to rigid transformations). Then one can construct a perturbed shape optimization problem by removing from $\mathcal{A}$ an arbitrarily small neighbourhood around all other minimizers of the perimeter (in the Hausdorff sense). For this new shape optimization problem any sequence of maximizers would converge to the now unique minimizer of the perimeter.

1.3. Related results and further questions. Similar results in asymptotics of extremal domains have recently been obtained in several different settings. The most commonly studied problem is that of finding a domain asymptotically minimizing $\lambda_k(\Omega)$ among Ω in a certain class of admissible domains. That is, given an admissible class of domains $\mathcal{A}$ one wants to find a domain Ω_∞ such that the extremizers of the problem

$$\inf\{\lambda_k(\Omega) \colon \Omega \in \mathcal{A}\} \tag{7}$$

converge to Ω_∞ as k goes to infinity.

The first result in this direction is due to Antunes and Freitas who proved that if $\mathcal{A}$ is the set of rectangles with area one, then any sequence of extremal sets converges to the unit square as k goes to infinity [2]. In [7] van den Berg and Gittins proved the corresponding result in three dimensions, and in [22] the result was obtained in general dimension. In the class $\mathcal{A}$ of sets of the form $(0, a_1) \times \cdots \times (0, a_n) \subset \mathbb{R}^n$ of unit measure any sequence of minimizers of the k-th eigenvalue converges to the unit cube in $\mathbb{R}^n$ as $k \to \infty$. In [6, 22] the corresponding results were proved to hold also if one instead considers eigenvalues of the Neumann Laplacian on the same class of domains, in which case the natural problem is to maximize the eigenvalues.

The idea of Antunes and Freitas [2] was to reformulate the problem of minimizing eigenvalues as a maximization problem for the counting function (1) and exploit the explicit structure of Laplacian eigenvalues on rectangles. This effectively reformulates the problem as an optimization problem in the setting of geometric lattice point counting. For fixed $\Lambda \geq 0$ find the ellipses among those on the form $(x/a)^2 + (ay)^2 \leq \Lambda/\pi^2$ which contain the greatest number of positive integer lattice points. The asymptotic problem translates into studying the shape of such ellipses in the limit $\Lambda \to \infty$.

The lattice point problem which arose in the work of Antunes and Freitas has since then seen several generalizations. Laugesen and Liu [38] and Ariturk and Laugesen [3] consider a similar problem but replace the bounding region, which in [2] was given by a quarter of an ellipse, by the region under the graph of a decreasing concave or convex function f. The optimization problem studied can then be phrased as follows. Given $r > 0$ find $s > 0$ realizing

$$\sup_{s>0} \#\{(j,k) \in \mathbb{N}^2 : k \le rsf(js/r)\}.$$

The main results of [3, 38] are that under weak assumptions on f the optimal values of s tend to a unique limit as $r \to \infty$. Moreover, the limit can be explicitly expressed in terms of f. More recently these results have been generalized to allow for a shift of the lattice, that is replacing the standard lattice by $(\mathbb{N}+\sigma) \times (\mathbb{N}+\tau)$, see [39]. Also higher dimensional versions of this problem have been studied by Marshall, and Guo and Wang in [24, 42].

A particularly interesting case of the lattice point optimization problem is to consider $f(x) = 1-x$. In this case the behaviour of maximizing values s is highly erratic, and it was proved by Marshall and Steinerberger [43] that there is no unique limit as r tends to infinity. In fact they prove that there are infinitely many values of s which are optimal for arbitrarily large r, which proves a conjecture of Laugesen and Liu in [38]. Recently a related problem but in the setting of Riesz means was studied in [37].

In the same direction as the work of Antunes and Freitas, one can consider the shape optimization problem (7) as k tends to infinity but with the measure constraint replaced by different ones, see [5, 11, 20]. In particular, Bucur and Freitas considered the problem in $\mathbb{R}^2$ under a constraint on the measure of the perimeter and prove that any sequence of optimal domains converges to the disk [11]: if $\Omega_k \subset \mathbb{R}^2$ is a domain realizing the infimum

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open}, |\partial\Omega| \le |\partial B|\},$$

then, up to translation, $\lim_{k\to\infty} \Omega_k = B$ in the Hausdorff topology. In [20] Freitas considered the problem of minimizing the average (or equivalently the sum) of the m first eigenvalues in the limit as m tends to infinity under a constraint on either the measure or the perimeter. In the former case he obtains that the extremal averages are in a certain sense sub-additive and compute their leading order asymptotic behaviour. In the latter he proves that the extremal sets converge to a ball in the limit $m \to \infty$.

The fact that the problem studied here allows the same type of analysis as in the results discussed above under the constraint of fixed measure, and in large classes of convex sets is the main reason that we find it noteworthy. Moreover, after this article was completed it has been proved that the a priori regularity assumptions on $\mathcal{A}$ needed to identify the asymptotically maximizing domains as minimizers of the perimeter can be removed. That these assumptions can be dropped is a consequence of the results in [19] where two-term asymptotic formulas for $\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-$ are obtained in the semi-classical limit $\Lambda \to \infty$, under the assumption that $\Omega \subset \mathbb{R}^n$ has Lipschitz-regular boundary (in particular this covers all convex domains).

A natural further question is of course whether the convexity assumption can be dropped, and instead consider the optimization problem (3) in the class of quasi-open sets. As mentioned in the introduction the existence of optimizers for this problem with $\gamma \geq 1$ is covered by the results in [9, 13, 45, 46] (see also [44, 55]). The results in these articles consider the case of functions of a *fixed* number of eigenvalues which is not the case for (2). However, using the Li–Yau inequality [40],

$$\lambda_k(\Omega) \geq \Gamma\Big(\frac{n}{2}+1\Big)^{2/n} \frac{4\pi n}{n+2}\Big(\frac{k}{|\Omega|}\Big)^{2/n}, \tag{8}$$

we can bound the number of eigenvalues present in the sum (2) and thus reduce our problem to this situation. If $\Lambda > \lambda_1(B)$ and $\gamma \geq 1$ then the functional $\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma$ is Lipschitz continuous as a function of the eigenvalues and weakly strictly decreasing in a neighbourhood of any maximizer (see [45]). Hence the problem is covered by the existing results for such cost functions. However, in terms of what happens as $\Lambda \to \infty$ these results yield little information. To analyse the asymptotic behaviour of maximizers we here use inequalities for $\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma$ in terms of geometric quantities of Ω, see Theorem 2.4. Similar inequalities have in recent years been obtained in a variety of different forms, see e.g. [21, 25, 26, 31, 48, 56]. These inequalities indeed provide geometric information about maximizers in a more general setting. However, without the convexity assumption it is unclear whether this information is sufficient to prove that maximizers cannot degenerate as $\Lambda \to \infty$.

1.4. Structure of the paper. The remainder of the paper is structured as follows. In Section 2 we introduce some notation, recall some known results, and prove a number of inequalities needed in the sequel. Section 3 is devoted to proving that given an admissible class of convex domains $\mathcal{A}$ the shape optimization problem (3) has at least one extremal domain for fixed values of Λ and γ. In Section 4 we establish that for $\gamma \geq 1$ any sequence of extremal domains has a convergent subsequence, and show that under an additional assumption on the class $\mathcal{A}$ any limit point of the sequence must be a minimizer of the perimeter. In Section 5 we show that the tools developed to prove our main theorems also allow us to deduce the corresponding results when minimizing the sum of the first m eigenvalues among convex domains. Section 6 is devoted to studying the asymptotic behaviour of (2) as $\Lambda \to \infty$, and proving that the assumption from Section 4 holds true in $\mathcal{P}_m$. That the same assumption is true in $\mathcal{K}^n_\omega$ is a consequence of the results in [16, 17] (see Lemma 2.2). We end the paper by proving that our results generalize to the case when the admissible domains are allowed to consist of disjoint unions of convex domains, see Section 7.

2. Notation and preliminaries

We denote by $\operatorname{dist}(\cdot\,,\cdot)$ the distance between two sets in $\mathbb{R}^n$ (possibly singletons):

$$\operatorname{dist}(A, B) := \inf_{x\in A, y\in B} |x - y|.$$

We will in several places make use of the fact that if $\partial\Omega$ is Lipschitz regular then $\operatorname{dist}(\cdot\,, \partial\Omega)$ satisfies

$$|\nabla \operatorname{dist}(x, \partial\Omega)| = 1 \tag{9}$$

for almost every $x \in \mathbb{R}^n$. In particular this holds true as soon as Ω is convex.

2.1. Preliminary convex geometry.

We continue by recalling some basic definitions from convex geometry and introducing the notation we use. For more details and a general treatment of classical convex geometry we refer the reader to the books [23, 51].

Let $\Omega \in \mathcal{K}^n$. We define the *inradius*, *diameter*, and *minimal width* of Ω by

$$r(\Omega) := \sup_{x\in\Omega} \operatorname{dist}(x, \Omega^c),$$

$$D(\Omega) := \sup_{x,y\in\Omega} |x - y|,$$

resp.

$$w(\Omega) := \inf_{\nu\in\mathbb{S}^{n-1}} \big(\sup_{x\in\Omega} x\cdot\nu - \inf_{x\in\Omega} x\cdot\nu\big).$$

We note that r is the radius of the largest ball contained in Ω, and w is the smallest distance such that Ω is contained between two parallel hyperplanes separated by this distance.

Clearly $2r(\Omega) \le w(\Omega)$. Less clear is that also a reversed inequality holds [53]: there exists a dimensional constant $c > 0$ such that, for all $\Omega \in \mathcal{K}^n$,

$$cw(\Omega) \le r(\Omega). \tag{10}$$

The *inner* and *outer parallel sets* of Ω at distance $t \in (0, \infty)$ are defined by

$$\Omega_t := \Omega \sim B_t = \{x \in \Omega : \mathrm{dist}(x, \Omega^c) > t\},$$

$$\Omega^t := \Omega + B_t = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) < t\}.$$

The notation $+, \sim$ comes from the concepts of Minkowski addition and the Minkowski difference [51].

We let $W : (\mathcal{K}^n)^n \to \mathbb{R}$ denote the unique symmetric function (with respect to permutations of its arguments) such that

$$|\eta_1\Omega_1 + \cdots + \eta_m\Omega_m| = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \eta_{j_1} \cdots \eta_{j_n} W(\Omega_{j_1}, \dots, \Omega_{j_n}), \tag{11}$$

for any $\Omega_1, \dots, \Omega_m \in \mathcal{K}^n$ and $\eta_1, \dots, \eta_m \ge 0$ [51]. The quantity $W(\Omega_1, \dots, \Omega_n)$ is called the *mixed volume* of $\Omega_1, \dots, \Omega_n \in \mathcal{K}^n$. Here we will only need the following elementary properties of W (see [51]).

1. $W(\Omega_1, \dots, \Omega_n) > 0$ for $\Omega_1, \dots, \Omega_n \in \mathcal{K}^n$.
2. W is a multilinear function with respect to Minkowski addition.
3. W is increasing with respect to inclusions.
4. W is invariant under translations in each argument.
5. The volume and perimeter of $\Omega \in \mathcal{K}^n$ can be written in terms of W:

$$|\Omega| = W(\Omega, \dots, \Omega) \quad \text{and} \quad |\partial\Omega| = nW(\Omega, \dots, \Omega, B_1).$$

In what follows we shall need to bound certain of these quantities in terms of others; these and similar bounds can be found in the literature but we include proofs for completeness. To this end we recall the main result of [35]: for $t \ge 0$ and any $\Omega \in \mathcal{K}^n$ it holds that

$$|\partial\Omega_t| \ge |\partial\Omega|\Big(1 - \frac{t}{r(\Omega)}\Big)_+^{n-1}. \tag{12}$$

Since the measure of the perimeter of convex sets is decreasing under set inclusion we also have that $|\partial\Omega_t| \le |\partial\Omega|$.

Using the co-area formula and (9) we have that

$$|\Omega| = \int_0^{r(\Omega)} |\partial\Omega_t|\, dt.$$

By the upper, respectively lower, bound on $|\partial\Omega_t|$ above we find, after integrating and rearranging, that

$$\frac{|\Omega|}{|\partial\Omega|} \le r(\Omega) \le n\frac{|\Omega|}{|\partial\Omega|}. \tag{13}$$

Furthermore, it is not difficult to deduce an upper bound for $D(\Omega)$ in terms of $r(\Omega)$ and $|\Omega|$. After translation and rotation we may assume that the ball $B_{r(\Omega)}(0) \subset \Omega$ and that $x_0 = (0, \dots, 0, R) \in \Omega$. By convexity the cone V with vertex x_0 and base $\{x \in \mathbb{R}^n : x_n = 0,\ x_1^2 + \cdots + x_{n-1}^2 = r(\Omega)^2\}$ is contained in Ω. The volume of this cone is equal to

$$|V| = cr(\Omega)^{n-1}\int_0^R \Big(1 - \frac{x_n}{R}\Big)^{n-1} dx_n = cr(\Omega)^{n-1}R.$$

Thus we have a contradiction if $cr(\Omega)^{n-1}R \ge |\Omega|$ and hence $R \le c\frac{|\Omega|}{r(\Omega)^{n-1}}$. Consequently there is a constant $c > 0$, depending only on n, such that

$$D(\Omega) \le c\frac{|\Omega|}{r(\Omega)^{n-1}}. \tag{14}$$

2.2. Weyl asymptotics. From the classical Weyl asymptotics for the Dirichlet eigenvalues (see [57]) it follows that the Riesz means for $\gamma \ge 0$ obey the asymptotic formula

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L_{\gamma,n}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+n/2} + o(\Lambda^{\gamma+n/2}) \quad \text{as } \Lambda \to \infty.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded and open set and $L_{\gamma,n}^{\mathrm{cl}}$ denotes the semi-classical Lieb–Thirring constant:

$$L_{\gamma,n}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{(4\pi)^{n/2}\Gamma(\gamma+1+n/2)}.$$

If in addition Ω satisfies certain regularity properties the following two-term asymptotic formula holds:

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L_{\gamma,n}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+n/2} - \frac{L_{\gamma,n-1}^{\mathrm{cl}}}{4}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \tag{15}$$

as $\Lambda \to \infty$. This refined asymptotic formula was conjectured by Weyl in [57].

Under the sole assumption of convexity we prove that the asymptotic behaviour does not lie below that suggested by the Weyl conjecture.

Lemma 2.1 (one-sided two-term asymptotics). *Let $\Omega \in \mathcal{K}^n$. Then, for $\gamma \geq 1$,*

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \geq L^{\mathrm{cl}}_{\gamma,n}|\Omega|\Lambda^{\gamma+n/2} - \frac{1}{4}L^{\mathrm{cl}}_{\gamma,n-1}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}), \tag{16}$$

as $\Lambda \to \infty$. Moreover, the error term is uniform on compact subsets of $\mathcal{K}^n$.

In [30] Ivrii proved that (15) holds for $\gamma = 0$ under the assumptions that $\partial\Omega$ is smooth and the measure of the periodic billiards in Ω is zero. By the Aizenman–Lieb identity it follows that the expansion holds for all $\gamma > 0$ under the same assumptions. More recently, Frank and Geisinger proved that (15) is true for $\gamma = 1$ if the boundary of Ω is $C^{1,\alpha}$-regular [16]. In [17] the same authors treat the case of Robin boundary conditions and show that their method also covers C^1-domains. Again the Aizenman–Lieb identity implies that (15) is valid under the same assumptions for all $\gamma \geq 1$. In particular, the results of [16, 17] imply that the expansion (15) holds uniformly on compact subsets of $\mathcal{K}^n_\omega$.

Lemma 2.2 ([16, Theorem 1.1], [17, Theorem 1.3]). *Let $\Omega \in \mathcal{K}^n_\omega$. Then, for $\gamma \geq 1$,*

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L^{\mathrm{cl}}_{\gamma,n}|\Omega|\Lambda^{\gamma+n/2} - \frac{L^{\mathrm{cl}}_{\gamma,n-1}}{4}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}),$$

as $\Lambda \to \infty$. Moreover, the error term is uniform on compact subsets of $\mathcal{K}^n_\omega$.

That the error term in the above expansion is uniform on compact subsets follows from the methods of Frank and Geisinger, in fact the uniform C^1-modulus of continuity of $\partial\Omega$ together with upper and lower bounds on $|\Omega|$ and $|\partial\Omega|$ suffices. This uniformity is not explicitly stated in their results but it is nonetheless possible to track the geometric dependence through their proof and conclude that this is the case. However, this is not an entirely trivial task. To see how this can be done we refer the reader to [19] where the same construction is used and the error term is tracked explicitly.

In Section 6 we shall prove that (15) holds uniformly also for Ω in compact subsets of $\mathcal{P}_m$.

Lemma 2.3. *Let $\Omega \in \mathcal{P}_m$. Then, for $\gamma \geq 1$,*

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = L^{\mathrm{cl}}_{\gamma,n}|\Omega|\Lambda^{\gamma+n/2} - \frac{L^{\mathrm{cl}}_{\gamma,n-1}}{4}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}),$$

as $\Lambda \to \infty$. Moreover, the error term is uniform on compact subsets of $\mathcal{P}_m$.

The reason that we here need to further restrict our admissible classes of convex domains is that, prior to the results in [19], (15) was not known to hold uniformly in compact subsets of $\mathcal{K}^n$, for $\gamma \geq 1$.

The refined asymptotics (15) combined with the isoperimetric inequality indicates that if we can prove that an asymptotically optimal shape exists, it is likely the ball. This is indeed the heuristic idea behind the belief that maximizers of the Riesz means, or for that matter minimizers of the eigenvalues, should be well behaved in the limit $\Lambda \to \infty$.

2.3. A two-term Berezin inequality. A key ingredient in our proof here will be the following two-term bound for the Riesz means of order $\gamma \geq 1$ when $\Omega \subset \mathbb{R}^n$ is convex. This result was first obtained for $\gamma \geq 3/2$ in the planar case in [21] under an additional geometric assumption. In [35] it was proved that this additional assumption was true in general, and in [36] this was used to generalize the bound for $\gamma \geq 3/2$ to any dimension and arbitrary convex domains. The extension to $1 \leq \gamma < 3/2$ was until recently unknown to us but follows as a simple corollary of an inequality due to Harrell and Stubbe [25], which reduces the problem to considering a domain of the form $(0, a_1) \times \cdots \times (0, a_n) \subset \mathbb{R}^n$. For domains on this form precise bounds for Riesz means were proved in [22].

Theorem 2.4 ([21, Corollary 3.4], [36, Theorem 1.1], [25], [22]). *Let $\Omega \in \mathcal{K}^n$. For $\gamma \geq 1$ there exists a constant $c(\gamma, n) > 0$ such that*

- *if $\Lambda \leq \frac{\pi^2}{4r(\Omega)^2}$, then*
$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = 0;$$
- *if $\Lambda > \frac{\pi^2}{4r(\Omega)^2}$, then*
$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L^{\mathrm{cl}}_{\gamma,n}|\Omega|\Lambda^{\gamma+n/2} - c(\gamma, n)L^{\mathrm{cl}}_{\gamma,n-1}|\partial\Omega|\Lambda^{\gamma+(n-1)/2}.$$

Proof of Theorem 2.4. The first part of the theorem is a direct consequence of the bound $\lambda_1(\Omega) \geq \frac{\pi^2}{4r(\Omega)^2}$ proved in [29, 49]. For the second part we, without loss of generality, assume that $\Omega \subseteq (0, 2w(\Omega)) \times (0, a_2) \times \cdots \times (0, a_n) =: R$, where $0 < 2w(\Omega) \leq a_2 \leq \cdots \leq a_n < \infty$.

By equation 4.3 in [25] it follows that, for all $\Lambda \geq 0$,

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_- \leq \frac{|\Omega|}{|R|} \operatorname{Tr}(-\Delta_R - \Lambda)_-.$$

By an application of the Aizenman–Lieb identity [1] (see also Section 6 below) we also have that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq \frac{|\Omega|}{|R|} \operatorname{Tr}(-\Delta_R - \Lambda)_-^\gamma,$$

for any $\gamma \geq 1$ and $\Lambda \geq 0$.

By Lemma 4.4 in [22] and the behaviour of Laplacian eigenvalues under scaling, $\lambda_k(t\Omega) = t^{-2}\lambda_k(\Omega)$ for $t > 0$, we find that for all $\gamma \geq 1$ there exist positive constants c_1, c_2, b_0 such that

$$\begin{aligned}
&\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \\
&\quad \leq \frac{|\Omega|}{|R|} \operatorname{Tr}(-\Delta_R - \Lambda)_-^\gamma \\
&\quad \leq L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} - \frac{c_1 b L_{\gamma,n-1}^{\mathrm{cl}}}{2w(\Omega)} |\Omega| \Lambda^{\gamma+(n-1)/2} + \frac{c_2 b^2 L_{\gamma,n-2}^{\mathrm{cl}}}{4w(\Omega)^2} |\Omega| \Lambda^{\gamma+(n-2)/2},
\end{aligned} \tag{17}$$

for all $\Lambda \geq 0$ and $b \in [0, b_0]$.

For $\Lambda \geq \frac{\pi^2}{4r(\Omega)^2}$ we find that

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2} - c L_{\gamma,n-1}^{\mathrm{cl}} \frac{|\Omega|}{w(\Omega)} \Lambda^{\gamma+(n-1)/2},$$

where we used $w(\Omega) \geq 2r(\Omega)$ and that we can choose b arbitrarily small. The claimed bound follows since $\frac{|\Omega|}{w(\Omega)} \geq c|\partial\Omega|$ by combining (10) and (13). $\square$

The bound in Theorem 2.4 is an improvement of an inequality going back to Berezin [4] which states that for the convex Riesz means, i.e. when $\gamma \geq 1$, the first term in the Weyl asymptotic formula always overestimates the eigenvalue mean:

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L_{\gamma,n}^{\mathrm{cl}} |\Omega| \Lambda^{\gamma+n/2}. \tag{18}$$

This inequality should more correctly be attributed to Berezin, Lieb, and Li and Yau [4, 40, 41]. Berezin and Lieb both proved inequalities of which (18) is a special case, while Li and Yau proved an inequality for the sum of the first m eigenvalues which is equivalent to (18) (see [34]).

We emphasize that the second term appearing in the bound of Theorem 2.4 is up to a constant the same as that appearing in the refined Weyl asymptotic formula (this is essential in proving the boundedness of the maximizers).

3. Existence of extremal domain

For any fixed $\gamma \geq 1$ and Λ large enough, we have that the existence of a maximizer in the class of quasi-open sets follows from known results [9, 13, 45, 46]. However, the methods used in these articles do not take into account that we wish to stay within our class of convex domains. But, as this is already a very nice class of sets, proving the existence of a maximizer for our problem is not difficult.

Lemma 3.1 (Existence of maximizers). *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^n$. Then, for any $\gamma \geq 0$ and $\Lambda \geq 0$ there exists a domain $\Omega_{\Lambda,\gamma} \in \mathcal{A}$ realizing the supremum*

$$\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\}. \tag{19}$$

Moreover, if $\mathcal{A} = \mathcal{K}^n$, $\gamma \geq 1$, and $\Lambda > \lambda_1(B)$ then any such domain has C^1-regular boundary.

Proof of Lemma 3.1. For fixed $\Lambda > \lambda_1(B)$ and $\gamma \geq 1$ our functional is weakly strictly decreasing [45], that is if $\lambda_k(\Omega) < \lambda_k(\widetilde{\Omega})$ for all $k \geq 1$ then

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma > \operatorname{Tr}(-\Delta_{\widetilde{\Omega}} - \Lambda)_-^\gamma.$$

Moreover, by the Li–Yau inequality (8), our functional is for any fixed Λ a finite sum of Lipschitz functions and hence Lipschitz. Thus the last part of the lemma is a direct consequence of Theorem 3.4 in [8].

If $\Lambda \leq \inf\{\lambda_1(\Omega) : \Omega \in \mathcal{A}, |\Omega| = 1\}$ then the supremum in (19) is zero and hence any domain in $\mathcal{A}$ is a maximizer. If this is not the case we let $\{\Omega_k\}_{k\geq 1} \subset \mathcal{A}$, with $|\Omega_k| = 1$, be a maximizing sequence for (19). Without loss of generality we assume that $\operatorname{Tr}(-\Delta_{\Omega_k} - \Lambda)_-^\gamma > 0$. In particular, we must have that $\lambda_1(\Omega_k) < \Lambda$ for all k. Hence the inequality $\lambda_1(\Omega) \geq \frac{\pi^2}{4r(\Omega)^2}$, for $\Omega \subset \mathcal{K}^n$, due to Hersch in $\mathbb{R}^2$ and Protter in $\mathbb{R}^n$ [29, 49] implies that $r(\Omega_k) > \frac{\pi}{2\sqrt{\Lambda}}$. Since $|\Omega_k| = 1$ for each k we by (14) obtain an upper bound for $D(\Omega_k)$ which is independent of k.

As our functional is invariant under translation we may translate each Ω_k so that it has barycentre at the origin and obtain a new maximizing sequence which is uniformly bounded. By the Blaschke selection theorem [51, Theorem 1.8.7] we can extract a subsequence which converges in $\mathcal{K}^n$, and hence in $\mathcal{A}$. Abusing notation denote this subsequence by $\{\Omega_k\}_{k\geq 1}$ and let Ω_∞ denote its limit. Since the eigenvalues of the Dirichlet Laplacian are lower-semi continuous with respect to the topology on $\mathcal{K}^n$ [28] we find that Ω_∞ realizes the supremum in (19). $\square$

4. Convergence of maximizers

In this section we prove that for any sequence $\{\Lambda_j\}_{j\geq 1}$ tending to infinity the corresponding sequence of maximizers $\Omega_{\Lambda_j,\gamma}(\mathcal{A})$ has a convergent subsequence. Moreover, if $\mathcal{A}$ satisfies an additional assumption we characterize the possible limit points of such subsequences. Our main objective is to prove the following proposition:

Proposition 4.1. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^n$. Fix $\gamma \geq 1$ and let $\Omega_{\Lambda,\gamma}(\mathcal{A})$ denote any extremal domain for the shape optimization problem*

$$\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma : \Omega \in \mathcal{A}, |\Omega| = 1\}. \tag{20}$$

Then the following statements hold.

(i) *For any sequence $\{\Lambda_j\}_{j\geq 1}\uparrow \infty$ the corresponding sequence $\{\Omega_{\Lambda_j,\gamma}(\mathcal{A})\}_{j\geq 1}$ has a subsequence which, up to translation, converges in $\mathcal{A}$. Moreover, Ω_∞ the limit of such a subsequence has unit measure.*

(ii) *Under the additional assumption that*

$$\begin{aligned}\operatorname{Tr}&(-\Delta_\Omega - \Lambda)_-^\gamma \\ &= L_{\gamma,n}^{\mathrm{cl}}|\Omega|\Lambda^{\gamma+n/2} - \frac{1}{4}L_{\gamma,n-1}^{\mathrm{cl}}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}),\end{aligned}$$

as $\Lambda \to \infty$, uniformly for Ω in compact subsets of $\mathcal{A}$, then the limit Ω_∞ also minimizes the perimeter in $\mathcal{A}$:

$$|\partial\Omega_\infty| = \inf\{|\partial\Omega| : \Omega \in \mathcal{A}, |\Omega| = 1\}. \tag{21}$$

Remark 4.2. As a consequence of the results in [19] we know that the assumption in the second part of the theorem is redundant, the expansion holds uniformly on any compact subset of $\mathcal{K}^n$. As a consequence the conclusions of Proposition 4.1 remain true without it and hence extends Theorems 1.1 and 1.2 to *any* admissible class of convex domains $\mathcal{A}$ (see [19]).

With Proposition 4.1 in hand it is straightforward to prove our main theorems.

Proof of Theorems 1.1 and 1.2. By Lemmas 2.2 and 2.3 the classes $\mathcal{K}_\omega^n$ and $\mathcal{P}_m$ satisfy the assumptions of (i) and (ii) of Proposition 4.1, and thus the theorems follow as special cases thereof. $\square$

Proof of Proposition 4.1. The proof follows closely the strategy of Antunes and Freitas [2] (see also [6, 7, 11, 22] for applications in very similar settings). Using the bound of Theorem 2.4 one readily obtains that the sequence of maximizers have uniformly bounded perimeters. Using the inequalities of Section 2.1 we can conclude that the sequence is uniformly bounded, and thus extract a convergent subsequence. The final ingredient is to use the uniform asymptotic expansions in (ii) to identify the limiting domains as minimizers of the perimeter.

Fix $\mathcal{A}$ and $\gamma \geq 1$. For notational convenience we will for a maximizer of (20) write simply Ω_Λ instead of $\Omega_{\Lambda,\gamma}(\mathcal{A})$. Without loss of generality we throughout the proof assume that the barycentre of each maximizer is the origin. The idea used to prove the existence of a convergent subsequence of $\Omega_{\Lambda,\gamma}(\mathcal{A})$ is to use the maximality of $\operatorname{Tr}(-\Delta_{\Omega_\Lambda}-\Lambda)_-^\gamma$ and compare it with the corresponding Riesz mean for some fixed domain $\Omega_0 \in \mathcal{A}$ with $|\Omega_0| = 1$.

Assume that $\Lambda > \inf\{\lambda_1(\Omega) \colon \Omega \in \mathcal{A}, |\Omega| = 1\}$. Then, by the maximality of Ω_Λ,

$$0 < \operatorname{Tr}(-\Delta_{\Omega_0} - \Lambda)_-^\gamma \leq \operatorname{Tr}(-\Delta_{\Omega_\Lambda} - \Lambda)_-^\gamma.$$

Using Theorem 2.4 and Lemma 2.1 this inequality implies that

$$\begin{aligned} L^{\mathrm{cl}}_{\gamma,n}\Lambda^{\gamma+n/2} - \frac{L^{\mathrm{cl}}_{\gamma,n-1}}{4}|\partial\Omega_0|\Lambda^{\gamma+(n-1)/2} + o(\Lambda^{\gamma+(n-1)/2}) \\ \leq L^{\mathrm{cl}}_{\gamma,n}\Lambda^{\gamma+n/2} - c(\gamma,n)L^{\mathrm{cl}}_{\gamma,n-1}|\partial\Omega_\Lambda|\Lambda^{\gamma+(n-1)/2}. \end{aligned} \tag{22}$$

Rearranging (22) yields

$$|\partial\Omega_\Lambda| \leq \frac{|\partial\Omega_0|}{4c(\gamma,n)} + o(1),$$

as $\Lambda \to \infty$, and thus the perimeter of the maximizers remains uniformly bounded in Λ. By (13) and (14) we conclude that Ω_Λ remains uniformly bounded with respect to Λ. Thus we can for any sequence $\{\Lambda_j\}_{j\geq 1}$ tending to infinity extract a subsequence of $\{\Omega_{\Lambda_j}\}_{j\geq 1}$ which converges to a domain $\Omega_\infty \in \mathcal{A}$. Since $|\Omega|$ and $|\partial\Omega|$ are continuous with respect to the topology of $\mathcal{K}^n$ we find that $|\Omega_\infty| = 1$ and $|\partial\Omega_\infty| \leq \frac{|\partial\Omega_0|}{4c(\gamma,n)}$, this completes the proof of (i).

With a slight abuse of notation we let $\{\Lambda_j\}_{j\geq 1}$ denote the subsequence along which $\{\Omega_{\Lambda_j}\}_{j\geq 1}$ converges to Ω_∞. For each $j \geq 1$ we have, by the maximality of Ω_{Λ_j}, that

$$\frac{\operatorname{Tr}(-\Delta_{\Omega_0} - \Lambda_j)_-^\gamma - L^{\mathrm{cl}}_{\gamma,n}\Lambda_j^{\gamma+n/2}}{\Lambda_j^{\gamma+(n-1)/2}} \leq \frac{\operatorname{Tr}(-\Delta_{\Omega_{\Lambda_j}} - \Lambda_j)_-^\gamma - L^{\mathrm{cl}}_{\gamma,n}\Lambda_j^{\gamma+n/2}}{\Lambda_j^{\gamma+(n-1)/2}}.$$

Assume now that $\mathcal{A}$ satisfies the assumption in (ii). Using that our sequence of maximizers $\{\Omega_{\Lambda_j}\}_{j\geq 1}$ is bounded, and hence contained in a compact subset of $\mathcal{A}$, to uniformly control the error terms, one finds that

$$|\partial\Omega_{\Lambda_j}| \leq |\partial\Omega_0| + o(1),$$

as $j \to \infty$. Since the sequence Ω_{Λ_j} converges to Ω_∞ and the measure of the perimeter is continuous in the topology of $\mathcal{K}^n$, we obtain that $|\partial\Omega_\infty| \leq |\partial\Omega_0|$. Choosing Ω_0 to realize the infimum in (21) concludes the proof. □

Remark 4.3. We note that in the proof of (i) we do not require the full statement of Lemma 2.1 only that there exists *one* domain $\Omega_0 \in \mathcal{A}$ with $|\Omega_0| = 1$ for which the second term of the asymptotic expansion of the Riesz mean is of the correct order $\sim \Lambda^{\gamma+(n-1)/2}$.

5. Sums of eigenvalues

In this section we prove that our techniques allow us also to study the behaviour of convex domains realizing the infimum

$$\inf\Big\{\frac{1}{m}\sum_{k=1}^{m}\lambda_k(\Omega) \colon \Omega \in \mathcal{A}, |\Omega| = 1\Big\} \tag{23}$$

in the limit $m \to \infty$. This problem, but without the convexity assumptions, was recently studied by Freitas [20].

Theorem 5.1. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^n$ satisfying the assumption in (ii) of Proposition 4.1. Let $\Omega_m(\mathcal{A})$ denote any extremal domain for the shape optimization problem (23). Then the sequence $\{\Omega_m\}_{m\geq 1}$ has a subsequence which, up to translations, converges in $\mathcal{A}$. Moreover, Ω_∞ the limit of such a subsequence has unit measure and minimizes the perimeter in $\mathcal{A}$:*

$$|\partial\Omega_\infty| = \inf\{|\partial\Omega| \colon \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Remark 5.2. Again we note that the extra assumption on $\mathcal{A}$ can be dropped in light of the results in [19].

The proof of Theorem 5.1 is based on our tools developed for Riesz means and the close connection between sums of eigenvalues and Riesz means of order $\gamma = 1$. In particular, via the Legendre transform the asymptotic expansion for $\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-$ implies a similar two-term expansion for the sum (see, for instance, [18, Appendix A]): for $\Omega \subset \mathbb{R}^n$ such that (15) holds

$$\frac{1}{m}\sum_{k=1}^m \lambda_k(\Omega) = A_n\Big(\frac{m}{|\Omega|}\Big)^{2/n} + B_n\frac{|\partial\Omega|}{|\Omega|}\Big(\frac{m}{|\Omega|}\Big)^{1/n} + o(m^{1/n}),$$

as $m \to \infty$. The constants A_n, B_n are explicitly given by

$$A_n = \frac{4\pi n\Gamma\big(\frac{n}{2}+1\big)^{2/n}}{n+2}, \qquad B_n = \frac{2\pi\Gamma\big(\frac{n}{2}+1\big)^{1+1/n}}{(n+1)\Gamma\big(\frac{n+1}{2}\big)}.$$

It should also be noted that the Legendre transform switches the direction of inequalities. In particular, the lower bound for the Riesz mean asymptotics provided by Lemma 2.1 turns into a corresponding upper bound for the asymptotics of the sum.

If we can prove a bound similar to Theorem 2.4 in the setting of eigenvalue sums then it is straightforward to follow the strategy in the proofs of Lemma 3.1 and Proposition 4.1 to prove first the existence and uniform boundedness of the minimizers.

Corollary 5.3 (Improved Li–Yau inequality). *Let $\Omega \in \mathcal{K}^n$. There exists a positive constant $c(n)$ such that, for all $m \geq 1$,*

$$\frac{1}{m}\sum_{k=1}^m \lambda_k(\Omega) \geq A_n\Big(\frac{m}{|\Omega|}\Big)^{2/n} + c(n)B_n\frac{|\partial\Omega|}{|\Omega|}\Big(\frac{m}{|\Omega|}\Big)^{1/n}.$$

Proof of Corollary 5.3. By (17) there are positive constants c_1, c_2, b_0 such that

$$\begin{aligned}\sup_{\Lambda\geq 0}\Big(m\Lambda - \sum_{k:\lambda_k(\Omega)<\Lambda}(\Lambda-\lambda_k(\Omega))\Big)&\\ \geq \sup_{\Lambda\geq 0}\Big(m\Lambda - L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} + \frac{c_1 b L^{\mathrm{cl}}_{1,n-1}}{w(\Omega)}|\Omega|\Lambda^{(n+1)/2}&\\ - \frac{c_2 b^2 L^{\mathrm{cl}}_{1,n-2}}{w(\Omega)^2}|\Omega|\Lambda^{n/2}\Big),&\end{aligned} \tag{24}$$

for all $m \geq 1$ and $b \in [0, b_0]$.

It is well known that the left-hand side of (24) is equal to the sum of the m first eigenvalues, this follows from studying the sign of the derivative of the expression in the parenthesis with respect to Λ on intervals where $N_\Omega(\Lambda)$ is constant. Moreover, since the supremum on the right-hand side is larger than its value at any fixed Λ we obtain a valid inequality by simply choosing a $\Lambda \geq 0$. Specifically we choose $\Lambda = \frac{4^{1/n}}{(n+2)^{2/n}(L^{\mathrm{cl}}_{1,n})^{2/n}}\bigl(\frac{m}{|\Omega|}\bigr)^{2/n}$ which leads to

$$\begin{aligned}
&\sup_{\Lambda\geq 0}\Bigl(m\Lambda - L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} + \frac{c_1 b L^{\mathrm{cl}}_{1,n-1}}{w(\Omega)}|\Omega|\Lambda^{(n+1)/2} - \frac{c_2 b^2 L^{\mathrm{cl}}_{1,n-2}}{w(\Omega)^2}|\Omega|\Lambda^{n/2}\Bigr)\\
&\quad\geq mA_n|\Omega|\Bigl(\frac{m}{|\Omega|}\Bigr)^{2/n} + m\frac{c_1' b}{w(\Omega)}\Bigl(\frac{m}{|\Omega|}\Bigr)^{1/n} - m\frac{c_2' b^2}{w(\Omega)^2}\\
&\quad\geq mA_n|\Omega|\Bigl(\frac{m}{|\Omega|}\Bigr)^{2/n} + mc(n)B_n\frac{|\partial\Omega|}{|\Omega|}\Bigl(\frac{m}{|\Omega|}\Bigr)^{1/n},
\end{aligned}$$

where we in the final step used (10) and (13), $m \geq 1$, and that we can choose b as small as we wish. □

Proof of Theorem 5.1. The claim follows by mimicking the proofs of Lemmas 3.1 and Proposition 4.1. The use of the asymptotic bound of Lemma 2.1 should be replaced by its corresponding Legendre transform, and the use of Theorem 2.4 by Corollary 5.3. □

6. Uniform two-term asymptotics

In this section we use the methods of Frank and Geisinger [16, 17] to prove Lemmas 2.1 and 2.3. The proof of Lemma 2.1 will complete the proof of Proposition 4.1, which in combination with Lemma 2.2 and Lemma 2.3 proves Theorem 1.2 and Theorem 1.1, respectively.

To match the notation used in [16, 17] we here consider the asymptotics of $\operatorname{Tr}(-h^2\Delta_\Omega - 1)_-^\gamma$ as $h \to 0^+$. By a simple calculation (15) is equivalent to

$$\operatorname{Tr}(-h^2\Delta_\Omega - 1)_-^\gamma = L^{\mathrm{cl}}_{\gamma,n}|\Omega|h^{-n} - \frac{L^{\mathrm{cl}}_{\gamma,n-1}}{4}|\partial\Omega|h^{-n+1} + o(h^{-n+1}), \quad \text{as } h \to 0^+,$$

and (16) to the corresponding inequality.

In [16, 17] the authors consider only the case $\gamma = 1$ but it can be lifted to larger γ using the Aizenman–Lieb identity [1]: if $\gamma_1 \geq 0$ and $\gamma_2 > \gamma_1$, then

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^{\gamma_2} = B(1+\gamma_1, \gamma_2-\gamma_1)^{-1}\int_0^\infty \tau^{-1+\gamma_2-\gamma_1}\operatorname{Tr}(-\Delta_\Omega - (\Lambda-\tau))_-^{\gamma_1}\,d\tau,$$

where B denotes the Euler Beta function. It thus suffices to prove Lemmas 2.1 and 2.3 in the case $\gamma = 1$.

The proof relies on localizing the operator into balls whose sizes vary depending on the distance to the complement of Ω. The asymptotic contributions from each of these localizations is then analysed separately.

Using Theorem 22 in [52] the localization is constructed by introducing a length-scale $l(u)$ and functions $\phi_u \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ with support in $B_{l(u)}(u) = \{x \in \mathbb{R}^n \colon |x - u| < l(u)\}$, satisfying

$$\|\phi_u\|_\infty \leq c, \quad \|\nabla \phi_u\|_\infty \leq c\, l(u)^{-1} \tag{25}$$

and for any $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} \phi_u^2(x) l(u)^{-n}\, du = 1. \tag{26}$$

Here and in what follows c will denote a positive constant which may change from line to line, but which depends only on the dimension and the choice of $l(u), \phi_u$. Following [16, 17] we set

$$l(u) := \frac{1}{2}\big(1 + (\operatorname{dist}(u, \Omega^c)^2 + l_0^2)^{-1/2}\big)^{-1}, \tag{27}$$

where $l_0 \in (0, 1)$ is a parameter depending only on h which will tend to zero as $h \to 0^+$.

We will use the following results from [16, 17]:

Lemma 6.1 ([16, Proposition 1.1]). *For* $0 < l_0 < 1$ *and* $0 < h \leq M l_0$ *we have that*

$$\begin{aligned}\Big|\operatorname{Tr}(-h^2\Delta_\Omega - 1)_- - \int_{\mathbb{R}^n} \operatorname{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n}\, du\Big|\\ \leq c \int_{\Omega^*} l(u)^{-2}\, du\, h^{-n+2},\end{aligned}$$

where $\Omega^* := \{u \in \mathbb{R}^n \colon \operatorname{supp} \phi_u \cap \Omega \neq \emptyset\}$ *and the constant* c *depends only on* M *and those in* (25).

Remark 6.2. In Proposition 1.1 in [16] the integral on the right-hand side is in the final step of the proof bounded in terms of l_0^{-1}. As we here wish to keep track of how the remainder depends on Ω we choose to keep it in integral form. Moreover, in [52] the function l is assumed to be C^1-regular, however, a Lipschitz assumption is sufficient (see [19]).

Remark 6.3. We also note that $\Omega \subsetneq \Omega^* \subseteq \Omega^t$, for $t = \frac{l_0}{2+2l_0}$. If the functions ϕ_u are chosen so that $\operatorname{supp} \phi_u = B_{l(u)}(u)$ then equality holds in the second inclusion.

Lemma 6.4 ([16, Proposition 1.1] and [17, Proposition 2.3]). *Let $\phi \in C^\infty(\mathbb{R}^n)$ be supported in a ball of radius $l > 0$ and satisfy*

$$\|\nabla\phi\|_\infty \le c l^{-1}. \tag{28}$$

Assume that the intersection $\partial\Omega \cap \operatorname{supp}\phi$ is C^1 with modulus of continuity $\omega\colon [0, L) \to \mathbb{R}$, with $L \ge 2l$, in the sense of (B).

Then if l is so small that $\omega(l) \le c_n$, where c_n depends only on the dimension, it holds for $0 < h \le l$ that

$$\Big|\operatorname{Tr}(\phi(-h^2\Delta_\Omega - 1)\phi)_- - L^{\mathrm{cl}}_{1,n}\int_\Omega \phi^2(x)\,dx\,h^{-n} \\ + \frac{1}{4}L^{\mathrm{cl}}_{1,n-1}\int_{\partial\Omega}\phi^2(x)\,d\sigma(x)\,h^{-n+1}\Big| \le r(l,h),$$

where $d\sigma$ denotes the $(n-1)$-dimensional Lebesgue measure on $\partial\Omega$ and the remainder satisfies

$$r(l,h) \le c\Big(\frac{l^{n-2}}{h^{n-2}} + \frac{\omega(l)^2 l^{n-1}}{h^{n-1}} + \frac{\omega(l) l^n}{h^n}\Big),$$

where the constant c depends only on that in (28).

Remark 6.5. Here we shall only make use of Lemma 6.4 when the boundary of Ω is either $C^{1,1}$-regular or when $\Omega \in \mathcal{P}_m$ and the boundary is locally a hyperplane, in the latter case we can take $\omega \equiv 0$. In [16, 17] it is stated that the smallness assumption on l may depend on Ω, this is however not necessary the relevant local geometry is encoded by ω. Inspection of the proofs in [16, 17] yields that one can can take $c_n = \frac{1}{4(n-1)}$.

We shall also need the following lemma which can be viewed as a local version of (18).

Lemma 6.6 ([16, Lemma 2.1]). *For any $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h > 0$ we have that*

$$\operatorname{Tr}(\phi(-h^2\Delta_\Omega - 1)\phi)_- \le L^{\mathrm{cl}}_{1,n}\int_\Omega \phi^2(x)\,dx\,h^{-n}.$$

To prove Lemma 2.1 we will need a more refined version of this inequality when the support of ϕ is disjoint from the boundary of Ω.

Lemma 6.7 ([16, Proposition 1.2]). *Let $\phi \in C_0^\infty(\Omega)$ be supported in a ball of radius $l > 0$ and satisfy*

$$\|\nabla\phi\|_\infty \le c l^{-1}. \tag{29}$$

Then, for all $h > 0$,

$$\left|\operatorname{Tr}(\phi(-h^2\Delta_\Omega - 1)\phi)_- - L^{\mathrm{cl}}_{1,n}\int_\Omega \phi^2(x)\,dx\,h^{-n}\right| \le c\,l^{n-2}h^{-n+2},$$

where the constant c depends only on that in (29).

To control the error terms coming from the applications of the local bounds above we shall need the following inequalities which appear in [16] (or with explicitly stated geometric dependence in [19]): for $\Omega \in \mathcal{K}^n$ and $\alpha \in \mathbb{R}$ it holds that

$$\int_{\Omega_*} l(u)^{-2}\,du \le c(|\Omega| + |\partial\Omega|)l_0^{-1}, \tag{30}$$

$$\int_{\Omega^*\setminus\Omega_*} l(u)^{\alpha}\,du \le c|\partial\Omega| l_0^{1+\alpha}, \tag{31}$$

where Ω^* is defined as in Lemma 6.1, $\Omega^* = \{u \in \mathbb{R}^n : \operatorname{supp}\phi_u \cap \Omega \neq \emptyset\}$, and similarly $\Omega_* := \{u \in \Omega : \operatorname{supp}\phi_u \subset \Omega\}$. As noted above Ω^* is essentially an outer parallel set of Ω. Similarly Ω_* is essentially an inner parallel set. In particular we note the inclusions $\Omega_* \subset \Omega \subset \Omega^*$.

Using the above we are ready to prove Lemmas 2.1 and 2.3.

Proof of Lemma 2.1. The proof is based on constructing a nested family of regular convex domains $\Omega(\varepsilon) \in \mathcal{K}^n$, for $\varepsilon > 0$, such that $\Omega(0) = \Omega$ and $\Omega(\varepsilon) \subset \Omega(\varepsilon')$ if $\varepsilon > \varepsilon'$.

Define, in the notation introduced in Section 2.1, $\Omega(\varepsilon) := (\Omega_\varepsilon)^\varepsilon = (\Omega \sim B_\varepsilon) + B_\varepsilon$, that is the outer parallel set of the inner parallel set of Ω at distance $\varepsilon > 0$. For $0 \le \varepsilon < r(\Omega)$ it is clear from the construction that $\Omega(\varepsilon)$ are non-empty and nested as described above. We also see that $\Omega(\varepsilon)$ satisfies an ε-inner ball condition, and hence its boundary is $C^{1,1}$-regular (see, for instance, [14]). Furthermore, it holds that $D(\Omega(\varepsilon)) \le D(\Omega)$ and $r(\Omega(\varepsilon)) = r(\Omega)$.

By (11) and the properties of mixed volumes listed in Section 2.1 we have

$$\big||\Omega^\varepsilon| - |\Omega| - \varepsilon|\partial\Omega|\big| = \sum_{j=2}^{n}\binom{n}{j}\varepsilon^j W(\underbrace{\Omega,\dots,\Omega}_{n-j},\underbrace{B_1,\dots,B_1}_{j})$$

and

$$\begin{aligned}
&\sum_{j=2}^{n}\binom{n}{j}\varepsilon^{j}W(\underbrace{\Omega,\dots,\Omega}_{n-j},\underbrace{B_1,\dots,B_1}_{j})\\
&\quad=\sum_{j=2}^{n}\binom{n}{j}\frac{\varepsilon^{j}}{D(\Omega)^{j}}W(\underbrace{\Omega,\dots,\Omega}_{n-j},\underbrace{D(\Omega)B_1,\dots,D(\Omega)B_1}_{j})\\
&\quad\le\sum_{j=2}^{n}\binom{n}{j}\frac{\varepsilon^{j}}{D(\Omega)^{j}}D(\Omega)^{n}|B_1|\le cD(\Omega)^{n-2}\varepsilon^{2}.
\end{aligned}$$

Similarly

$$\big||\partial\Omega^{\varepsilon}|-|\partial\Omega|\big|=n\sum_{j=1}^{n-1}\binom{n-1}{j}\varepsilon^{j}W(\underbrace{\Omega,\dots,\Omega}_{n-j-1},\underbrace{B_1,\dots,B_1}_{j+1})\le cD(\Omega)^{n-2}\varepsilon.$$

Hence we can conclude that

$$|\Omega^{\varepsilon}|=|\Omega|+\varepsilon|\partial\Omega|+O(\varepsilon^{2})\quad\text{and}\quad|\partial\Omega^{\varepsilon}|=|\partial\Omega|+O(\varepsilon),\tag{32}$$

where both error terms are uniform on compact subsets of $\mathcal{K}^n$.

Moreover, by (12) and the corresponding upper bound

$$|\partial\Omega_{\varepsilon}|=|\partial\Omega|+O(\varepsilon),\tag{33}$$

where the implicit constant can be bounded from above by a constant times $|\partial\Omega|/r(\Omega)$.

Combining $\Omega(\varepsilon)\subseteq\Omega$ with (32), (33) and the inequality $|\Omega_{\varepsilon}|\ge|\Omega|-\varepsilon|\partial\Omega|$ yields that

$$|\Omega(\varepsilon)|=|\Omega|+O(\varepsilon^{2})\quad\text{and}\quad|\partial\Omega(\varepsilon)|=|\partial\Omega|+O(\varepsilon),\tag{34}$$

where again both the error terms are uniform on compact subsets of $\mathcal{K}^n$.

By the monotonicity of Dirichlet eigenvalues under domain inclusion

$$\operatorname{Tr}(-h^{2}\Delta_{\Omega}-1)_{-}\ge\operatorname{Tr}(-h^{2}\Delta_{\Omega(\varepsilon)}-1)_{-}.\tag{35}$$

The idea is now to apply the methods of [16] to each $\Omega(\varepsilon)$, keeping track of how the error terms depend on ε, and in the final step choose ε appropriately depending on h.

We first observe that if $x \in \partial\Omega(\varepsilon)$ and $\delta \leq \varepsilon/2$ then the set $\partial\Omega(\varepsilon) \cap B_\delta(x)$ is (in the sense above) locally a graph of a $C^{1,1}$-function f satisfying

$$|\nabla f(x') - \nabla f(y')| \leq \frac{c}{\varepsilon}|x' - y'|, \tag{36}$$

where c is a dimensional constant. Indeed, by convexity and the fact that Ω satisfies a uniform ε-inner ball condition it follows that f is $C^{1,1}$-regular from Propositions 1.1 and 1.2 in [14]. That the constant in the $C^{1,1}$-estimate (36) behaves like ε^{-1} is a consequence of scaling: If $f(x)$ can be touched from above and below at each point by a ball of radius ε then $g(x) := f(\varepsilon x)/\varepsilon$ can at each point be touched from above and below by a ball of radius 1.

Let $l(u)$ be defined as in (27) with respect to the set $\Omega(\varepsilon)$ with $l_0 \in (0, 1)$ to be chosen later, and ϕ_u be the corresponding family of functions (we emphasize that this definition depends on ε even though this is not reflected in our notation).

Consider the set

$$\Omega^*(\varepsilon) := \{u \in \mathbb{R}^n : \operatorname{supp}\phi_u \cap \Omega(\varepsilon) \neq \emptyset\},$$

we note again that $\Omega^*(\varepsilon)$ contains points in the complement of $\Omega(\varepsilon)$. This is precisely the set of $u \in \mathbb{R}^n$ where $\operatorname{Tr}(\phi_u(-h^2\Delta_{\Omega(\varepsilon)} - 1)\phi_u)_-$ is non-zero. We split $\Omega^*(\varepsilon)$ into the sets $\Omega_*(\varepsilon) := \{u \in \Omega^*(\varepsilon) : \operatorname{supp}\phi_u \subset \Omega(\varepsilon)\}$ and $\Omega_b(\varepsilon) := \Omega^*(\varepsilon) \setminus \Omega_*(\varepsilon)$. The set $\Omega_*(\varepsilon)$ is precisely the set of $u \in \Omega^*(\varepsilon)$ such that $\operatorname{supp}\phi_u \cap \partial\Omega(\varepsilon) = \emptyset$, and $\Omega_b(\varepsilon)$ is the set where the same intersection is non-empty.

Let t^* solve the equation $t = \frac{1}{2}(1 + (t^2 + l_0^2)^{-1/2})^{-1} = l(u)\big|_{\operatorname{dist}(u,\Omega^c)=t}$. By observing that $l(u)\big|_{u\in\Omega^c} = \frac{l_0}{2l_0+2}$ and $0 \leq \frac{d}{dt}\big(\frac{1}{2}(1 + (t^2 + l_0^2)^{-1/2})^{-1}\big) \leq \frac{1}{2}$ it is clear that t^* is unique, and moreover that $t^* \leq l_0/\sqrt{3}$ since

$$\frac{1}{2}(1 + ((l_0/\sqrt{3})^2 + l_0^2)^{-1/2})^{-1} = \frac{l_0}{\sqrt{3} + 2l_0} \leq \frac{l_0}{\sqrt{3}}.$$

By the remarks above $l(u) \geq l_0/4$ for all $u \in \mathbb{R}^n$, and moreover since $\Omega_b(\varepsilon)$ is precisely the set where $l(u) \geq \operatorname{dist}(u, \partial\Omega)$ we find that if $u \in \Omega_b(\varepsilon)$ then $l(u) \leq l_0/\sqrt{3}$.

By Lemma 6.1 and (30) we have for $0 < h \leq Ml_0$ and $\varepsilon \in [0, r(\Omega))$ that

$$\begin{aligned}\operatorname{Tr}(-h^2\Delta_{\Omega(\varepsilon)} - 1)_- \geq \int_{\Omega_*(\varepsilon)} &\operatorname{Tr}(\phi_u(-h^2\Delta_{\Omega(\varepsilon)} - 1)\phi_u)_- l(u)^{-n}\,du \\ &+ \int_{\Omega_b(\varepsilon)} \operatorname{Tr}(\phi_u(-h^2\Delta_{\Omega(\varepsilon)} - 1)\phi_u)_- l(u)^{-n}\,du \\ &- c(|\Omega| + |\partial\Omega|)l_0^{-1}h^{-n+2},\end{aligned} \tag{37}$$

where the constant in the error term can be chosen independent of ε due to (34).

If $u \in \Omega_b(\varepsilon)$ then $\operatorname{dist}(u, \partial\Omega) \le l(u) \le l_0/\sqrt{3}$ and thus $B_{l(u)}(u) \subset B_{2l_0/\sqrt{3}}(x)$ for some $x \in \partial\Omega$. If we assume that $l_0 \le c\varepsilon$ then by the observation that $\partial\Omega(\varepsilon)$ is $C^{1,1}$-regular with the explicit estimate (36) we can apply Lemma 6.4 to the second integrand of (37), assuming c is small enough (depending only on dimension). By also applying Lemma 6.7 to the first integrand in (37) this yields that

$$\begin{aligned}
&\operatorname{Tr}(-h^2\Delta_{\Omega(\varepsilon)} - 1)_-\\
&\quad\ge L^{\mathrm{cl}}_{1,n}\int_{\Omega_*(\varepsilon)}\int_{\Omega(\varepsilon)}\phi_u^2(x) l(u)^{-n}\,dx du\, h^{-n}\\
&\qquad+\int_{\Omega_b(\varepsilon)}\Big(L^{\mathrm{cl}}_{1,n}\int_{\Omega(\varepsilon)}\phi_u^2(x)\,dx\,h^{-n}\\
&\qquad\qquad-\frac14 L^{\mathrm{cl}}_{1,n-1}\int_{\partial\Omega(\varepsilon)}\phi_u^2(x)\,d\sigma(x)\,h^{-n+1}\Big) l(u)^{-n}\,du\\
&\qquad- ch^{-n+1}\int_{\Omega_b(\varepsilon)}\Big(hl(u)^{-2} + l(u)\varepsilon^{-2} + \varepsilon^{-1}h^{-1}l(u)\Big)du\\
&\qquad- c(|\Omega| + |\partial\Omega|) l_0^{-1} h^{-n+2},
\end{aligned}\tag{38}$$

where we used the C^1-modulus of continuity for $\partial\Omega$ in (36), and (30) and (31) to bound the error terms coming from Lemmas 6.1 and 6.7.

Using (26), and (31) we find that (38) implies

$$\begin{aligned}
\operatorname{Tr}(-h^2\Delta_{\Omega(\varepsilon)} - 1)_- &\ge L^{\mathrm{cl}}_{1,n}|\Omega(\varepsilon)|h^{-n} - \frac{L^{\mathrm{cl}}_{1,n-1}}{4}|\partial\Omega(\varepsilon)|h^{-n+1}\\
&\quad - c(|\Omega| + |\partial\Omega|)(hl_0^{-1} + l_0^2\varepsilon^{-2} + l_0^2 h^{-1}\varepsilon^{-1})h^{-n+1}\\
&= L^{\mathrm{cl}}_{1,n}|\Omega|h^{-n} - \frac{L^{\mathrm{cl}}_{1,n-1}}{4}|\partial\Omega|h^{-n+1}\\
&\quad + (hl_0^{-1} + l_0^2\varepsilon^{-2} + l_0^2h^{-1}\varepsilon^{-1} + h^{-1}\varepsilon^2 + \varepsilon)O(h^{-n+1}),
\end{aligned}\tag{39}$$

where we in the second step also use (34). The final error term of (39) is uniform on compact subsets of $\mathcal{K}^n$ since this is the case for all the error terms leading up to the estimate.

In the construction above we have required that $h \le Ml_0$ and $l_0/c \le \varepsilon < r(\Omega)$, for a dimensional constant c, and $l_0 < 1$. Setting $l_0 = ch^\alpha$, $M = 1/c$, and $\varepsilon = h^\beta$ for some $0 < \beta \le \alpha < 1$ we find that our assumptions are satisfied for all $0 < h < \min\{1, r(\Omega)^{1/\beta}\}$. With these choices the expression in the parenthesis of the last term in (39) becomes

$$hl_0^{-1} + l_0^2\varepsilon^{-2} + l_0^2h^{-1}\varepsilon^{-1} + h^{-1}\varepsilon^2 + \varepsilon \lesssim h^{1-\alpha} + h^{2\alpha-2\beta} + h^{2\alpha-1-\beta} + h^{2\beta-1} + h^\beta.$$

Choosing $\alpha = 6/7, \beta = 4/7$ we find

$$h^{1-\alpha} + h^{2\alpha-2\beta} + h^{2\alpha-1-\beta} + h^{2\beta-1} + h^{\beta} \lesssim h^{1/7}.$$

By (35), and since the the error term in (39) is uniform on compact subsets of $\mathcal{K}^n$, this completes the proof of Lemma 2.1 for $\gamma = 1$. As noted above the statement for $\gamma > 1$ follows from an application of the Aizenman–Lieb identity. □

Proof of Lemma 2.3. Fix $\Omega \in \mathcal{P}_m$. By Lemma 2.1 we only need to prove the corresponding upper bound for $\mathrm{Tr}(-h^2\Delta_\Omega - 1)_-$. The main idea of the proof is similar to that used above for the regular sets $\Omega(\varepsilon)$. However, since the boundary is now not regular enough to use Lemma 6.4 close to every point we split our domain of integration into three parts. Define

$$\begin{aligned}
\Omega^* &:= \{u \in \mathbb{R}^n : \operatorname{supp} \phi_u \cap \Omega \neq \emptyset\},\\
\Omega_* &:= \{u \in \mathbb{R}^n : \operatorname{supp} \phi_u \subset \Omega\},\\
\Omega_b &:= \{u \in \Omega^* : \operatorname{supp} \phi_u \cap \partial\Omega \text{ is a piece of a hyperplane}\},\\
\Omega_s &:= \Omega^* \setminus (\Omega_* \cup \Omega_b).
\end{aligned}$$

The set Ω^* is again the set of $u \in \mathbb{R}^n$ where $\mathrm{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_-$ is non-zero. The set Ω_* is the bulk of Ω, where the effect from the boundary is not felt. Finally Ω_b and Ω_s are the remaining parts of Ω^*. The first set Ω_b is where the intersection of $\operatorname{supp} \phi_u$ with the boundary consists of part of a single face of Ω, and hence we can apply Lemma 6.4 with $\omega \equiv 0$. The second set Ω_s is where the intersection of $\operatorname{supp} \phi_u$ with the boundary contains pieces of several faces of Ω, we shall show that the contribution from this set is negligible in the limit $h \to 0^+$.

By Lemma 6.1 and (30) we have that, for $0 < h \le l_0$,

$$\begin{aligned}
&\mathrm{Tr}(-h^2\Delta_\Omega - 1)_-\\
&\quad \le \int_{\Omega_*} \mathrm{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n}\, du\\
&\qquad + \int_{\Omega_b} \mathrm{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n}\, du\\
&\qquad + \int_{\Omega_s} \mathrm{Tr}(\phi_u(-h^2\Delta_\Omega - 1)\phi_u)_- l(u)^{-n}\, du + c(|\Omega| + |\partial\Omega|) l_0^{-1} h^{-n+2}.
\end{aligned}$$

We estimate the first and third terms using Lemma 6.6, and apply Lemma 6.4 with $\omega \equiv 0$ to the integrand of the second, this yields

$$\begin{aligned}
&\mathrm{Tr}(-h^2\Delta_\Omega - 1)_- \\
&\quad\le L^{\mathrm{cl}}_{1,n}\int_{\Omega^*}\int_\Omega \phi_u(x)^2 l(u)^{-n}\,dx du\,h^{-n} \\
&\qquad - \frac{L^{\mathrm{cl}}_{1,n-1}}{4}h^{-n+1}\int_{\Omega_b\cup\Omega_s}\int_{\partial\Omega}\phi_u(x)^2 l(u)^{-n}\,d\sigma(x)du \\
&\qquad + \frac{L^{\mathrm{cl}}_{1,n-1}}{4}h^{-n+1}\int_{\Omega_s}\int_{\partial\Omega}\phi_u(x)^2 l(u)^{-n}\,d\sigma(x)du + c(|\Omega|+|\partial\Omega|)l_0^{-1}h^{-n+2}.
\end{aligned}$$

Here we have added and subtracted the boundary term integrated over Ω_s, and used (31) to bound the remainder from our application of Lemma 6.4. Using (26) we obtain that

$$\begin{aligned}
\mathrm{Tr}(-h^2\Delta_\Omega - 1)_- \le L^{\mathrm{cl}}_{1,n}|\Omega|h^{-n} &- \frac{L^{\mathrm{cl}}_{1,n-1}}{4}|\partial\Omega|h^{-n+1} \\
&+ \frac{L^{\mathrm{cl}}_{1,n-1}}{4}h^{-n+1}\int_{\Omega_s}\int_{\partial\Omega}\phi_u(x)^2 l(u)^{-n}\,d\sigma(x)du \\
&+ c(|\Omega|+|\partial\Omega|)l_0^{-1}h^{-n+2}.
\end{aligned} \tag{40}$$

Using (25) and the convexity of Ω it holds that

$$\begin{aligned}
\int_{\Omega_s}\int_{\partial\Omega}\phi_u(x)^2 l(u)^{-n}\,d\sigma(x)du &\le c\int_{\Omega_s}|\operatorname{supp}\phi_u\cap\partial\Omega|\,l(u)^{-n}\,du \\
&\le c\int_{\Omega_s} l(u)^{-1}\,du \le c|\Omega_s|l_0^{-1},
\end{aligned}$$

where we used that $l(u)\ge l_0/4$. We want to prove that we can choose l_0 such that

$$h^{-n+1}l_0^{-1}|\Omega_b| + (|\Omega|+|\partial\Omega|)l_0^{-1}h^{-n+2} = o(h^{-n+1}) \tag{41}$$

uniformly for Ω in compact subsets of $\mathcal{P}_m$. If we can prove that such a choice is possible the combination of (40) and Lemma 2.1 implies the claimed asymptotic expansion for $\gamma = 1$. As above an application of the Aizenman–Lieb identity completes the proof for all $\gamma > 1$.

Our aim is to show that $|\Omega_b|$ is small, specifically we shall show that it is $\sim l_0^2$. To this end we shall prove that Ω_b is contained in an l_0-neighbourhood of the $(n-2)$-dimensional faces of Ω.

Take $u \in \Omega_s$. By definition there are two points $x_1, x_2 \in B_{l(u)}(u) \cap \partial\Omega$ such that x_1, x_2 belong to two different faces of Ω (otherwise u would be in Ω_b). Let x_0 be a point in Ω such that $B_{r(\Omega)}(x_0) \subset \Omega$. Consider the plane spanned by the points x_0, x_1, x_2, noting that x_0, x_1, x_2 cannot lie on a line since by convexity this would imply that $x_0 \in \Omega^c$ which is a contradiction. Without loss of generality we can assume that x_0 is the origin. Since $|x_1 - x_2| \le 2l_0/\sqrt{3}$ we can if $l_0 \le r(\Omega)$ also assume that x_1, x_2 are in the same half-plane H.

Let Ω' be the polygon obtained as the intersection of Ω with this plane. Clearly $r(\Omega') \ge r(\Omega)$ and $D(\Omega') \le D(\Omega)$. We also note that the segment of $\partial\Omega' \cap H$ connecting x_1, x_2 must contain a point belonging to an $(n-2)$-dimensional face of Ω. Let x' be any such point. By convexity Ω' contains the open triangle which has one vertex at x' and the other two on $\partial B_{r(\Omega)}(x_0) \cap L$, where L is the line through x_0 perpendicular to that through x_0 and x'. In other words we consider the isosceles triangle with one side being a diameter of the disk $B_{r(\Omega)}(x_0)$ and symmetry axis being the segment from x_0 to x'. As $x_1, x_2 \in \partial\Omega$ they are necessarily in the complement of this triangle. Since $|x_1 - x_2| \le 2l_0/\sqrt{3}$ and $|x_0 - x'| \le D(\Omega)$ the convexity of Ω' and elementary trigonometry gives us that

$$\max\{|x_1 - x'|, |x_2 - x'|\} \le \frac{cD(\Omega)}{r(\Omega)} l_0.$$

We can thus conclude that Ω_b is contained in a $\frac{cD(\Omega)}{r(\Omega)} l_0$-neighbourhood of the $(n-2)$-dimensional faces of Ω. Let $\{F_k\}_k$ denote the collection of these faces. There are fewer than $\binom{m}{2}$ such faces and each of them is contained in a subset of an $(n-2)$-dimensional affine subspace of $\mathbb{R}^n$ whose diameter is less than $D(\Omega)$. Hence we find that

$$\begin{aligned}
|\Omega_b| &\le \left|\left\{u \in \mathbb{R}^n : \operatorname{dist}(u, \cup_k F_k) \le \frac{cD(\Omega)}{r(\Omega)} l_0\right\}\right| \\
&\le \sum_k \left|\left\{u \in \mathbb{R}^n : \operatorname{dist}(u, F_k) \le \frac{cD(\Omega)}{r(\Omega)} l_0\right\}\right| \\
&\le \binom{m}{2} \left|\left\{u \in \mathbb{R}^n : \operatorname{dist}(u, \hat{F}) < \frac{cD(\Omega)}{r(\Omega)} l_0\right\}\right| \\
&\le \frac{cD(\Omega)^n}{r(\Omega)^2} l_0^2,
\end{aligned}$$

where

$$\hat{F} = \{u \in \mathbb{R}^n : u_1 = u_2 = 0, |u| \le 2D(\Omega)\}.$$

Returning to (41) we can conclude that, with $l_0 = h^{1/3}$,

$$h^{-n+1}(l_0^{-1}|\Omega_b| + (|\Omega| + |\partial\Omega|)hl_0^{-2}) \le ch^{-n+4/3}\Big(\frac{D(\Omega)^n}{r(\Omega)^2} + |\Omega| + |\partial\Omega|\Big).$$

As the choice of l_0 clearly fulfils the requirements $h \le l_0 \le \min\{1, r(\Omega)\}$ as soon as $h \le \min\{1, r(\Omega)^3\}$ this completes the proof of Lemma 2.3. $\square$

7. Maximizing Riesz means over disjoint unions of convex domains

In this section we show that our results are unchanged if one allows also for disjoint unions of convex domains. We begin by proving that the result remains true if one allows two convex components.

Lemma 7.1. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^n$ which is invariant under dilations and satisfies the assumption in (ii) of Proposition 4.1. Fix $\gamma \ge 1$ and let $\Omega_{\Lambda,\gamma}(\mathcal{A}^2)$ denote any extremal domain of the shape optimization problem*

$$\begin{aligned}\sup\{\mathrm{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \colon |\Omega| = 1,\\ \Omega = \Omega_1 \cup \Omega_2,\\ \Omega_1 \cap \Omega_2 = \emptyset,\\ \Omega_j \in \mathcal{A} \text{ or } \Omega_j = \emptyset\}.\end{aligned}$$

Let also Ω_Λ^1 denote the largest of the two components of $\Omega_{\Lambda,\gamma}(\mathcal{A}^2)$.

For any sequence $\{\Lambda_j\}_{j\ge1} \uparrow \infty$ the corresponding sequence $\{\Omega_{\Lambda_j}^1\}_{j\ge1}$ has a subsequence which, up to rigid transformations, converges in $\mathcal{A}$. Moreover, Ω_∞ the limit of such a subsequence has unit measure and minimizes the perimeter in $\mathcal{A}$:

$$|\partial\Omega_\infty| = \inf\{|\partial\Omega| \colon \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Proof of Lemma 7.1. Fix $\gamma \ge 1$ and let $\Omega_{\Lambda,\gamma}(\mathcal{A}^2) = \Omega_\Lambda = \Omega_\Lambda^1 \cup \Omega_\Lambda^2$. Assume without loss of generality that $|\Omega_\Lambda^1| \ge 1/2$. Since the Riesz mean is additive under disjoint unions the two components must be maximizers for the shape optimization problems among domains in $\mathcal{A}$ of their respective measure. After rescaling to unit measure one finds that Ω_Λ^1 solves the one-component optimization problem at $\Lambda' = \Lambda|\Omega_\Lambda^1|^{2/n}$. Thus by Proposition 4.1 we are done as soon as we can show that $|\Omega_{\Lambda_j}^1| \to 1$ as $j \to \infty$.

After possibly passing to a subsequence of $\{\Lambda_j\}_{j\geq 1}$ we have two possibilities.

Case 1: $|\Omega^2_{\Lambda_j}| \to 0$ as $j \to \infty$. In which case we are done.

Case 2: $|\Omega^2_{\Lambda_j}| \geq c > 0$. Since Riesz means are additive under disjoint unions we have that the bound in Theorem 2.4 holds also in our current setting: Sum the corresponding bounds for the components of the disjoint union. Hence by arguing as in the first part of the proof of Proposition 4.1 we find that $|\partial\Omega_{\Lambda_j}| \leq c$. Thus both sequences $\{\Omega^1_{\Lambda_j}\}_{j\geq 1}, \{\Omega^2_{\Lambda_j}\}_{j\geq 1}$ are after translation contained in a compact subset of $\mathcal{K}^n$. Hence our assumptions imply that

$$\begin{aligned}
&\mathrm{Tr}(-\Delta_{\Omega_{\Lambda_j}} - \Lambda_j)^\gamma_-\\
&\quad= \mathrm{Tr}(-\Delta_{\Omega^1_{\Lambda_j}} - \Lambda)^\gamma_- + \mathrm{Tr}(-\Delta_{\Omega^2_{\Lambda_j}} - \Lambda_j)^\gamma_-\\
&\quad= L^{\mathrm{cl}}_{\gamma,n}|\Omega_{\Lambda_j}|\Lambda_j^{\gamma+n/2} - \frac{L^{\mathrm{cl}}_{\gamma,n-1}}{4}|\partial\Omega_{\Lambda_j}|\Lambda_j^{\gamma+(n-1)/2} + o(\Lambda_j^{\gamma+(n-1)/2}),
\end{aligned}$$

as $j \to \infty$. Arguing as in the proof of Proposition 4.1 we find that Ω_{Λ_j} converges to a domain which minimizes the perimeter among domains with at most two components, each of which is in $\mathcal{A}$. If $\Omega' = \Omega'_1 \cup \Omega'_2$ with $\Omega'_j \in \mathcal{A}$ it is clear that the perimeter of Ω' is minimal when the perimeter of the two components are minimizers of the perimeter in $\mathcal{A}$ among sets of their respective measure. By scaling we find that

$$|\partial\Omega'| = (|\Omega'_1|^{(n-1)/n} + |\Omega'_2|^{(n-1)/n}) \inf\{|\partial\Omega| \colon \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Since $\eta^{(n-1)/n} + (1-\eta)^{(n-1)/n} \geq 1$ with equality if and only if $\eta = 0$ or 1 we find that any domain minimizing the perimeter must have only one component. This contradicts the assumption that $|\Omega^2_{\Lambda_j}| \geq c$, and hence completes the proof. $\square$

Using the same idea as above it is not difficult to prove the corresponding result when any fixed and finite number of components is allowed. However, our goal is here to show that this restriction is in fact not necessary and we can allow for an arbitrary number of components. The only reason to first prove the two-component case is that it will be used in the proof of the general result.

Corollary 7.2. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}^n$ which is invariant under dilations and satisfies the assumption in (ii) of Proposition 4.1. Fix $\gamma \geq 1$ and let $\Omega_{\Lambda,\gamma}(\mathcal{A}^\infty)$ denote any extremal domain of the shape optimization problem*

$$\sup\{\mathrm{Tr}(-\Delta_\Omega - \Lambda)^\gamma_- \colon |\Omega| = 1,\ \Omega = \textstyle\bigcup_{k\geq 1}\Omega_k,\ \Omega_k \in \mathcal{A},\ \Omega_k \cap \Omega_{k'} = \emptyset \text{ if } k \neq k'\}.$$

Let also Ω^1_Λ denote the largest of the components of $\Omega_{\Lambda,\gamma}(\mathcal{A}^\infty)$.

For any sequence $\{\Lambda_j\}_{j\geq 1} \uparrow \infty$ *the corresponding sequence* $\{\Omega^1_{\Lambda_j}\}_{j\geq 1}$ *has a subsequence which, up to rigid transformations, converges in* $\mathcal{A}$. *Moreover,* Ω_∞ *the limit of such a subsequence has unit measure and minimizes the perimeter in* $\mathcal{A}$:

$$|\partial\Omega_\infty| = \inf\{|\partial\Omega| \colon \Omega \in \mathcal{A}, |\Omega| = 1\}.$$

Remark 7.3. We note that Corollary 7.2 can be interesting even in extremely simple cases. For instance, it implies that among unions of disjoint balls the maximizers will as $\Lambda \to \infty$ converge to a *single* ball of unit measure.

Proof of Corollary 7.2. Again we can argue as in Proposition 4.1 to find that

$$|\partial\Omega_{\Lambda,\gamma}(\mathcal{A}^\infty)| \leq c. \tag{42}$$

Moreover, by Faber–Krahn's inequality we know that each component of a maximizer $\Omega_{\Lambda,\gamma}(\mathcal{A}^\infty)$ has measure greater than $c\Lambda^{-n/2}$. Indeed, the Riesz mean is zero for any component with smaller measure, which contradicts the maximality of $\Omega_{\Lambda,\gamma}(\mathcal{A}^\infty)$ since we can remove such components and rescale the remaining domain to have measure one and in the process increasing the Riesz mean.

Let $\Omega_{\Lambda_j,\gamma}(\mathcal{A}^\infty) = \bigcup_{k\geq 1}\Omega^k_\Lambda$ be a maximizer, where we assume $|\Omega^k_{\Lambda_j}| \geq |\Omega^{k'}_{\Lambda_j}|$ if $k < k'$. Fix $\{\Lambda_j\}_{j\geq 1} \uparrow \infty$. After possibly passing to a subsequence we can assume that $|\Omega^1_{\Lambda_j}| < 1-\varepsilon$, for some $\varepsilon > 0$. If this is not the case we are already done.

Step 1. We first exclude that all components have size $\sim \Lambda_j^{-n/2}$. Assume that along the sequence Λ_j (or a subsequence thereof) we have that $|\Omega^1_{\Lambda_j}| \leq c\Lambda_j^{-n/2}$ for some $c > 0$. Due to the measure constraint we must have $\sim \Lambda_j^{n/2}$ components. By the isoperimetric inequality

$$|\partial\Omega_{\Lambda_j}| = \sum_{k\geq 1}|\partial\Omega^k_{\Lambda_j}| \gtrsim \Lambda_j^{n/2}\Lambda_j^{-(n-1)/2} = \Lambda_j^{1/2} \to \infty,$$

which contradicts (42).

Step 2. The set $\Omega^1_{\Lambda_j} \cup \Omega^2_{\Lambda_j}$ is a maximizer for the problem

$$\begin{aligned}\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda_j)_-^\gamma \colon & |\Omega| = m_j,\\ & \Omega = \Omega_1 \cup \Omega_2,\\ & \Omega_1 \cap \Omega_2 = \emptyset,\\ & \Omega_j \in \mathcal{A} \text{ or } \Omega_j = \emptyset\},\end{aligned}$$

with $m_j = |\Omega^1_{\Lambda_j}| + |\Omega^2_{\Lambda_j}|$. By Step 1 we can assume that $m_j \Lambda_j^{n/2} \to \infty$ and hence this problem is, after rescaling to unit measure, equivalent to that considered in Lemma 7.1. Hence we find that for j large enough $|\Omega^2_{\Lambda_j}| \leq c|\Omega^1_{\Lambda_j}|$ for any $0 < c < 1$ to be chosen later.

Step 3. Similarly, $\widehat{\Omega}_1 = \bigcup_{k\geq 2} \Omega^k_{\Lambda_j}$ is a maximizer for the problem

$$\begin{aligned}\sup\{\operatorname{Tr}(-\Delta_\Omega - \Lambda_j)^\gamma_-\colon |\Omega| = |\widehat{\Omega}_1|,&\\ \Omega = \textstyle\bigcup_{k\geq 1}\Omega_k,&\\ \Omega_k \in \mathcal{A},&\\ \Omega_k \cap \Omega_{k'} = \emptyset \text{ if } k \neq k'\}.&\end{aligned}$$

Since $|\widehat{\Omega}_1| = 1 - |\Omega^1_{\Lambda_j}| > \varepsilon$ this problem is again in the asymptotic regime and we can argue as in Steps 1 and 2 and find that $|\Omega^3_{\Lambda_j}| \leq c|\Omega^2_{\Lambda_j}|$ for any $0 < c < 1$ if j is large enough.

Step 4. Set $c = \frac{\varepsilon}{2-\varepsilon}$. We can then iterate the arguments above. For each $l > 1$ we have $\left|\bigcup_{k\geq l} \Omega^k_{\Lambda_j}\right| = 1 - \left|\bigcup_{k=1}^{l-1} \Omega^k_{\Lambda_j}\right| \geq 1 - |\Omega^1_{\Lambda_j}| \sum_{k=1}^{l-1} \frac{\varepsilon^{k-1}}{(2-\varepsilon)^{k-1}} > \varepsilon/2$, this ensures that the maximization problem which $\widehat{\Omega}_l = \bigcup_{k\geq l} \Omega^k_{\Lambda_j}$ solves is still in the asymptotic regime when $j \to \infty$. Hence, by Steps 1-3 it holds that $|\Omega^{k+1}_{\Lambda_j}| \leq |\Omega^k_{\Lambda_j}| \frac{\varepsilon}{2-\varepsilon}$, for all $k \geq 1$, provided that Λ_j is large enough (depending only on ε).

Step 5. Calculate the measure of $\Omega_{\Lambda_j,\gamma}(\mathcal{A}^\infty)$:

$$\begin{aligned}|\Omega_{\Lambda_j,\gamma}(\mathcal{A}^\infty)| &= \sum_{k\geq 1} |\Omega^k_{\Lambda_j}|\\ &\leq |\Omega^1_{\Lambda_j}| \sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{(2-\varepsilon)^{k-1}}\\ &\leq (1-\varepsilon) \sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{(2-\varepsilon)^{k-1}}\\ &= 1 - \varepsilon/2,\end{aligned}$$

which is a contradiction for all $\varepsilon > 0$ and hence $|\Omega^1_{\Lambda_j}| \to 1$ as $j \to \infty$.

By Proposition 4.1 we can conclude that $\Omega^1_{\Lambda_j}$ converges to a domain which minimizes the perimeter among domains of unit measure in $\mathcal{A}$. This completes the proof. □

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References

[1] M. Aizenman and E. H. Lieb, On semiclassical bounds for eigenvalues of Schrödinger operators. *Phys. Lett. A* **66** (1978), no. 6, 427–429. MR 0598768 Zbl

[2] P. R. S. Antunes and P. Freitas, Optimal spectral rectangles and lattice ellipses. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **469** (2013), no. 2150, 20120492, 15 pp. MR 3001382 Zbl 1371.49040

[3] S. Ariturk and R. S. Laugesen, Optimal stretching for lattice points under convex curves. *Port. Math.* **74** (2017), no. 2, 91–114. MR 3734407 Zbl 1386.35289

[4] F. A. Berezin, Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1134–1167. In Russian. English translation, *Math. USSR-Izv.* **6** (1972), 1117–1151. MR 0350504 Zbl 0259.47004

[5] M. van den Berg, On the minimization of Dirichlet eigenvalues. *Bull. Lond. Math. Soc.* **47** (2015), no. 1, 143–155. MR 3312973 Zbl 1317.49052

[6] M. van den Berg, D. Bucur, and K. Gittins, Maximising Neumann eigenvalues on rectangles. *Bull. Lond. Math. Soc.* **48** (2016), no. 5, 877–894. MR 3556370 Zbl 1358.35078

[7] M. van den Berg and K. Gittins, Minimizing Dirichlet eigenvalues on cuboids of unit measure. *Mathematika* **63** (2017), no. 2, 469–482. MR 3706591 Zbl 1378.35091

[8] D. Bucur, Regularity of optimal convex shapes. *J. Convex Anal.* **10** (2003), no. 2, 501–516. MR 2044433 Zbl 1057.49029

[9] D. Bucur, Minimization of the k-th eigenvalue of the Dirichlet Laplacian. *Arch. Ration. Mech. Anal.* **206** (2012), no. 3, 1073–1083. MR 2989451 Zbl 1254.35165

[10] D. Bucur and G. Buttazzo, *Variational methods in shape optimization problems.* Progress in Nonlinear Differential Equations and their Applications, 65. Birkhäuser Boston, Boston, MA, 2005. MR 2150214 Zbl 1117.49001

[11] D. Bucur and P. Freitas, Asymptotic behaviour of optimal spectral planar domains with fixed perimeter. *J. Math. Phys.* **54** (2013), no. 5, 053504, 6 pp. MR 3098942 Zbl 1290.35164

[12] D. Bucur, D. Mazzoleni, A. Pratelli, and B. Velichkov, Lipschitz regularity of the eigenfunctions on optimal domains. *Arch. Ration. Mech. Anal.* **216** (2015), no. 1, 117–151. MR 3305655 Zbl 1319.49067

[13] G. Buttazzo and G. Dal Maso, An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.* **122** (1993), no. 2, 183–195. MR 1217590 Zbl 0811.49028

[14] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations.* American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, R.I., 1995. MR 1351007 Zbl 0834.35002

[15] G. Faber, Beweis, daß unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. *Sitz. Ber. Bayer. Akad. Wiss.* **1923**, 169–172. JFM 49.0342.03

[16] R. L. Frank and L. Geisinger, Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain. In P. Exner (ed.), *Mathematical results in quantum physics.* Proceedings of the QMath11 Conference held in Hradec Králové, September 6–10, 2010. World Scientific, Hackensack, N.J., 2011, 138–147. MR 2885166 Zbl 1253.47031

[17] R. L. Frank and L. Geisinger, Semi-classical analysis of the Laplace operator with Robin boundary conditions. *Bull. Math. Sci.* **2** (2012), no. 2, 281–319. MR 2994205 Zbl 1253.35090

[18] R. L. Frank and L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator. *J. Reine Angew. Math.* **712** (2016), 1–37. MR 3466545 Zbl 1337.35163

[19] R. L. Frank and S. Larson, Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain. arXiv:1901.09771 [math.SP]

[20] P. Freitas, Asymptotic behaviour of extremal averages of Laplacian eigenvalues. *J. Stat. Phys.* **167** (2017), no. 6, 1511–1518. MR 3652523 Zbl 1372.35198

[21] L. Geisinger, A. Laptev, and T. Weidl, Geometrical versions of improved Berezin–Li–Yau inequalities. *J. Spectr. Theory* **1** (2011), no. 1, 87–109. MR 2820887 Zbl 1257.35138

[22] K. Gittins and S. Larson, Asymptotic behaviour of cuboids optimising Laplacian eigenvalues. *Integral Equations Operator Theory* **89** (2017), no. 4, 607–629. MR 3735512 Zbl 1388.35016

[23] P. M. Gruber, *Convex and discrete geometry.* Grundlehren der Mathematischen Wissenschaften, 336. Springer, Berlin, 2007. MR 2335496 Zbl 1139.52001

[24] J. Guo and W. Wang, Lattice points in stretched model domains of finite type in $\mathbb{R}^d$. *J. Number Theory* **191** (2018), 273–288. MR 3825471 Zbl 06901230

[25] E. M. Harrell II and J. Stubbe, Two-term asymptotically sharp estimates for eigenvalue means of the Laplacian. *J. Spectr. Theory* **8** (2018), 1529–1550.

[26] E. M. Harrell II, L. Provenzano, and J. Stubbe, Complementary asymptotically sharp estimates for eigenvalue means of Laplacians. Preprint, 2018. arXiv:1806.10366 [math.SP]

[27] A. Henrot (ed.), *Shape optimization and spectral theory.* De Gruyter Open, Warsaw, 2017. MR 3681143 Zbl 1369.49004

[28] A. Henrot and M. Pierre, *Variation et optimisation de formes.* Mathématiques & Applications (Berlin), 48. Springer, Berlin, 2005. MR 2512810 Zbl 1098.49001

[29] J. Hersch, Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum. *Z. Angew. Math. Phys.* **11** (1960), 387–413. MR 0125319 Zbl 0104.41403

[30] V. Y. Ivrii, On the second term of the spectral asymptotics for the Laplace–Beltrami operator on manifolds with boundary and for elliptic operators acting in fiberings. *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 2, 25–34. In Russian. English translation, *Functional Anal. Appl.* **14** (1980), no. 2, 98–106. MR 0575202 Zbl 0448.58024

[31] H. Kovařík, S. Vugalter, and T. Weidl, Two-dimensional Berezin–Li–Yau inequalities with a correction term. *Comm. Math. Phys.* **287** (2009), no. 3, 959–981. MR 2486669 Zbl 1197.35182

[32] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.* **94** (1925), no. 1, 97–100. MR 1512244 JFM 51.0356.05

[33] E. Krahn, Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen. *Acta Univ. Dorpat A* **9** (1926), 1–44. JFM 52.0510.03

[34] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.* **151** (1997), no. 2, 531–545. MR 1491551 Zbl 0892.35115

[35] S. Larson, A bound for the perimeter of inner parallel bodies. *J. Funct. Anal.* **271** (2016), no. 3, 610–619. MR 3506959 Zbl 1341.52005

[36] S. Larson, On the remainder term of the Berezin inequality on a convex domain. *Proc. Amer. Math. Soc.* **145** (2017), no. 5, 2167–2181. MR 3611329 Zbl 1377.35206

[37] S. Larson, Maximizing Riesz means of anisotropic harmonic oscillators. To appear in *Ark. Mat.* Preprint, 2017. arXiv:1712.10247 [math.SP]

[38] R. S. Laugesen and S. Liu, Optimal stretching for lattice points and eigenvalues. *Ark. Mat.* **56** (2018), no. 1, 111–145. MR 3800462 Zbl 1390.35206

[39] R. S. Laugesen and S. Liu, Shifted lattices and asymptotically optimal ellipses. *J. Anal.* **26** (2018), no. 1, 71–102. MR 3825687 Zbl 1391.35300

[40] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.* **88** (1983), no. 3, 309–318. MR 0701919 Zbl 0554.35029

[41] E. H. Lieb, The classical limit of quantum spin systems. *Comm. Math. Phys.* **31** (1973), 327–340. MR 0349181 Zbl 1125.82305

[42] N. F. Marshall, Stretching convex domains to capture many lattice points. *Int. Math. Res. Not. IMRN* (2018), published online, rny102.

[43] N. F. Marshall and S. Steinerberger, Triangles capturing many lattice points. *Mathematika* **64** (2018), no. 2, 551–582. MR 3798612 Zbl 06908365

[44] D. Mazzoleni, Recent existence results for spectral problems. In A. Pratelli and G. Leugering (eds.), *New trends in shape optimization.* Papers based on the workshop held at Friedrich-Alexander University Erlangen-Nürnberg, Erlangen, September 2013. International Series of Numerical Mathematics, 166. Birkhäuser/Springer, Cham, 2015, 199–215. MR 3467381 Zbl 1329.49089

[45] D. Mazzoleni, Boundedness of minimizers for spectral problems in $\mathbb{R}^N$. *Rend. Semin. Mat. Univ. Padova* **135** (2016), 207–221. MR 3506069 Zbl 1343.49071

[46] D. Mazzoleni and A. Pratelli, Existence of minimizers for spectral problems. *J. Math. Pures Appl.* (9) **100** (2013), no. 3, 433–453. MR 3095209 Zbl 1296.35100

[47] D. Mazzoleni, S. Terracini, and B. Velichkov, Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.* **27** (2017), no. 2, 373–426. MR 3626615 Zbl 1368.49047

[48] A. D. Melás, A lower bound for sums of eigenvalues of the Laplacian. *Proc. Amer. Math. Soc.* **131** (2003), no. 2, 631–636. MR 1933356 Zbl 1015.58011

[49] M. H. Protter, A lower bound for the fundamental frequency of a convex region. *Proc. Amer. Math. Soc.* **81** (1981), no. 1, 65–70. MR 0589137 Zbl 0476.35060

[50] J. W. S. Rayleigh, *The theory of sound.* Macmillan, London, 1877.

[51] R. Schneider, *Convex bodies: the Brunn–Minkowski theory.* Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014. MR 3155183 Zbl 1287.52001

[52] J. P. Solovej and W. L. Spitzer, A new coherent states approach to semiclassics which gives Scott's correction. *Comm. Math. Phys.* **241** (2003), no. 2-3, 383–420. MR 2013804 Zbl 1098.81040

[53] P. Steinhagen, Über die größte Kugel in einer konvexen Punktmenge. *Abh. Math. Sem. Univ. Hamburg* **1** (1921), no. 1, 15–26. MR 3069385 JFM 48.0837.03

[54] G. Szegő, Inequalities for certain eigenvalues of a membrane of given area. *J. Rational Mech. Anal.* **3** (1954). 343–356. MR 0061749 Zbl 0055.08802

[55] B. Velichkov, *Existence and regularity results for some shape optimization problems.* Tesi. Scuola Normale Superiore di Pisa (Nuova Series), 19. Edizioni della Normale, Pisa, 2015. MR 3309888 Zbl 1316.49006

[56] T. Weidl, Improved Berezin–Li–Yau inequalities with a remainder term. In T. Suslina and D. Yafaev (eds.), *Spectral theory of differential operators.* M. Sh. Birman 80$^{\text{th}}$ anniversary collection. American Mathematical Society Translations, Series 2, 225. Advances in the Mathematical Sciences, 62. American Mathematical Society, Providence, R.I., 2008, 253–263. MR 2509788 Zbl 1168.35404

[57] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.* **71** (1912), no. 4, 441–479. MR 1511670 JFM 43.0436.01

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Asymptotic Behaviour of Cuboids Optimising Laplacian Eigenvalues

Katie Gittins and Simon Larson

Abstract. We prove that in dimension $n \geq 2$, within the collection of unit-measure cuboids in $\mathbb{R}^n$ (i.e. domains of the form $\prod_{i=1}^{n}(0, a_n)$), any sequence of minimising domains $R_k^{\mathcal{D}}$ for the Dirichlet eigenvalues λ_k converges to the unit cube as $k \to \infty$. Correspondingly we also prove that any sequence of maximising domains $R_k^{\mathcal{N}}$ for the Neumann eigenvalues μ_k within the same collection of domains converges to the unit cube as $k \to \infty$. For $n = 2$ this result was obtained by Antunes and Freitas in the case of Dirichlet eigenvalues and van den Berg, Bucur and Gittins for the Neumann eigenvalues. The Dirichlet case for $n = 3$ was recently treated by van den Berg and Gittins. In addition we obtain stability results for the optimal eigenvalues as $k \to \infty$. We also obtain corresponding shape optimisation results for the Riesz means of eigenvalues in the same collection of cuboids. For the Dirichlet case this allows us to address the shape optimisation of the average of the first k eigenvalues.

Mathematics Subject Classification. 35J20, 35P99.

Keywords. Spectral optimisation, Laplacian, Eigenvalues, Asymptotics, Cuboids.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open set with finite Lebesgue measure $|\Omega| < \infty$. Then the spectrum of the Dirichlet Laplace operator $-\Delta_\Omega^{\mathcal{D}}$ acting on $L^2(\Omega)$ is discrete and its eigenvalues can be written in a non-decreasing sequence, repeating each eigenvalue according to its multiplicity,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots,$$

with $\lambda_1(\Omega) > 0$. Moreover, the sequence accumulates only at infinity.

If in addition the boundary of Ω is Lipschitz regular, then the spectrum of the Neumann Laplace operator $-\Delta_\Omega^{\mathcal{N}}$ is discrete and its eigenvalues can be written in a non-decreasing sequence, repeating each eigenvalue according to its multiplicity,

$$0 = \mu_0(\Omega) \le \mu_1(\Omega) \le \cdots \le \mu_k(\Omega) \le \cdots$$

Again the sequence accumulates only at infinity.

1.1. Optimising Laplacian Eigenvalues with a Measure Constraint

For $k \in \mathbb{N}$ and fixed $c > 0$, the existence of sets $\Omega_k^{\mathcal{D}}$ and $\Omega_k^{\mathcal{N}}$ which realise the infimum respectively the supremum in the optimisation problems

$$\lambda_k(\Omega_k^{\mathcal{D}}) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n \text{ open}, |\Omega| = c\},$$
$$\mu_k(\Omega_k^{\mathcal{N}}) = \sup\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^n \text{ open and Lipschitz}, |\Omega| = c\}$$

has received a great deal of attention throughout the last century.

It was shown by Faber [10] in $\mathbb{R}^2$ and Krahn [22,23] in any dimension that the first Dirichlet eigenvalue is minimised by the ball of measure c. Furthermore, Krahn [23] proved that the disjoint union of two balls each of measure $\frac{c}{2}$ minimises λ_2. In the Neumann case it was shown by Szegő [35] and Weinberger [39] that the ball of measure c maximises μ_1. Girouard et al. [12] proved that amongst all bounded, open, planar, simply connected sets of area c, the maximum of μ_2 is realised by a sequence of sets which degenerates to the disjoint union of two discs each of area $\frac{c}{2}$.

For $k \ge 3$, it is known that a minimiser of λ_k exists in the collection of quasi-open sets, see [7,32]. But whether these minimisers are open is currently unresolved. In general minimisers of λ_3 are not known to date, but there are some conjectures for them, for example see [15,33]. In the plane, and with $k \ge 5$, it is known that neither a disc nor a disjoint union of discs minimises λ_k [6]. In addition, for some values of $k \ge 3$, numerical evidence suggests that minimisers of λ_k might not have any natural symmetries, see [2].

In the Neumann case the existence of a maximising set which realises the above supremum remains open to date (see, for example, [8, Sect. 7.4]).

1.2. Asymptotic Shape Optimisation

An idea brought forward by Antunes and Freitas [3] was to consider the behaviour of minimisers of λ_k at the other end of the spectrum. That is, for a collection of sets in which a minimiser $\Omega_k^{\mathcal{D}}$ of λ_k exists for all $k \in \mathbb{N}$, to determine the limiting shape of a sequence of minimising sets $(\Omega_k^{\mathcal{D}})_k$ as $k \to \infty$. Analogously, if a maximiser $\Omega_k^{\mathcal{N}}$ of μ_k exists in some collection of sets, then one can consider the asymptotic behaviour of a sequence of maximising sets $(\Omega_k^{\mathcal{N}})_k$ as $k \to \infty$.

It was shown in [9] that the statement that $\lambda_k(\Omega_k^{\mathcal{D}})$ resp. $\mu_k(\Omega_k^{\mathcal{N}})$ is asymptotically equal to $4\pi^2\omega_n^{-2/n}(\frac{k}{c})^{2/n}$ as $k \to \infty$, where ω_n is the measure of the unit ball in $\mathbb{R}^n$, is equivalent to Pólya's conjecture: for $k \in \mathbb{N}$ and any bounded, open set $\Omega \subset \mathbb{R}^n$ of measure c,

$$\begin{aligned} \lambda_k(\Omega) &\ge 4\pi\Gamma\Big(\frac{n}{2}+1\Big)^{2/n}\Big(\frac{k}{c}\Big)^{2/n}, \\ \mu_k(\Omega) &\le 4\pi\Gamma\Big(\frac{n}{2}+1\Big)^{2/n}\Big(\frac{k}{c}\Big)^{2/n}. \end{aligned} \tag{1}$$

Note that the right-hand side of (1) is precisely the quantity we would like to find as $k \to \infty$, since $4\pi^2\omega_n^{-2/n} = 4\pi\Gamma(\frac{n}{2}+1)^{2/n}$. These inequalities were shown to hold for tiling domains by Pólya [34], see also [21]. In particular, they hold for $\Omega = \prod_{i=1}^{n}(0, a_i)$.

In [3], it was shown that amongst all planar rectangles of unit area, any sequence of minimising rectangles for λ_k converges to the unit square as $k \to \infty$. In [37] it was shown that the corresponding result holds in the Neumann case. Furthermore, the analogous result for the Dirichlet eigenvalues in three dimensions was proven in [38]. That is, amongst all cuboids in $\mathbb{R}^3$ of unit volume, any sequence of cuboids minimising λ_k converges to the unit cube as $k \to \infty$. For the Dirichlet eigenvalues, it was conjectured in [4] that the analogous result also holds in dimensions $n \geq 4$, and some support for this conjecture was obtained there (see [4, Sect. 2]). Similar arguments also suggest that the corresponding result holds for the Neumann eigenvalues in dimensions $n \geq 3$ (by invoking [4, Theorem 4] instead of [4, Theorem 1]).

The goal of this paper is to generalise the results of [3,37,38] to arbitrary dimensions. To that end, throughout the paper we let $R = R_{a_1,\dots,a_n}$ denote an n-dimensional cuboid of unit measure, that is a domain of the form $\prod_{i=1}^{n}(0, a_i) \subset \mathbb{R}^n$ where $a_1, \dots, a_n \in \mathbb{R}_+$ are such that $\prod_{i=1}^{n} a_i = 1$. Without loss of generality we will always label the a_i so that $a_1 \leq \dots \leq a_n$. Moreover, we let Q denote the n-dimensional unit cube.

For $k \in \mathbb{N}$, $\lambda_k(R)$ and $\mu_k(R)$ obey the two-term asymptotic formulae

$$\begin{aligned} \lambda_k(R) &= 4\pi\Gamma\Big(\frac{n}{2}+1\Big)^{2/n} k^{2/n} + \frac{2\pi\Gamma(\frac{n}{2}+1)^{1+1/n}}{n\Gamma(\frac{n+1}{2})}|\partial R| k^{1/n} + o(k^{1/n}), \\ \mu_k(R) &= 4\pi\Gamma\Big(\frac{n}{2}+1\Big)^{2/n} k^{2/n} - \frac{2\pi\Gamma(\frac{n}{2}+1)^{1+1/n}}{n\Gamma(\frac{n+1}{2})}|\partial R| k^{1/n} + o(k^{1/n}), \end{aligned} \tag{2}$$

as $k \to \infty$ (see [20] or Sect. 2.3). Here, and in what follows, $|\partial R|$ denotes the perimeter of R. Corresponding two-term asymptotic formulae were conjectured by Weyl for more general domains $\Omega \subset \mathbb{R}^n$, and under certain regularity assumptions the conjecture was proven by Ivrii in [20].

Since the cube in $\mathbb{R}^n$ has smallest perimeter in the collection of n-dimensional cuboids, (2) suggests that the cube is the limiting domain of a sequence of optimising cuboids in this collection as $k \to \infty$. However, this argument does not provide a proof as we are not considering a fixed cuboid R and then letting $k \to \infty$. The minimising or maximising cuboids themselves depend upon k (see, for instance, [4]).

1.3. Eigenvalues of Cuboids

For a cuboid R as above, the Laplacian eigenvalues are given by

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \dots + \frac{\pi^2 i_n^2}{a_n^2}, \tag{3}$$

where $i_1, \dots, i_n$ are positive integers in the Dirichlet case and non-negative integers in the Neumann case.

From (3) we see that a minimising cuboid of unit measure for λ_k, $k \in \mathbb{N}$, must exist. Indeed, as in [38], we consider a minimising sequence for λ_k where one side-length is blowing up. Then another side-length must be shrinking in order to preserve the measure constraint. However, this shrinking side would give rise to large eigenvalues, whilst for the unit cube Q we have that $\lambda_k(Q) \le n\pi^2 k^2 < \infty$, contradicting the minimality of the sequence. To emphasise the optimality, when referring to a cuboid which minimises λ_k we will write $R_k^{\mathcal{D}}$ and denote its side-lengths by $a_{1,k}^*, \ldots, a_{n,k}^*$.

Similarly we see that a maximising cuboid of unit measure for μ_k, $k \in \mathbb{N}$, exists. As in [37], if $(R_\ell)_{\ell\in\mathbb{N}} = (R_{a_1^\ell,\ldots,a_n^\ell})_{\ell\in\mathbb{N}}$ is a maximising sequence for μ_k with $a_n^\ell \to \infty$ as $\ell \to \infty$, then for sufficiently large ℓ

$$\mu_k(R_\ell) \le \frac{\pi^2 k^2}{(a_n^\ell)^2}$$

and so $\mu_k(R_\ell) \to 0$ as $\ell \to \infty$. For the unit cube Q we have that $\mu_k(Q) > \pi^2$, contradicting the maximality of the sequence. When referring to a cuboid which maximises μ_k we will write $R_k^{\mathcal{N}}$ and let $a_{1,k}^*, \ldots, a_{n,k}^*$ denote its side-lengths.

1.4. Main Results

Before we state our results we need the following definition which plays a central role in what follows.

Definition 1.1. *For $n \ge 2$, we define θ_n as any exponent such that for all $a_1, \ldots, a_n \in \mathbb{R}_+$,*

$$\#\{z \in \mathbb{Z}^n : a_1^{-2} z_1^2 + \cdots + a_n^{-2} z_n^2 \le t^2\} - \omega_n t^n \prod_{i=1}^n a_i = O(t^{\theta_n}), \tag{4}$$

as $t \to \infty$, uniformly for a_i on compact subsets of $\mathbb{R}_+$.

Geometrically θ_n describes the asymptotic order of growth of the difference between the number of integer lattice points in the ellipsoid $a_1^{-2}x_1^2 + \cdots + a_n^{-2}x_n^2 \le t^2$ and its volume. Finding the optimal order of growth in the case $n = 2$ and $a_1 = a_2 = 1$ is the well-known, and still open, Gauss circle problem (see [19] and references therein).

If (4) is not required to hold uniformly for different a_i, then estimates for θ_n are well-known (see, for instance, [14,18,19]). However, with the additional requirement of a uniform remainder term the literature is less extensive. For $n \ge 5$, $\theta_n = n - 2$ is known to hold and to be optimal [13]. As far as the authors are aware, the smallest known value, for $n = 3, 4$, is $\theta_n = \frac{n(n-1)}{n+1}$ which is due to Herz [16]. For $n = 2$ it holds that $\theta_2 \le \frac{46}{73} + \varepsilon$, for any $\varepsilon > 0$, due to Huxley [17]. In all dimensions, $\theta_n < n - 1$.

The main aim of this paper is to prove the following theorems, and thereby extend the results of [3,37,38] to all dimensions.

Theorem 1.1. *Let $n \ge 2$. For $k \in \mathbb{N}$, let $R_k^{\mathcal{D}}$ denote an n-dimensional unit-measure cuboid which minimises λ_k. Then, as $k \to \infty$, we have that*

$$a_{n,k}^* = 1 + O(k^{(\theta_n-(n-1))/(2n)}),$$

where θ_n is as defined in (4). That is, any sequence of minimising cuboids $(R_k^{\mathcal{D}})_k$ for λ_k converges to the n-dimensional unit cube as $k \to \infty$.

Theorem 1.2. *Let $n \geq 2$. For $k \in \mathbb{N}$, let $R_k^{\mathcal{N}}$ denote an n-dimensional unit-measure cuboid which maximises μ_k. Then, as $k \to \infty$, we have that*

$$a_{n,k}^* = 1 + O(k^{(\theta_n - (n-1))/(2n)}),$$

where θ_n is as defined in (4)*. That is, any sequence of maximising cuboids $(R_k^{\mathcal{N}})_k$ for μ_k converges to the n-dimensional unit cube as $k \to \infty$.*

A further interesting question is what this implies for the difference between $\lambda_k^* = \lambda_k(R_k^{\mathcal{D}})$ and $\lambda_k(Q)$, resp. $\mu_k^* = \mu_k(R_k^{\mathcal{N}})$ and $\mu_k(Q)$. By Pólya's inequalities (1), and the leading order asymptotics of $\lambda_k(Q), \mu_k(Q)$, we see that $|\lambda_k(Q) - \lambda_k^*| = O(k^{1/n})$ and $|\mu_k(Q) - \mu_k^*| = O(k^{1/n})$. By a more detailed analysis, we obtain the following.

Theorem 1.3. *As $k \to \infty$,*

$$|\lambda_k(Q) - \lambda_k^*| = O(k^{(\theta_n - (n-2))/n}),$$
$$|\mu_k(Q) - \mu_k^*| = O(k^{(\theta_n - (n-2))/n}),$$

where θ_n is as defined in (4)*.*

Note that for $n \geq 5$ the above estimate states that the difference between the extremal eigenvalues and those of the unit cube remain bounded for all k, which we do not know to be the case for $n < 5$.

1.5. Strategy of Proof

Let $r \geq 0$ and let $R \subset \mathbb{R}^n$ be a cuboid of measure one. We define

$$E(r, R) := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} \leq \frac{r}{\pi^2} \right\}. \tag{5}$$

The set $E(r, R) \subset \mathbb{R}^n$ is an n-dimensional ellipsoid with radii $r_j = \frac{a_j r^{1/2}}{\pi}$, $j = 1, \ldots, n$, and measure $|E(r, R)| = \omega_n \prod_{j=1}^{n} r_j = \frac{\omega_n r^{n/2}}{\pi^n}$.

By (3), we see that the Dirichlet eigenvalues $\lambda_1(R), \ldots, \lambda_k(R)$ correspond to integer lattice points with positive coordinates that lie inside or on the ellipsoid $E(\lambda_k(R), R)$. In this setting, determining a cuboid of unit measure which minimises λ_k corresponds to determining the ellipsoid which contains k integer lattice points with positive coordinates and has minimal measure. Similarly, the Neumann eigenvalues $\mu_0(R), \mu_1(R), \ldots, \mu_k(R)$ correspond to integer lattice points with non-negative coordinates that lie inside or on the ellipsoid $E(\mu_k(R), R)$. Determining a cuboid of unit measure which maximises μ_k corresponds to determining the ellipsoid of maximal measure which contains fewer than $k + 1$ integer lattice points with non-negative coordinates.

This observation is used to prove Theorems 1.1 and 1.2 by following the strategy of [3] (see also [37,38]). In particular, we compare the number of lattice points that are inside or on a minimal, respectively maximal, ellipsoid to the number of lattice points that are inside or on the sphere with radius $\pi^{-1}(\lambda_k^*)^{1/2}$, respectively $\pi^{-1}(\mu_k^*)^{1/2}$, and let $k \to \infty$. To make this comparison, we use known estimates for the number of integer lattice points that are inside or on an n-dimensional ellipsoid (this explains the appearance of the quantity θ_n in the above results). However, in order to use these estimates, we must first show that for any sequence of minimising or maximising cuboids, the corresponding side-lengths are bounded independently of k. The difficulty lies in obtaining a sufficiently good upper, resp. lower, bound for the Dirichlet, resp. Neumann, counting function which, for $\lambda, \mu \geq 0$ and $R, E(r, R)$ as above, we define as

$$\begin{aligned} N^{\mathcal{D}}(\lambda, R) &:= \#\{(i_1, \ldots, i_n) \in \mathbb{N}^n \cap E(\lambda, R)\}, \\ N^{\mathcal{N}}(\mu, R) &:= \#\{(i_1, \ldots, i_n) \in (\mathbb{N} \cup \{0\})^n \cap E(\mu, R)\}. \end{aligned} \tag{6}$$

In this paper, in order to obtain an upper bound for $N^{\mathcal{D}}(\lambda, R)$ and corresponding lower bound for $N^{\mathcal{N}}(\mu, R)$, we make use of an argument going back to Laptev [24] and the fact that cuboids satisfy Pólya's inequalities (1). This argument, together with an application of an identity due to Aizenman and Lieb (see [1] or (9) below), allows us to reduce the problem to estimating $\sum_k (\lambda - k^2)_+$, which arises as the Riesz mean of the Laplacian on an interval.

The approach taken in [3,37,38] to prove the two- and three-dimensional versions of Theorems 1.1 and 1.2 makes use of the fact that the functions $i \mapsto (y - i^2)^{m/2}$, for $m = 1, 2$, are concave on $[0, y^{1/2}]$. However, for $m \geq 3$, this concavity fails and hence this approach cannot be used to deal with the higher-dimensional cases, see [38]. To use the same approach as in [37] to deal with the case $n = 3$, it would also be necessary to show that $\limsup_{k\to\infty} (a^*_{1,k})^{-1} (\mu^*_k)^{-1/2} < \infty$ (compare with [37, Lemma 2.3]). The approach taken for the Neumann case here allows us to obtain a two-term lower bound for $N^{\mathcal{N}}(\mu, R)$ which enables us to avoid such considerations. This issue was also avoided when the two-dimensional case was proven in [28]. Nonetheless, in any dimension it is possible to obtain a bound for the quantity $\limsup_{k\to\infty} (a^*_{1,k})^{-1} (\mu^*_k)^{-1/2}$ by exploiting that if $a^*_{1,k} = o((\mu^*_k)^{-1/2})$ then all $\mu_l(R^{\mathcal{N}}_k)$, for $l < k$, must be of the form $\pi^2 \sum_{j=2}^n i_j^2 (a^*_{j,k})^{-2}$ and by the maximality of μ^*_k the domain $\prod_{j=2}^n (0, a^*_{j,k})$ must be a maximiser of μ_k amongst cuboids in $\mathbb{R}^{n-1}$ of measure $1/a^*_{1,k}$.

1.6. Additional Remarks

Our approach naturally lifts to considering the shape optimisation problems of maximising, resp. minimising, the Riesz means of Dirichlet, resp. Neumann, eigenvalues, which for $\lambda, \mu \geq 0$ and $\gamma \geq 0$ are defined by

$$\operatorname{Tr}(-\Delta^{\mathcal{D}}_\Omega - \lambda)^\gamma_- = \sum_{k=1}^\infty (\lambda - \lambda_k(\Omega))^\gamma_+,$$

resp.

$$\operatorname{Tr}(-\Delta^{\mathcal{N}}_\Omega - \mu)^\gamma_- = \sum_{k=0}^\infty (\mu - \mu_k(\Omega))^\gamma_+.$$

For $\Omega \subset \mathbb{R}^n$ and $\gamma \geq 3/2$ the Dirichlet case of this problem was addressed in [27], where it was shown that amongst collections of convex sets of unit measure, satisfying certain additional regularity assumptions, the extremal sets converge to the ball as $\lambda \to \infty$. Within the collection of n-dimensional cuboids we obtain the corresponding result for all $\gamma \geq 0$ in both the Dirichlet and Neumann cases, that is, any sequence of optimal cuboids converges to the unit cube as $\lambda, \mu \to \infty$ (see Propositions 4.1 and 4.2 below).

A problem which is closely related to that considered here was recently studied by Laugesen and Liu [28]. In this article the authors consider a collection of concave, planar curves that lie in the first quadrant and have intercepts $(L, 0)$ and $(0, M)$. They fix such a curve and scale it in the x direction by s^{-1} and in the y direction by s, as well as radially by r. Their goal is to determine the curve which contains the most integer lattice points in the first quadrant as $r \to \infty$. Under certain assumptions on the curve they prove that the optimal stretch factor $s(r) \to 1$ as $r \to \infty$. In particular, they recover the result of Antunes and Freitas [3], and, in a similar way, that of van den Berg et al. [37]. They also obtain analogous results for p-ellipses where $1 < p < \infty$. The case where $0 < p < 1$ has recently been addressed by Ariturk and Laugesen [4]. As mentioned above, the results of that paper lend some support to Theorem 1.1 in the case where $n \geq 5$ (see [4, Sect. 2]). Recently the case $p = 1$ was treated by Marshall and Steinerberger [31]. In contrast to the case $p \neq 1$, the set of maximising s in this setting does not converge when $r \to \infty$ and in fact there is an infinite set of limit points. After the first version of this paper appeared Marshall generalised the results of Laugesen and Liu to an n-dimensional setting [30]. The results of that paper include the convergence results of Theorems 1.1 and 1.2 as special cases.

The plan for the remainder of the paper is as follows. In Sect. 2.1 we obtain bounds for the eigenvalue counting functions $N^{\mathcal{D}}, N^{\mathcal{N}}$. We continue in Sect. 2.2 by applying the obtained bounds to prove that the side-lengths of a sequence of minimising, respectively maximising, cuboids $(R_k^{\mathcal{D}})_k$, $(R_k^{\mathcal{N}})_k$ are bounded independently of k. In Sect. 2.3 we prove uniform asymptotic expansions for the counting functions $N^{\mathcal{D}}(\lambda, R), N^{\mathcal{N}}(\mu, R)$. All the above is combined in Sect. 3 in order to prove Theorems 1.1, 1.2 and 1.3. Finally, in Sect. 4 we apply our methods to the shape optimisation problems of maximising, resp. minimising, the Riesz means of Dirichlet, resp. Neumann, eigenvalues and minimising the average of the first k Dirichlet eigenvalues. For both problems we obtain analogous results to those obtained in the case of individual eigenvalues.

2. Preliminaries

We begin this section by establishing three- respectively two-term bounds for the eigenvalue counting functions for the Dirichlet and Neumann Laplacians on an arbitrary cuboid. These bounds will allow us to prove that the sequence of extremal cuboids remains uniformly bounded, i.e. does not degenerate, as k tends to infinity (see Sect. 2.2).

We end this section by obtaining precise and uniform asymptotic expansions for the eigenvalue counting functions on the sequence of extremal cuboids.

Here and in what follows we let $L^{\mathrm{cl}}_{\gamma,m}$ denote the semi-classical Lieb–Thirring constant

$$L^{\mathrm{cl}}_{\gamma,m} = \frac{\Gamma(\gamma+1)}{(4\pi)^{m/2}\Gamma(\gamma+\frac{m}{2}+1)}.$$

For $x \in \mathbb{R}$ we also define the positive and negative parts of x by $x_\pm = (|x| \pm x)/2$.

2.1. Asymptotically Sharp Bounds for the Eigenvalue Counting Functions

In this section we prove a three-term upper bound for the counting function $N^{\mathcal{D}}(\lambda, R)$ and a two-term lower bound for the counting function $N^{\mathcal{N}}(\mu, R)$. More specifically we prove the following lemmas.

Lemma 2.1. *For $n \geq 2$, there exist positive constants c_1, c_2 and b_0 such that, for any cuboid $R \subset \mathbb{R}^n$ with $|R| = 1$, the bound*

$$N^{\mathcal{D}}(\lambda, R) \leq L^{\mathrm{cl}}_{0,n}\lambda^{n/2} - \frac{c_1 b L^{\mathrm{cl}}_{0,n-1}}{a_1}\lambda^{(n-1)/2} + \frac{c_2 b^2 L^{\mathrm{cl}}_{0,n-2}}{a_1^2}\lambda^{(n-2)/2},$$

holds for all $\lambda \geq 0$ and $b \in \big[0, b_0\big]$.

Lemma 2.2. *For $n \geq 2$, there exists a constant $c_1 > 0$ such that for any cuboid $R \subset \mathbb{R}^n$, with $|R| = 1$, the bound*

$$N^{\mathcal{N}}(\mu, R) \geq L^{\mathrm{cl}}_{0,n}\mu^{n/2} + \frac{c_1 L^{\mathrm{cl}}_{0,n-1}}{a_1}\mu^{(n-1)/2},$$

holds for all $\mu \geq 0$.

Remark 2.3. The parameter b in our bounds for $N^{\mathcal{D}}(\lambda, R)$ allows us to tune whether we wish the bound to be more accurate near the bottom of the spectrum or asymptotically as $\lambda \to \infty$. This flexibility will be of importance for us when we prove the uniform boundedness of the extremal cuboids for the Dirichlet problem, see Sect. 2.2.

It should be noted that for large λ the third term is not fundamental and could be absorbed by the second one. For instance, when $n \geq 5$ a bound similar to Lemma 2.1 was obtained in [26, Corollary 1.2] without the third term by instead requiring that λ is large enough. Similarly a two-term bound in the two-dimensional case was obtained in [28, Proposition 10]. However, the procedure of lifting the above bounds to Riesz means is much simplified if the bounds are valid for all $\lambda \geq 0$, and correspondingly $\mu \geq 0$ (see Sect. 4).

Proof of Lemmas 2.1 and 2.2. The main idea of the proof is to reduce the problem to proving one-dimensional estimates. To this end we follow an idea due to Laptev [24], which uses the fact that cuboids satisfy Pólya's inequalities (1) and the product structure of the domains. Let $R' = (0, a_2) \times \cdots \times (0, a_n)$ and write

$$\begin{aligned} N^{\mathcal{D}}(\lambda, R) &= \sum_{k:\lambda_k(R)\leq\lambda} (\lambda - \lambda_k(R))^0 \\ &= \sum_{k,l:\lambda_k(R')+\lambda_l((0,a_1))\leq\lambda} (\lambda - \lambda_l((0,a_1)) - \lambda_k(R'))^0 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l:\lambda_l((0,a_1))\le\lambda} \; \sum_{k:\lambda_k(R')\le\lambda-\lambda_l((0,a_1))} ((\lambda-\lambda_l((0,a_1)))-\lambda_k(R'))^0 \\
&= \sum_{l:\lambda_l((0,a_1))\le\lambda} N^{\mathcal{D}}((\lambda-\lambda_l((0,a_1)))_+, R'),
\end{aligned}
$$

where we use the convention "$0^0 = 1$". The above could be done with strict inequalities to avoid this issue, but to match (5), (6) we also wish to count the eigenvalues that are equal to λ. Applying Pólya's inequality for the counting function on R', which says $N^{\mathcal{D}}(\lambda, R') \le L^{\mathrm{cl}}_{0,n-1}|R'|\lambda^{(n-1)/2}$ (see [34]), yields that

$$
\begin{aligned}
N^{\mathcal{D}}(\lambda, R) &\le \sum_{l:\lambda_l((0,a_1))\le\lambda} L^{\mathrm{cl}}_{0,n-1}|R'|(\lambda-\lambda_l((0,a_1)))_+^{(n-1)/2} \\
&= L^{\mathrm{cl}}_{0,n-1}|R'|\operatorname{Tr}(-\Delta^{\mathcal{D}}_{(0,a_1)}-\lambda)_-^{(n-1)/2}. \qquad (7)
\end{aligned}
$$

Analogously, with the only difference being that Pólya's inequality goes in the opposite direction, one finds that

$$
N^{\mathcal{N}}(\mu, R) \ge L^{\mathrm{cl}}_{0,n-1}|R'|\operatorname{Tr}(-\Delta^{\mathcal{N}}_{(0,a_1)}-\mu)_-^{(n-1)/2}. \qquad (8)
$$

The Aizenman–Lieb Identity [1] asserts that if $\gamma_1 \ge 0$ and $\gamma_2 > \gamma_1$, then, for $\eta \ge 0$,

$$
\operatorname{Tr}(-\Delta_\Omega-\eta)_-^{\gamma_2} = B(1+\gamma_1,\gamma_2-\gamma_1)^{-1}\int_0^\infty \tau^{-1+\gamma_2-\gamma_1}\operatorname{Tr}(-\Delta_\Omega-(\eta-\tau))_-^{\gamma_1}\,d\tau, \qquad (9)
$$

where B denotes the Euler Beta function:

$$
B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.
$$

The identity follows immediately from linearity and that, for any $a \in \mathbb{R}$,

$$
\begin{aligned}
\int_0^\infty \tau^{-1+\gamma_2-\gamma_1}(a+\tau)_-^{\gamma_1}\,d\tau &= \int_0^{a_-} \tau^{-1+\gamma_2-\gamma_1}(a+\tau)_-^{\gamma_1}\,d\tau \\
&= a_-^{\gamma_2} B(1+\gamma_1,\gamma_2-\gamma_1),
\end{aligned}
$$

by the change of variables $t = \frac{(a+\tau)_-}{a_-}$.

Thus we can write the bounds (7) and (8) in the form

$$
N^{\mathcal{D}}(\lambda, R) \le \frac{L^{\mathrm{cl}}_{0,n-1}|R'|}{B(1+\gamma,\frac{n-1}{2}-\gamma)}\int_0^\lambda \tau^{-1+\frac{n-1}{2}-\gamma}\operatorname{Tr}(-\Delta^{\mathcal{D}}_{(0,a_1)}-(\lambda-\tau))_-^{\gamma}\,d\tau, \qquad (10)
$$

$$
N^{\mathcal{N}}(\mu, R) \ge \frac{L^{\mathrm{cl}}_{0,n-1}|R'|}{B(1+\gamma,\frac{n-1}{2}-\gamma)}\int_0^\mu \tau^{-1+\frac{n-1}{2}-\gamma}\operatorname{Tr}(-\Delta^{\mathcal{N}}_{(0,a_1)}-(\mu-\tau))_-^{\gamma}\,d\tau, \qquad (11)
$$

where we are free to choose $\gamma \in [0,(n-1)/2)$. By choosing suitable γ and appropriate one-dimensional estimates it is possible to obtain a variety of bounds for the counting functions. The bounds that we make use of here are proven in the appendix (see Lemmas A.1 and A.2).

For $n = 3$ we do not use the Aizenman–Lieb Identity. Applying the bounds of Lemmas A.1 and A.2 to the one-dimensional traces of (7) resp. (8) yields the claimed bounds. For dimensions $n \geq 4$ choose $\gamma = 1$ in (10) and (11) and apply Lemma A.1 resp. Lemma A.2. Computing the resulting integrals one obtains the claimed bounds.

In the two-dimensional Neumann case a bound of the required form was obtained in [28, Proposition 14]. Moreover, the two-dimensional Dirichlet case follows almost directly from Proposition 10 of the same paper. This proposition states that, for $\lambda \geq 1/a_1^2$,

$$N^{\mathcal{D}}(\lambda, R) \leq \frac{\lambda}{4\pi} - \frac{c\lambda^{1/2}}{a_1}, \tag{12}$$

for some constant $c > 0$. We aim for a bound of the form $N^{\mathcal{D}}(\lambda, R) \leq \frac{1}{4\pi}(\sqrt{\lambda} - b/a_1)^2$. Note that the bound is trivially true for $\lambda < \pi^2/a_1^2$. Note also that for $b \leq \pi$ the right-hand side is pointwise decreasing in b, hence if it holds true for some b_0 it holds for all $b \in [0, b_0]$. Therefore, using (12) it suffices to prove that

$$\frac{\lambda}{4\pi} - \frac{c\lambda^{1/2}}{a_1} \leq \frac{\lambda}{4\pi} - \frac{b\lambda^{1/2}}{2\pi a_1} + \frac{b^2}{4\pi a_1^2}$$

for all $\lambda > \pi^2/a_1^2$, which is clearly true if and only if $b \leq 2\pi c$. □

2.2. Extremal Cuboids are Uniformly Bounded

In this section we obtain a uniform lower bound for the shortest side-length of the extremal cuboids $R_k^{\mathcal{D}}$ and $R_k^{\mathcal{N}}$.

As the proof is almost precisely the same for the Dirichlet and the Neumann cases we only write out the former in full. The only difference between the two cases is that an element of the proof in the Dirichlet case is not present in the proof of the Neumann result. This difference stems from the fact that in the Dirichlet case we have a three-term bound and so we need to bound the quantity that this extra term gives rise to.

For $n \geq 2$ let $R_k^{\mathcal{D}}$, $k \geq 1$, be a sequence of unit measure cuboids minimising λ_k, i.e. such that $\lambda_k(R_k^{\mathcal{D}}) = \lambda_k^*$, and as usual we assume that $a_{1,k}^* \leq \cdots \leq a_{n,k}^*$. By optimality $\lambda_k^* \leq \lambda_k(Q)$ and so $\lambda_k^* - \varepsilon < \lambda_k(Q)$, for any $0 < \varepsilon < 1$, which implies that

$$N^{\mathcal{D}}(\lambda_k^* - \varepsilon, Q) \leq k - 1 < k \leq N^{\mathcal{D}}(\lambda_k^*, R_k^{\mathcal{D}}).$$

The two-term asymptotics for the Dirichlet eigenvalue counting function on the cube (see [20] or Sect. 2.3) combined with Lemma 2.1 then yield that

$$\begin{aligned}
L_{0,n}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{n/2} &- \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(n-1)/2} + o((\lambda_k^* - \varepsilon)^{(n-1)/2}) \\
&\leq L_{0,n}^{\mathrm{cl}}(\lambda_k^*)^{n/2} - \frac{c_1 b L_{0,n-1}^{\mathrm{cl}}}{a_{1,k}^*}(\lambda_k^*)^{(n-1)/2} + \frac{c_2 b^2 L_{0,n-2}^{\mathrm{cl}}}{(a_{1,k}^*)^2}(\lambda_k^*)^{(n-2)/2}.
\end{aligned}$$

Rearranging and taking $\varepsilon = \frac{1}{2}$ we find that

$$\frac{b}{a_{1,k}^*}\left(1 - \frac{c_2 b L_{0,n-2}^{\mathrm{cl}}(\lambda_k^*)^{-1/2}}{c_1 L_{0,n-1}^{\mathrm{cl}} a_{1,k}^*}\right) \leq \frac{n}{2c_1} + o(1).$$

Since $\lambda_k^* = \lambda_k(R_k^{\mathcal{D}}) \geq \lambda_1(R_k^{\mathcal{D}}) > \pi^2(a_{1,k}^*)^{-2}$, we have that $-(\lambda_k^*)^{-1/2} \geq -a_{1,k}^*/\pi$. Hence

$$\frac{b}{a_{1,k}^*}\left(1 - \frac{c_2 b L_{0,n-2}^{\mathrm{cl}}}{c_1 \pi L_{0,n-1}^{\mathrm{cl}}}\right) \leq \frac{n}{2c_1} + o(1).$$

We now choose $b \in (0, b_0]$, where b_0 is as defined in Lemma 2.1, small enough so that the left-hand side is positive. Then the above implies that there exists a $C > 0$ such that

$$\frac{1}{a_{1,k}^*} \leq C + o(1), \tag{13}$$

which in turn implies that

$$a_{n,k}^* \leq \left(\frac{1}{a_{1,k}^*}\right)^{n-1} \leq C^{n-1} + o(1).$$

Thus $\liminf_{k\to\infty} a_{1,k}^* \geq 1/C > 0$ and $\limsup_{k\to\infty} a_{n,k}^* < \infty$ so the side-lengths of a minimising sequence of cuboids are uniformly bounded away from zero and infinity. For dimensions $n = 2, 3$, the corresponding result was obtained, through a slightly different argument, in [3,38].

To prove the corresponding result for the Neumann problem one can take the same approach. Observe that $N^{\mathcal{N}}(\mu_k^* - \varepsilon, R_k^{\mathcal{N}}) \leq k - 1 < k \leq N^{\mathcal{N}}(\mu_k(Q), Q) \leq N^{\mathcal{N}}(\mu_k^*, Q)$, for $k \geq 1$ and any $0 < \varepsilon < 1$, apply the lower bound of Lemma 2.2 to the left-hand side and expand the right-hand side using its two-term asymptotic expansion. Rearranging the obtained inequality yields a bound of the form (13).

2.3. Precise Asymptotics for Eigenvalue Counting Functions

Let $\lambda, \mu, r \geq 0$ and $E(r, R), N^{\mathcal{D}}(\lambda, R)$ and $N^{\mathcal{N}}(\mu, R)$ be as defined in Sect. 1. Assume that R has bounded side-lengths so that the ellipsoid $E(r, R)$ has positive Gaussian curvature. In this section, we obtain two-term asymptotic expansions for $N^{\mathcal{D}}(\lambda, R)$ and $N^{\mathcal{N}}(\mu, R)$ with remainder estimates which are uniform in the side-lengths of R. As the calculations for the Dirichlet and Neumann problems are almost identical, we will write out the argument in full only for the Dirichlet case and indicate what differences appear for the Neumann case. Specifically we prove the following.

Lemma 2.4. *For $n \geq 2$ and $R = \prod_{i=1}^n (0, a_i) \subset \mathbb{R}^n$, with $a_i > 0$,*

$$N^{\mathcal{D}}(\lambda, R) = L_{0,n}^{\mathrm{cl}}|R|\lambda^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial R|\lambda^{(n-1)/2} + O(\lambda^{\theta_n/2}), \tag{14}$$

$$N^{\mathcal{N}}(\mu, R) = L_{0,n}^{\mathrm{cl}}|R|\mu^{n/2} + \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial R|\mu^{(n-1)/2} + O(\mu^{\theta_n/2}), \tag{15}$$

as $\lambda, \mu \to \infty$, where θ_n is as defined in (4). Moreover, the remainder terms are uniform on any collection of cuboids with side-lengths contained in a compact subset of $\mathbb{R}_+$.

Similar two-term asymptotic expansions for the counting function of the Dirichlet, resp. Neumann, Laplacian are known to hold for more general domains than cuboids (see, for example, [20]). However, to obtain the orders of convergence in Theorems 1.1, 1.2 and 1.3 we require a better remainder estimate than what is possible in general.

Proof of Lemma 2.4. The proof is based on the inclusion-exclusion principle. For notational simplicity, in what follows we will write $N^{\mathcal{D}}(\lambda), N^{\mathcal{N}}(\mu)$ and $E(r)$ with the dependence on R being implicit.

By symmetry of the ellipsoid $E(r)$ we have that

$$\begin{aligned}\#\{\mathbb{Z}^n \cap E(r)\} &= 2^n \#\{\mathbb{N}^n \cap E(r)\} \\ &\quad + \#\{(x_1, \dots, x_n) \in \mathbb{Z}^n \cap E(r) : \exists i \text{ for which } x_i = 0\}.\end{aligned} \tag{16}$$

Let $E_i(r)$ denote the set $E(r) \cap \{x_i = 0\}$. As the second term in the right-hand side of (16) is the union of the sets $E_i(r) \cap \mathbb{Z}^n$ we can apply the inclusion–exclusion principle

$$\begin{aligned}&\#\{\cup_{i=1}^n (E_i(r) \cap \mathbb{Z}^n)\} \\ &\quad = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1 \le i_1 < \cdots < i_k \le n} \#\{E_{i_1}(r) \cap \cdots \cap E_{i_k}(r) \cap \mathbb{Z}^n\} \right).\end{aligned} \tag{17}$$

The set $E_{i_1}(r) \cap \cdots \cap E_{i_k}(r)$ is naturally identified with an ellipsoid in $\mathbb{R}^{n-k}$, namely

$$E_I(r) = \left\{ (x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k} : \sum_{j \notin I} \frac{x_j^2}{a_j^2} \le \frac{r^2}{\pi^2} \right\},$$

where $I = \{i_1, \dots, i_k\}$. Moreover, we have that

$$\#\{E_{i_1}(r) \cap \cdots \cap E_{i_k}(r) \cap \mathbb{Z}^n\} = \#\{E_I(r) \cap \mathbb{Z}^{n-k}\}.$$

Since $N^{\mathcal{D}}(R, \lambda) = \#\{\mathbb{N}^n \cap E(\lambda)\}$, we find from (16), (17) and (4) that

$$\begin{aligned}N^{\mathcal{D}}(R, \lambda) &= \frac{\omega_n \lambda^{n/2}}{2^n \pi^n} - \frac{\omega_{n-1} \lambda^{(n-1)/2}}{2^n \pi^{n-1}} \sum_{i=1}^n \prod_{j \ne i} a_j \\ &\quad + O(\lambda^{\theta_n/2} + \lambda^{\theta_{n-1}/2} + \lambda^{(n-2)/2}) \\ &= L_{0,n}^{\mathrm{cl}} \lambda^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4} |\partial R| \lambda^{(n-1)/2} + O(\lambda^{\theta_n/2}).\end{aligned}$$

In the final step we used that $2 \sum_i \prod_{j \ne i} a_j = |\partial R|$ and $\theta_m \in [m-2, m-1)$ for all m. The uniformity of the remainder follows directly from Definition 1.1.

To obtain the corresponding expansion in the Neumann case, one writes the lattice points in $E(r)$ as the union of reflected copies of the lattice points in $E(r) \cap (\mathbb{N} \cup \{0\})^n$ and then applies the inclusion–exclusion principle to this union. $\square$

3. Geometric Convergence and Spectral Stability

In this section, we prove Theorems 1.1, 1.2 and 1.3. As the proofs of the Dirichlet and the Neumann cases are almost identical, we again write out the former case in full and indicate the differences which occur in the proof of the latter.

Since the minimisers $R_k^{\mathcal{D}}$, respectively the maximisers $R_k^{\mathcal{N}}$, need not be unique, we consider an arbitrary subsequence of such extremal sets. By the results obtained in Sect. 2.2 (or the corresponding statements in [3,37,38]), we know that the extremal cuboids in any dimension are uniformly bounded in k, and thus the remainder terms in (14) and (15) are uniform with respect to $R_k^{\mathcal{D}}$ and $R_k^{\mathcal{N}}$, respectively.

Proof of Theorems 1.1 and 1.2. As in the proof of the uniform boundedness, $N(\lambda_k^*-\varepsilon, Q) < k \leq N(\lambda_k^*, R_k^{\mathcal{D}})$, for any $0<\varepsilon<1$. Plugging in the asymptotic expansion (14) on both sides, we have that

$$\begin{aligned} L_{0,n}^{\mathrm{cl}}(\lambda_k^*-\varepsilon)^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^*-\varepsilon)^{(n-1)/2} - O((\lambda_k^*-\varepsilon)^{\theta_n/2}) \\ \leq L_{0,n}^{\mathrm{cl}}(\lambda_k^*)^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial R_k^{\mathcal{D}}|(\lambda_k^*)^{(n-1)/2} + O((\lambda_k^*)^{\theta_n/2}). \end{aligned}$$

Rearranging and choosing $\varepsilon = \frac{1}{2}$, we obtain that

$$|\partial R_k^{\mathcal{D}}| - |\partial Q| \leq O((\lambda_k^*)^{(\theta_n-(n-1))/2}) = O(k^{(\theta_n-(n-1))/n}), \tag{18}$$

which, when combined with the isoperimetric inequality for cuboids, implies that

$$|\partial R_k^{\mathcal{D}}| = \sum_{i=1}^{n} \frac{2}{a_{i,k}^*} = 2n + O(k^{(\theta_n-(n-1))/n}). \tag{19}$$

By the arithmetic–geometric means inequality, with $a_{n,k}^* = 1+\delta_k > 1$, we find that

$$(n-1)(1+\delta_k)^{1/(n-1)} + \frac{1}{1+\delta_k} \leq \sum_{i=1}^{n} \frac{1}{a_{i,k}^*}. \tag{20}$$

Then, by (19) and (20),

$$(n-1)(1+\delta_k)^{n/(n-1)} + 1 \leq n + n\delta_k + O(k^{(\theta_n-(n-1))/n}). \tag{21}$$

For each $n \geq 2$, we know by the results in Sect. 2.2 (or from [3,38]) that there exists $T>0$ so that $\delta_k = a_{n,k}^* - 1 \leq T$. Hence, letting $c(T) = \frac{(1+T)^{n/(n-1)}-1-\frac{n}{n-1}T}{T^2} > 0$, we have that

$$(1+\delta_k)^{n/(n-1)} \geq 1 + \frac{n}{n-1}\delta_k + c(T)\delta_k^2.$$

By substituting this into (21), we deduce that $\delta_k = O(k^{(\theta_n-(n-1))/(2n)})$.

For the Neumann case one can argue almost identically by observing (as in the proof of the uniform boundedness of $R_k^{\mathcal{N}}$) that, for any $0<\varepsilon<1$,

$$N^{\mathcal{N}}(\mu_k^*-\varepsilon, R_k^{\mathcal{N}}) \leq N^{\mathcal{N}}(\mu_k^*, Q). \qquad \square$$

Remark 3.1. We remark that if we restrict the collection of cuboids to a sub-collection containing a unique minimiser of the perimeter, then the above arguments prove that any sequence of minimising, resp. maximising, cuboids converges to the cuboid of smallest perimeter in this sub-collection (in particular, replace Q by this cuboid in (18)). For example, in the sub-collection consisting of all unit-measure cuboids in $\mathbb{R}^n$ of the form $\prod_{i=1}^n (0, a_i)$ such that $0 < a_1 \leq \cdots \leq a_n$ and $ca_1 = a_2$, with $c \geq 1$, any sequence of optimisers converges to the cuboid with $a_1 = c^{-(n-1)/n}$ and $a_2 = \cdots = a_n = c^{1/n}$.

We now turn to the question of spectral stability and the proof of Theorem 1.3.

Proof of Theorem 1.3. As in the proof of Theorems 1.1 and 1.2, we have that, for any $0 < \varepsilon < 1$, $N_k^{\mathcal{D}}(\lambda_k^* - \varepsilon, Q) \leq k \leq N^{\mathcal{D}}(\lambda_k^*, R_k^{\mathcal{D}})$. By the asymptotic expansion (14) we thus find that

$$\begin{aligned} L_{0,n}^{\mathrm{cl}}(\lambda_k^* - \varepsilon)^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^* - \varepsilon)^{(n-1)/2} - O((\lambda_k^* - \varepsilon)^{\theta_n/2}) \\ \leq k \leq L_{0,n}^{\mathrm{cl}}(\lambda_k^*)^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial R_k^{\mathcal{D}}|(\lambda_k^*)^{(n-1)/2} + O((\lambda_k^*)^{\theta_n/2}). \end{aligned}$$

By the isoperimetric inequality for cuboids this also holds with $|\partial R_k^{\mathcal{D}}|$ replaced by $|\partial Q|$.

Choosing $\varepsilon = \frac{1}{2}$ yields that

$$k = L_{0,n}^{\mathrm{cl}}(\lambda_k^*)^{n/2} - \frac{L_{0,n-1}^{\mathrm{cl}}}{4}|\partial Q|(\lambda_k^*)^{(n-1)/2} + O((\lambda_k^*)^{\theta_n/2}), \tag{22}$$

as $k \to \infty$. From which we can conclude that

$$\lambda_k^* = 4\pi\Gamma\Big(\frac{n}{2} + 1\Big)^{2/n} k^{2/n} + \frac{2\pi\Gamma(\frac{n}{2}+1)^{1+1/n}}{n\Gamma(\frac{n+1}{2})}|\partial Q| k^{1/n} + O(k^{(\theta_n - (n-2))/n}), \tag{23}$$

as $k \to \infty$. Now (22) is the same two-term expansion as that for $N(\lambda, Q)$, so (23) must agree with the two-term expansion for $\lambda_k(Q)$. Thus we obtain that $|\lambda_k(Q) - \lambda_k^*| = O(k^{(\theta_n - (n-2))/n})$ as $k \to \infty$.

The approach to prove the Neumann case is identical except that one instead uses that, for any $0 < \varepsilon < 1$,

$$N^{\mathcal{N}}(\mu_k^* - \varepsilon, R_k^{\mathcal{N}}) \leq k \leq N^{\mathcal{N}}(\mu_k^*, Q). \qquad \square$$

4. Riesz Means and Eigenvalue Averages

Given the techniques and bounds obtained above, it is not difficult to obtain the corresponding shape optimisation results for the following problems:

(i) For $\gamma \geq 0$ and $\lambda, \mu \geq 0$,

$$\begin{aligned} &\sup\big\{\mathrm{Tr}(-\Delta_R^{\mathcal{D}} - \lambda)_-^\gamma : R \subset \mathbb{R}^n \text{ cuboid}, |R| = 1\big\}, \\ &\inf\big\{\mathrm{Tr}(-\Delta_R^{\mathcal{N}} - \mu)_-^\gamma : R \subset \mathbb{R}^n \text{ cuboid}, |R| = 1\big\}. \end{aligned}$$

(ii) For $k \in \mathbb{N}$,

$$\inf\Big\{\frac{1}{k}\sum_{i=1}^{k}\lambda_i(R) : R \subset \mathbb{R}^n \text{ cuboid}, |R| = 1\Big\}.$$

For the Riesz means we prove that:

Proposition 4.1. *Let $n \geq 2$ and $\gamma \geq 0$. For $\lambda > 0$, let $R_\lambda^{\mathcal{D}}$ denote any cuboid which maximises* $\mathrm{Tr}(-\Delta_R^{\mathcal{D}} - \lambda)_-^\gamma$ *amongst all cuboids R of unit measure. Then as $\lambda \to \infty$ we have that*

$$a_{n,\lambda}^* = 1 + O(\lambda^{(\theta_n - (n-1))/4}).$$

Proposition 4.2. *Let $n \geq 2$ and $\gamma \geq 0$. For $\mu > 0$, let $R_\mu^{\mathcal{N}}$ denote any cuboid which minimises* $\mathrm{Tr}(-\Delta_R^{\mathcal{N}} - \mu)_-^\gamma$ *amongst all cuboids R of unit measure. Then as $\mu \to \infty$ we have that*

$$a_{n,\mu}^* = 1 + O(\mu^{(\theta_n - (n-1))/4}).$$

In [11] Freitas studied problem (ii) in the more general setting of minimising amongst all bounded, open sets of fixed measure, and obtained the leading order behaviour of the extremal values as $k \to \infty$. By utilising a connection between Riesz means of order $\gamma = 1$ and the eigenvalue averages, we prove here that:

Proposition 4.3. *Let $n \geq 2$. For $k \in \mathbb{N}$, let $\overline{R}_k^{\mathcal{D}}$ denote any cuboid which minimises the average $\frac{1}{k}\sum_{i=1}^{k}\lambda_i(R)$ amongst all cuboids R of unit measure. Then as $k \to \infty$ we have that*

$$\bar{a}_{n,k}^* = 1 + O(k^{(\theta_n - (n-1))/(2n)}).$$

We believe that the corresponding result should also hold for the maximisation of the Neumann averages. However, we have been unable to solve an issue which appears when trying to pass from a bound for the Riesz means to a bound for the averages (see Remark 4.7 below).

In a similar manner as in Sect. 1.3 above (see also [27]), one can conclude that for any fixed λ, μ or $k \in \mathbb{N}$ each of these problems has at least one optimal cuboid. We denote any such optimal cuboid by $R_\lambda^{\mathcal{D}}, R_\mu^{\mathcal{N}}$ and $\overline{R}_k^{\mathcal{D}}$, respectively, where the bar is to distinguish from the minimisers of the individual eigenvalues.

The approach we take for (i) is to use the Aizenman–Lieb Identity to lift our bounds for the counting functions to higher order Riesz means. For $\gamma \geq 1$ this improves special cases of a pair of inequalities due to Berezin [5] (see also [24]). For (ii) we use an approach based on the close relationship between the sum of eigenvalues and the Riesz means of order $\gamma = 1$. This allows us to obtain a three-term bound for the sum of the first k eigenvalues, which improves a special case of a bound obtained by Li and Yau [29] (see Lemma 4.6 below).

Lemma 4.4. *Let $\gamma \geq 0$. There exist positive constants c_1, c_2 and b_0 such that, for any cuboid $R \subset \mathbb{R}^n$ with $|R| = 1$, the bound*

$$\mathrm{Tr}(-\Delta_R^{\mathcal{D}} - \lambda)_-^\gamma \leq L_{\gamma,n}^{\mathrm{cl}} \lambda^{\gamma+n/2} - \frac{c_1 b L_{\gamma,n-1}^{\mathrm{cl}}}{a_1} \lambda^{\gamma+(n-1)/2} + \frac{c_2 b^2 L_{\gamma,n-2}^{\mathrm{cl}}}{a_1^2} \lambda^{\gamma+(n-2)/2},$$

holds for all $\lambda \geq 0$ and $b \in \big[0, b_0\big]$.

Lemma 4.5. *Let $\gamma \geq 0$. There exists a constant $c_1 > 0$ such that, for any cuboid $R \subset \mathbb{R}^n$ with $|R| = 1$, the bound*

$$\mathrm{Tr}(-\Delta_R^{\mathcal{N}} - \mu)_-^\gamma \geq L_{\gamma,n}^{\mathrm{cl}} \mu^{\gamma+n/2} + \frac{c_1 L_{\gamma,n-1}^{\mathrm{cl}}}{a_1} \mu^{\gamma+(n-1)/2},$$

holds for all $\mu \geq 0$.

Proof of Lemmas 4.4 and 4.5. Apply the Aizenman–Lieb Identity (9) with $\gamma_1 = 0$ and $\gamma_2 = \gamma$ to both sides of Lemma 2.1, respectively Lemma 2.2. □

We note that by using the Laplace transform instead of the Aizenman–Lieb Identity, one could apply the above procedure to obtain a three-term bound for $\mathrm{Tr}(e^{t\Delta_R^{\mathcal{D}/\mathcal{N}}})$ valid for all cuboids $R \subset \mathbb{R}^n$. Moreover, using Theorem 1.1 of [26] one can obtain a tunable three-term bound (similar to Lemma 4.4) for any convex domain $\Omega \subset \mathbb{R}^n$ which could then, using the Laplace transform, be lifted to a corresponding bound for $\mathrm{Tr}(e^{t\Delta_\Omega^{\mathcal{D}}})$. A similar inequality was obtained by van den Berg [36] for the Dirichlet Laplacian on smooth convex domains. By using results from [25], the upper bound of [36] can be extended to all convex domains.

Lemma 4.6. *There exist positive constants c_1, c_2 and b_0 such that, for any cuboid $R \subset \mathbb{R}^n$ with $|R| = 1$, the bound*

$$\frac{1}{k} \sum_{i=1}^k \lambda_i(R) \geq \frac{4\pi n \Gamma(\frac{n}{2}+1)^{2/n}}{n+2} k^{2/n} + \frac{c_1 b}{a_1} k^{1/n} - \frac{c_2 b^2}{a_1^2},$$

holds for all $k \in \mathbb{N}$ and all $b \in \big[0, b_0\big]$.

Proof of Lemma 4.6. It is well known that the sum of eigenvalues and the order 1 Riesz means are related by the Legendre transform [24]. It is a small modification of this insight that will allow us to obtain the claimed bound from Lemma 4.4 with $\gamma = 1$.

By Lemma 4.4 there exist constants $c_1', c_2' > 0$ such that, for any $k \in \mathbb{N}$,

$$\sup_{\lambda \geq 0} \Big(k\lambda - \sum_{i:\lambda_i \leq \lambda} (\lambda - \lambda_i(R)) \Big) \geq \sup_{\lambda \geq 0} \Big(k\lambda - L_{1,n}^{\mathrm{cl}} \lambda^{1+n/2} + \frac{c_1' b L_{1,n-1}^{\mathrm{cl}}}{a_1} \lambda^{1+(n-1)/2} - \frac{c_2' b^2 L_{1,n-2}^{\mathrm{cl}}}{a_1^2} \lambda^{1+(n-2)/2} \Big). \tag{24}$$

The supremum on the left-hand side is achieved precisely at $\lambda = \lambda_k(R)$. Indeed, the function $f_k(\lambda) = k\lambda - \sum_{i:\lambda_i \leq \lambda}(\lambda - \lambda_i(R))$ is continuous, increasing for all λ for which $N(\lambda, R) < k$, and decreasing if $N(\lambda, R) > k$. Moreover, for λ such that $N(\lambda, R) = k$ we have that $f_k(\lambda) = \sum_{i=1}^{k} \lambda_i(R)$. Thus the left-hand side reduces to

$$\sup_{\lambda \geq 0}\Big(k\lambda - \sum_{i:\lambda_i \leq \lambda} (\lambda - \lambda_i(R))\Big) = \sum_{i=1}^{k} \lambda_i(R).$$

On the other hand, maximising the right-hand side of the inequality is slightly more difficult and there may also be a question of uniqueness of the maximum. However, on this side we may choose any $\lambda \geq 0$ and still obtain a valid inequality.

Choosing λ to maximise $k\lambda - L^{\mathrm{cl}}_{1,n}\lambda^{1+n/2}$, which corresponds to

$$\lambda = \left(\frac{k}{(\frac{n}{2}+1)L^{\mathrm{cl}}_{1,n}}\right)^{2/n} = 4\pi\Gamma\Big(\frac{n}{2}+1\Big)^{2/n} k^{2/n},$$

ensures that the leading order term has the sharp constant (this follows from the equivalence, via the Legendre transform, of the Li–Yau inequality for the sum of eigenvalues and the Berezin inequality for the Riesz mean of order $\gamma = 1$, see [24]). With the above choice of λ we obtain the claimed bound from (24). $\square$

Remark 4.7. If one attempts to apply the same technique as above to obtain a lower bound for the average of the Neumann eigenvalues from Lemma 4.5, the inequality after the Legendre transform is reversed. Therefore one cannot pick μ analogously to how we chose λ above. Instead one needs to prove an upper bound for

$$\sup_{\mu \geq 0}\Big(k\mu - L^{\mathrm{cl}}_{1,n}\mu^{1+n/2} - \frac{c_1 L^{\mathrm{cl}}_{1,n-1}}{a_1}\mu^{1+(n-1)/2}\Big),$$

which is sufficiently good to obtain the uniform boundedness of the extremal cuboids.

4.1. Proof of Propositions 4.1–4.3

With the above bounds in hand, and almost step-by-step following the proof in Sect. 2.2, or the corresponding proof in [27], one obtains that $R^{\mathcal{D}}_{\lambda}, R^{\mathcal{N}}_{\mu}$ and $\overline{R}^{\mathcal{D}}_{k}$ are uniformly bounded as λ, μ or k goes to infinity.

For the Riesz means, in both the Dirichlet case and the Neumann case, the proof is completely analogous to that in Sect. 2.2 by using Lemmas 4.4 and 4.5 and the asymptotic expansions one obtains from Lemma 2.4 via the Aizenman–Lieb Identity.

For the eigenvalue averages we require an upper bound for $\frac{1}{\overline{a}_{1,k}k^{1/n}}$, which can be obtained as follows. Since $\overline{R}^{\mathcal{D}}_{k}$ is a minimiser, we have that

$$\frac{k\pi^2}{\overline{a}^2_{1,k}} \leq k\lambda_1(\overline{R}^{\mathcal{D}}_{k}) \leq \sum_{i=1}^{k} \lambda_i(\overline{R}^{\mathcal{D}}_{k}) \leq \sum_{i=1}^{k} \lambda_i(Q).$$

Inserting that, as $k \to \infty$,

$$\sum_{i=1}^{k} \lambda_i(Q) = \frac{4\pi n \Gamma(\frac{n}{2}+1)^{2/n}}{n+2} k^{1+2/n} + \frac{2\pi \Gamma(\frac{n}{2}+1)^{1+1/n}}{(n+1)\Gamma(\frac{n+1}{2})} |\partial Q| k^{1+1/n} + o(k^{1+1/n})$$

and rearranging implies the required bound.

To find an asymptotic expansion for the eigenvalue averages, one can make use of the corresponding two-term expansions that we have for $\lambda_i(R)$ and calculate the asymptotics of the resulting sums (for instance using the Euler–Maclaurin formula).

In a similar manner as in the preceding section, for these problems one could also obtain estimates for the spectral stability, i.e. to what order in the respective parameters do the extremal eigenvalue means or averages approach those of the limiting domain Q. However, by finer analysis of the asymptotics, and not lifting the results for the counting function, it should be possible to obtain sharper estimates than what is obtained directly by the method outlined in the previous paragraph. This is due to the fact that in the above problems the erratic behaviour of the eigenvalues and counting function has in some sense been reduced by summing.

It is possible to analyse the asymptotic behaviour of the extremal averages of the first k Neumann eigenvalues amongst unit-measure cuboids by invoking Theorem 1.3. Indeed, by using that

$$\frac{1}{k}\sum_{i=0}^{k} \mu_i(Q) \le \sup \left\{ \frac{1}{k}\sum_{i=0}^{k} \mu_i(R) : R \subset \mathbb{R}^n \text{ cuboid}, |R| = 1 \right\} \le \frac{1}{k}\sum_{i=0}^{k} \mu_i^{\mathcal{N}}$$

and Theorem 1.3, one obtains precise two-term asymptotics for the extremal averages, and finds that they agree with the corresponding asymptotics for Q. However, as mentioned above we have been unable to obtain an inequality which is sharp enough to conclude that the sequence of extremal cuboids for this problem remains uniformly bounded as $k \to \infty$. Thus our approach yields nothing about the geometric convergence.

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Appendix A. One-Dimensional Bounds

Lemma A.1. *There exist constants $c_1, c_2, b_0 > 0$ such that, for all $\lambda \geq 0$ and $a > 0$,*

$$\operatorname{Tr}(-\Delta^{\mathcal{D}}_{(0,a)} - \lambda)_- = \sum_{k\geq 1}\Big(\lambda - \frac{\pi^2 k^2}{a^2}\Big)_+ \leq a L^{\mathrm{cl}}_{1,1}\lambda^{3/2} - b c_1 L^{\mathrm{cl}}_{1,0}\lambda + \frac{b^2 c_2}{a} L^{\mathrm{cl}}_{1,-1}\lambda^{1/2},$$

for all $b \in [0, b_0]$.

Lemma A.2. *There exists a constant $c_1 > 0$ such that, for all $\mu \geq 0$ and $a > 0$,*

$$\operatorname{Tr}(-\Delta^{\mathcal{N}}_{(0,a)} - \mu)_- = \sum_{k\geq 0}\Big(\mu - \frac{\pi^2 k^2}{a^2}\Big)_+ \geq a L^{\mathrm{cl}}_{1,1}\mu^{3/2} + c_1 L^{\mathrm{cl}}_{1,0}\mu.$$

Remark A.3. For our purposes it is essential that the leading order term agrees with the asymptotic one. The lower order terms are of less importance up to their behaviour in λ and a. However, in the Dirichlet case it is important that the third term can be dominated by the second one by choosing b sufficiently small.

We also emphasise that when applying the Aizenman–Lieb Identity (9) it simplifies matters if we have bounds valid for all $\lambda, \mu \geq 0$. This is the reason for proving the above inequalities for $\lambda, \mu \geq 0$ even though our main interest here is focused on large λ, μ.

Proof of Lemma A.1. By rescaling it suffices to prove that, for $\lambda \geq 0$ and small enough b,

$$\sum_{k\geq 1}(\lambda - k^2)_+ \leq \frac{2}{3}\lambda^{3/2} - b c_1 \lambda + \frac{4b^2 c_2}{\pi}\lambda^{1/2}.$$

We will prove this with $c_1 = \frac{4}{3}$, $c_2 = \frac{\pi}{6}$ and $b \leq 1 - \frac{1}{6}\sqrt{\frac{27+\sqrt{3}}{2}}$.

With $r = \sqrt{\lambda} - \lfloor\sqrt{\lambda}\rfloor$ we have that

$$\sum_{k\geq 1}(\lambda - k^2)_+ = \frac{2}{3}\lambda^{3/2} - \frac{\lambda}{2} + \Big(r - r^2 - \frac{1}{6}\Big)\lambda^{1/2} + \frac{1}{6}(r - 3r^2 + 2r^3).$$

Maximising the coefficient in front of $\lambda^{1/2}$ and the constant term with respect to $r \in [0, 1)$ we obtain

$$\sum_{k\geq 1}(\lambda - k^2)_+ \leq \frac{2}{3}\lambda^{3/2} - \frac{\lambda}{2} + \frac{\lambda^{1/2}}{12} + \frac{1}{36\sqrt{3}}. \tag{25}$$

We aim for a bound of the form $\sum_{k\geq 1}(\lambda - k^2)_+ \leq \frac{2}{3}\lambda^{1/2}(\sqrt{\lambda} - b)^2$, which holds for all $\lambda \geq 0$ and some $b > 0$. Note that this bound holds trivially for all $\lambda \leq 1$, and thus we only need to choose b so that it is valid for all $\lambda > 1$. Moreover, note that, for $b < 1$ and $\lambda > 1$, this bound is pointwise decreasing in b. Hence if we know the bound to hold for some b_0 then it holds for all $0 \leq b \leq b_0$.

Since we have an upper bound in terms of the polynomial in (25), it suffices to choose b so that, for all $\lambda > 1$,

$$\lambda^{3/2} - \frac{3}{4}\lambda + \frac{\lambda^{1/2}}{8} + \frac{1}{24\sqrt{3}} \leq \lambda^{1/2}(\sqrt{\lambda} - b)^2 = \lambda^{3/2} - 2b\lambda + b^2\lambda^{1/2}.$$

Rearranging we see that this is equivalent to

$$\Big(\frac{1}{8} - b^2\Big)\lambda^{1/2} + \frac{1}{24\sqrt{3}} \leq \Big(\frac{3}{4} - 2b\Big)\lambda,$$

and thus we must choose $b < 3/8$. If this is true then, since $\lambda > 1$,

$$\Big(\frac{3}{4} - 2b\Big)\lambda \geq \Big(\frac{3}{4} - 2b\Big)\lambda^{1/2}.$$

Thus it is sufficient to choose b satisfying

$$\Big(\frac{1}{8} - b^2\Big)\lambda^{1/2} + \frac{1}{24\sqrt{3}} \leq \Big(\frac{3}{4} - 2b\Big)\lambda^{1/2},$$

or equivalently so that

$$\frac{1}{24\sqrt{3}} \leq \Big(\frac{5}{8} - 2b + b^2\Big)\lambda^{1/2}.$$

This holds for all $\lambda > 1$ if and only if the inequality is valid at $\lambda = 1$. Thus we can choose $b \in [0, b_0]$ with $b_0 = 1 - \frac{1}{6}\sqrt{\frac{27+\sqrt{3}}{2}} < \frac{3}{8}$. $\square$

Proof of Lemma A.2. We shall prove that the claimed bound holds if and only if $c_1 \leq \frac{36-\sqrt{3}}{108}$. By scaling it is sufficient to prove that

$$\sum_{k\geq 0}(\mu - k^2)_+ \geq \frac{2}{3}\mu^{3/2} + c_1\mu. \tag{26}$$

Analogously to the Dirichlet case above

$$\sum_{k\geq 0}(\mu - k^2)_+ = \frac{2}{3}\mu^{3/2} + \frac{\mu}{2} + \Big(r - r^2 - \frac{1}{6}\Big)\mu^{1/2} + \frac{1}{6}(r - 3r^2 + 2r^3),$$

where $r := \sqrt{\mu} - \lfloor\sqrt{\mu}\rfloor$. Minimising the coefficient in front of $\mu^{1/2}$ and the constant term with respect to $r \in [0, 1)$, we find that

$$\sum_{k\geq 0}(\mu - k^2)_+ \geq \frac{2}{3}\mu^{3/2} + \frac{\mu}{2} - \frac{\mu^{1/2}}{6} - \frac{1}{36\sqrt{3}}.$$

For $\mu \geq 1$ it is easy to prove that

$$\frac{2}{3}\mu^{3/2} + \frac{\mu}{2} - \frac{\mu^{1/2}}{6} - \frac{1}{36\sqrt{3}} \geq \frac{2}{3}\mu^{3/2} + c_1\mu,$$

if and only if $c_1 \leq \frac{36-\sqrt{3}}{108}$.

What remains is to prove that the bound is valid for $\mu \in [0, 1)$. In this range the inequality (26) reduces to

$$\mu \geq \frac{2}{3}\mu^{3/2} + c_1\mu.$$

As the right-hand side is strictly convex and the bound is valid at $\mu = 0$ and $\mu = 1$ the proof is complete. □

References

[1] Aizenman, M., Lieb, E.H.: On semiclassical bounds for eigenvalues of Schrödinger operators. Phys. Lett. A **66**, 427–429 (1978)

[2] Antunes, P.R.S., Freitas, P.: Numerical optimisation of low eigenvalues of the Dirichlet and Neumann Laplacians. J. Optim. Theory Appl. **154**, 235–257 (2012)

[3] Antunes, P.R.S., Freitas, P.: Optimal spectral rectangles and lattice ellipses. Proc. R. Soc. A **469**, 20120492 (2013). 15 pp

[4] Ariturk, S., Laugesen, R.S.: Optimal stretching for lattice points under convex curves. Portugaliae Mathematica. Preprint: arXiv:1701.03217 [math.SP] (2017) **(to appear)**

[5] Berezin, F.A.: Covariant and contravariant symbols of operators. Izv. Akad. Nauk SSSR Ser. Mat. **36**, 1134–1167 (1972)

[6] Berger, A.: The eigenvalues of the Laplacian with Dirichlet boundary condition in $\mathbb{R}^2$ are almost never minimized by disks. Ann. Glob. Anal. Geom. **47**, 285–304 (2015)

[7] Bucur, D.: Minimization of the k-th eigenvalue of the Dirichlet Laplacian. Arch. Ration. Mech. Anal. **206**, 1073–1083 (2012)

[8] Bucur, D., Buttazzo, G.: Variational Methods in Shape Optimization Problems. Progress in Nonlinear Differential Equations and Their Applications 65. Birkhäuser, Boston (2005)

[9] Colbois, B., El Soufi, A.: Extremal eigenvalues of the Laplacian on Euclidean domains and closed surfaces. Math. Z. **278**, 529–546 (2014)

[10] Faber, G.: Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. Sitz. Ber. Bayer. Akad. Wiss. **342**, 169–172 (1923)

[11] Freitas, P.: Asymptotic behaviour of extremal averages of Laplacian eigenvalues. J. Stat. Phys. **167**, 1511–1518 (2017)

[12] Girouard, A., Nadirashvili, N., Polterovich, I.: Maximization of the second positive Neumann eigenvalue for planar domains. J. Differ. Geom. **83**, 637–661 (2009)

[13] Götze, F.: Lattice point problems and values of quadratic forms. Invent. Math. **157**, 195–226 (2004)

[14] Heath-Brown, D.R.: Lattice Points in the Sphere. Number Theory in Progress (Zakopane-Koscielisko, 1997), vol. 2, pp. 883–892. de Gruyter, Berlin (1999)

[15] Henrot, A.: Extremum Problems for Eigenvalues of Elliptic Operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2006)

[16] Herz, C.S.: On the number of lattice points in a convex set. Am. J. Math. **84**, 126–133 (1962)

[17] Huxley, M.N.: Area, Lattice Points and Exponential Sums. London Mathematical Society Monographs 13. Oxford University Press, New York (1996)

[18] Huxley, M.N.: Exponential sums and lattice points III. Proc. Lond. Math. Soc. **87**, 591–609 (2003)

[19] Ivić, A., Krätzel, E., Kühleitner, M., Nowak, W.G.: Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic. In: Elementare und analytische Zahlentheorie, vol. 20, pp. 89–128, Schr. Wiss. Ges. Johann Wolfgang Goethe University Frankfurt am Main. Franz Steiner Verlag Stuttgart, Stuttgart (2006)

[20] Ivrii, V.Y.: The second term of the spectral asymptotics for a Laplace–Beltrami operator on manifolds with boundary. Funktsional. Anal. i Prilozhen **14**, 25–34 (1980)

[21] Kellner, R.: On a theorem of Pólya. Am. Math. Mon. **73**, 856–858 (1966)

[22] Krahn, E.: Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. Math. Ann. **94**, 97–100 (1925)

[23] Krahn, E.: Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen. Acta Commun. Univ. Tartu (Dorpat) **A9**, 1–44 (1926)

[24] Laptev, A.: Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. J. Funct. Anal. **151**, 531–545 (1997)

[25] Larson, S.: A bound for the perimeter of inner parallel bodies. J. Funct. Anal. **271**, 610–619 (2016)

[26] Larson, S.: On the remainder term of the Berezin inequality on a convex domain. Proc. Am. Math. Soc. **145**, 2167–2181 (2017)

[27] Larson, S.: Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains. arXiv:1611.0568 [math.SP] (2016)

[28] Laugesen, R.S., Liu, S.: Optimal stretching for lattice points and eigenvalues, *Arkiv för Matematik*. Preprint: arXiv:1609.06172 [math.SP] (2016) **(to appear)**

[29] Li, P., Yau, S.T.: On the Schrödinger equation and the eigenvalue problem. Commun. Math. Phys. **88**, 309–318 (1983)

[30] Marshall, N.F.: Stretching convex domains to capture many lattice points. arXiv:1707.00682 [math.MG] (2017)

[31] Marshall, N. F., Steinerberger, S.: Triangles capturing many lattice points. arXiv:1706.04170 [math.CO] (2017)

[32] Mazzoleni, D., Pratelli, A.: Existence of minimizers for spectral problems. J. Math. Pures Appl. **100**, 433–453 (2013)

[33] Oudet, É., Vial, G.: Spectral optimization with ShapeBox: the (new) pinched ball conjecture. http://www-ljk.imag.fr/membres/Edouard.Oudet/research/ShapeBox_first/html/index_n.php

[34] Pólya, G.: On the eigenvalues of vibrating membranes. Proc. Lond. Math. Soc. **11**, 419–433 (1961)

[35] Szegő, G.: Inequalities for certain eigenvalues of a membrane of given area. J. Ration. Mech. Anal. **3**, 343–356 (1954)

[36] van den Berg, M.: A uniform bound on trace$\,(e^{t\Delta})$ for convex regions in $\mathbb{R}^n$ with smooth boundaries. Commun. Math. Phys. **92**, 525–530 (1984)

[37] van den Berg, M., Bucur, D., Gittins, K.: Maximising Neumann eigenvalues on rectangles. Bull. Lond. Math. Soc. **48**, 877–894 (2016)

[38] van den Berg, M., Gittins, K.: Minimising Dirichlet eigenvalues on cuboids of unit measure. Mathematika **63**, 469–482 (2017)

[39] Weinberger, H.F.: An isoperimetric inequality for the N-dimensional free membrane problem. J. Ration. Mech. Anal. **5**, 633–636 (1956)

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Paper E

Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain
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TWO-TERM SPECTRAL ASYMPTOTICS FOR THE DIRICHLET LAPLACIAN IN A LIPSCHITZ DOMAIN

RUPERT L. FRANK AND SIMON LARSON

Abstract. We prove a two-term Weyl-type asymptotic formula for sums of eigenvalues of the Dirichlet Laplacian in a bounded open set with Lipschitz boundary. Moreover, in the case of a convex domain we obtain a universal bound which correctly reproduces the first two terms in the asymptotics.

1. Introduction and main result

In this paper we investigate the asymptotic behavior of the eigenvalues of the Dirichlet Laplacian on domains with rough boundary. Besides being of intrinsic interest, this question is relevant for some problems in shape optimization, as we will explain below in some more detail.

One of the central results in the spectral theory of differential operators is Weyl's law [36]. It states that the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots ,$$

repeated according to multiplicities, of the Dirichlet Laplacian $-\Delta_\Omega$ in an open set $\Omega \subset \mathbb{R}^d$ of finite measure satisfy

$$\#\{\lambda_k < \lambda\} = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{d/2} + o(\lambda^{d/2}) \qquad \text{as } \lambda \to \infty\,, \tag{1}$$

where ω_d denotes the measure of the unit ball in $\mathbb{R}^d$. The fact that this asymptotic expansion holds without any regularity conditions on Ω was shown in [28].

In [37] Weyl conjectured that a refined version of the asymptotic formula (1) holds. Namely, he conjectured that

$$\#\{\lambda_k < \lambda\} = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{d/2} - \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}) \qquad \text{as } \lambda \to \infty\,. \tag{2}$$

Here $\mathcal{H}^{d-1}(\partial\Omega)$ denotes the $(d-1)$-dimensional Hausdorff measure of the boundary. This conjecture was proved by Ivrii in [18] under two additional assumptions. The first assumption is that the measure of all periodic billiards is zero and the second assumption is that

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the boundary of the set is smooth. It is believed, but only known in special cases [33, 34], that the first assumption is always satisfied. Concerning the second assumption, in a series of papers [7, 19, 20] Ivrii and co-workers have tried to lower the required assumptions on the boundary of the set. In particular, in [20] the asymptotics (2) are proved under the billiard assumption for C^1 domains such that the derivatives of the functions describing the boundary have a modulus of continuity $o(|\log r|^{-1})$. Without the billiard assumption it is shown that the left side of (2) differs from the first term on the right side by $O(\lambda^{(d-1)/2})$. This bound, in the smooth case, is originally due to Seeley [30, 31].

The goal of this paper is to show that an averaged version of the asymptotics (2) is valid for any bounded open set with Lipschitz boundary. In order to state this result precisely, we write $x_\pm = (|x| \pm x)/2$, so that

$$\mathrm{Tr}(-\Delta_\Omega - \lambda)_- = \sum_{\lambda_k < \lambda} (\lambda - \lambda_k)\,,$$

and abbreviate

$$L_d = \frac{2}{2+d} \frac{\omega_d}{(2\pi)^d}\,.$$

Our main result is

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with Lipschitz regular boundary. Then, as $\lambda \to \infty$,*

$$\mathrm{Tr}(-\Delta_\Omega - \lambda)_- = L_d|\Omega|\lambda^{1+d/2} - \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2} + o(\lambda^{1+(d-1)/2})\,. \tag{3}$$

We will discuss momentarily in which sense this theorem improves earlier results and sketch the strategy of its proof. Before doing so, we would like to emphasize that the methods that we develop in order to prove Theorem 1.1 can also be used to prove universal, that is, non-asymptotic bounds. For instance, for convex sets we obtain the following bound.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a convex bounded open set. Then, for all $\lambda > 0$,*

$$\begin{aligned}\Big|\mathrm{Tr}(-\Delta_\Omega - \lambda)_- - L_d|\Omega|\lambda^{1+d/2} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2}\Big| \\ \leq C\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2}\Big(r_{in}(\Omega)\sqrt{\lambda}\Big)^{-1/11},\end{aligned}$$

where the constant C depends only on the dimension.

By integration with respect to λ, Theorem 1.2 implies a corresponding inequality for $\mathrm{Tr}(e^{t\Delta_\Omega})$ which is valid uniformly for all $t > 0$. This improves an earlier result by van den Berg [5], where an additional bound on the curvatures was assumed.

In a similar manner, Theorem 1.2 implies universal upper and lower bounds for $\mathrm{Tr}(H_\Omega)_-^\gamma$ for all $\gamma \geq 1$. The resulting upper bound can be seen as an improvement of an inequality going back to work of Berezin [4] and Li–Yau [25]. Such improved versions of the Berezin–Li–Yau inequality have been the topic of several recent papers [13, 14, 15, 21, 23, 26, 35]. Lower bounds in the same spirit are contained in [16]. In contrast to our Theorem 1.2, however, none of these previous upper and lower bounds reproduces correctly the second term in the asymptotics.

A challenging open question from shape optimization theory, which, in part, motivated this work, is whether for fixed $\gamma \geq 0$, a family $(\Omega_{\lambda,\gamma})_{\lambda>0}$ of optimizers of the problem

$$\sup\{\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma : \Omega \subset \mathbb{R}^d \text{ open}, \ |\Omega| = 1\}$$

converges as $\lambda \to \infty$ to a ball of unit measure. We refer to [24] for more on this problem. The intuition for why the convergence to a ball might be true is that, while the leading term in the asymptotics of $\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma$ as $\lambda \to \infty$ is fixed due to the constraint $|\Omega| = 1$, maximizing the second term leads to minimizing $\mathcal{H}^{d-1}(\partial\Omega)$ under the constraint $|\Omega| = 1$. By the isoperimetric inequality the unique solution to this problem is a ball of unit measure. The difficulty with making this intuition rigorous is that one needs the asymptotics of $\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma$ not only for a fixed domain Ω, but rather for a family of domains $\Omega_{\lambda,\gamma}$ depending on λ with a priori no information concerning their geometry.

While we have not been able to answer this question in full generality, we did prove the corresponding result for a similar optimization problem with an additional convexity constraint and $\gamma \geq 1$. Namely, as a corollary of Theorem 1.2 we obtain

Corollary 1.3. *Let $\gamma \geq 1$. For $\lambda > 0$ let $\Omega_{\lambda,\gamma}$ denote any extremal domain of the shape optimization problem*

$$\sup\{\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma : \Omega \subset \mathbb{R}^d \text{ convex open}, \ |\Omega| = 1\}\,.$$

Then, up to translation, $\Omega_{\lambda,\gamma}$ converges in the Hausdorff metric to a ball of unit measure as $\lambda \to \infty$.

Proof. Let $\mathcal{K}$ be the set of all non-empty, bounded convex open sets in $\mathbb{R}^d$. This is a metric space with respect to the Hausdorff metric. In order to prove the corollary, by [24, Proposition 4.1] we only need to show that the asymptotic expansion

$$\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^\gamma = L_{\gamma,d}|\Omega|\lambda^{\gamma+d/2} - \frac{1}{4}L_{\gamma,d-1}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{\gamma+(d-1)/2} + o(\lambda^{\gamma+(d-1)/2})\,, \qquad (4)$$

as $\lambda \to \infty$, holds uniformly on compact subsets of $\mathcal{K}$. Here

$$L_{\gamma,d} = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)}\,.$$

Recall the Aizenman–Lieb identity [1]: for $0 \leq \gamma_1 < \gamma_2$ and $\lambda \geq 0$,

$$\mathrm{Tr}(-\Delta_\Omega - \lambda)_-^{\gamma_2} = B(1+\gamma_1, \gamma_2-\gamma_1)^{-1}\int_0^\lambda \tau^{\gamma_2-\gamma_1-1}\,\mathrm{Tr}(-\Delta_\Omega - (\lambda-\tau))_-^{\gamma_1}\,d\tau\,, \qquad (5)$$

where B denotes the Euler Beta function.

By (5) it suffices to prove the uniform asymptotics (4) for $\gamma = 1$. Since $|\Omega|$ and $\mathcal{H}^{d-1}(\partial\Omega)$ are continuous on $\mathcal{K}$, they are bounded on compact subsets of $\mathcal{K}$. Therefore it suffices to prove (4) uniformly for sets Ω with bounded $|\Omega|$ and $\mathcal{H}^{d-1}(\partial\Omega)$. This follows from Theorem 1.2 together with the fact that one can bound $r_{in}(\Omega)$ from below in terms of $|\Omega|$ and $\mathcal{H}^{d-1}(\partial\Omega)$, see (49). ∎

Remark 1.4. In fact, the convergence in Corollary 1.3 holds not only for maximizers, but also for almost-maximizers $(\Omega_{\lambda,\gamma})_{\lambda>0}$ in the sense that $\Omega_{\lambda,\gamma}\subset\mathbb{R}^d$ is convex, open with $|\Omega_{\lambda,\gamma}|=1$ and

$$\limsup_{\lambda\to\infty}\lambda^{-\gamma-(d-1)/2}\big(\mathrm{Tr}(-\Delta_{\Omega_{\lambda,\gamma}}-\lambda)_-^\gamma-S_\gamma\big)\geq 0\,,$$

where S_γ denotes the supremum in the corollary. This follows by a straightforward adaptation of the arguments above and in [24, Proposition 4.1].

Let us now return to discussing Theorem 1.1. This theorem improves earlier results from [10, 11] where the asymptotics were shown for sets with $C^{1,\alpha}$ and C^1 boundary, respectively. As we will explain below in more detail, the technique of flattening the boundary from [10, 11] cannot be used in the case of Lipschitz boundary, but a different and more robust technique is needed.

The Lipschitz condition on the boundary is essentially an optimal assumption. On the one hand, the result is optimal in the Hölder scale (because there are sets with $C^{0,\alpha}$ boundary for $\alpha<1$ for which $\mathcal{H}^{d-1}(\partial\Omega)$ is infinite) and on the other hand, the asymptotics (3) are not valid for arbitrary sets for which $\mathcal{H}^{d-1}(\partial\Omega)$ is finite (for instance, for a ball divided in two pieces by a hyperplane the piece of the hyperplane contributes once to the measure of the boundary, but should contribute twice to the asymptotics).

Moreover, within Lipschitz domains the error term $o(\lambda^{1+(d-1)/2})$ is the best possible on the algebraic scale: for any $\varepsilon>0$ one can construct a Lipschitz domain Ω such that

$$\limsup_{\lambda\to\infty}\lambda^{-1-(d-1)/2+\varepsilon}\Big|\mathrm{Tr}(-\Delta_\Omega-\lambda)_--L_d|\Omega|\lambda^{1+d/2}+\frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)\lambda^{1+(d-1)/2}\Big|=\infty\,.$$

This follows by integration with respect to λ from a construction mentioned in [8].

Two-term spectral asymptotics under a Lipschitz assumption go back to the work [8] by Brown, where it is shown that

$$\mathrm{Tr}\,e^{t\Delta_\Omega}=\sum_{k\geq1}e^{-t\lambda_k}=(4\pi t)^{-d/2}\Big(|\Omega|-\frac{\sqrt{\pi}}{2}\mathcal{H}^{d-1}(\partial\Omega)t^{1/2}+o(t^{1/2})\Big)\qquad\text{as } t\to0^+\,. \tag{6}$$

Note that (6) is an Abel-type average of (2), whereas (3) is a Cesàro-type average. It is well-known and easy to see that the asymptotics in (3) imply those in (6), but not vice versa. The key insight in [8] was to use ideas from geometric measure theory to decompose a neighborhood of the boundary into a 'good' part and a 'bad' part with sufficiently precise control on the size of the bad part. Inserting well-known pointwise bounds on the heat kernel into this decomposition one obtains (6). While Brown's decomposition of a neighborhood of the boundary also plays an important role in our proof of (3), we are facing the additional difficulty that we cannot work on a pointwise level. Thus, our main task is to show that Brown's geometric measure theory arguments can be combined with the technique of local trace asymptotics used in [10, 11].

Let us sketch the overall strategy of the proof. As in [10, 11] we first localize the operator $-\Delta_\Omega$ into balls whose size varies depending on the distance to Ω^c. (As an aside we point out that our choice of the size of the balls here differs from that in [10, 11]. It is both simpler and has a natural scaling behavior which is crucial for the proof of the uniform inequality in Theorem 1.2.) There are four different types of balls:

(i) $B \subset \Omega$, i.e. we have localized in the bulk of Ω.
(ii) $B \cap \Omega$ is empty, i.e. we have localized outside Ω (here the localized operator is trivially zero).
(iii) $B \cap \partial\Omega$ is non-empty and is in a certain sense well-behaved.
(iv) cases (i)-(iii) fail, i.e. the set $B \cap \partial\Omega$ is non-empty and fails to be well-behaved in the sense of (iii).

Balls of type (i) are handled as in [10, 11] and those of type (ii) are trivial. The precise sense in which balls of type (iii) and (iv) are distinguished follows the geometric construction due to Brown [8].

Our analysis diverges from that in [10, 11] when it comes to treating the region near the boundary. In [10, 11] the types (iii) and (iv) were not distinguished. There, the bounds rely on the fact that if the boundary is sufficiently regular, then one can locally make a change of coordinates mapping the boundary to a hyperplane while retaining control of how the Laplacian is perturbed under this mapping. For Lipschitz boundaries this method cannot work; flattening the boundary requires a Lipschitz change of coordinates and can thus result in large perturbations of the Laplacian.

The idea of distinguishing types (iii) and (iv) is in the spirit of Brown's decomposition of a neighborhood of the boundary into a large 'good' and a small 'bad' part. Essentially, Brown's geometric construction tells us in a quantitative manner that at a sufficiently small scale, the boundary is in most regions well approximated by a hyperplane. For these approximating hyperplanes we can proceed as in the smooth case. However, we are still left with controlling the error from the hyperplane approximation. This is dealt with by proving precise local spectral asymptotics for circular cones (which are the content of Lemma 2.10).

This concludes our sketch of the proof of Theorem 1.1. We would like to emphasize that the methods that we develop in this paper are not limited to the situation at hand. In particular, the following three generalizations seem possible:

(1) For our proof it is not crucial that the boundary around *any* point can be represented as a Lipschitz graph. For instance, we could treat domains with a finite number of cusps and also domains with slits (the second term in the asymptotics (3) should be modified so that the measure of a slit is counted twice).

(2) Uniform inequalities similar to that in Theorem 1.2 are probably valid also for other classes of domains. The essential ingredients here are Lemmas 5.3 and 5.4. For example, analogues of these lemmas can probably be established for sets satisfying a uniform inner and outer ball condition. For such sets uniform bounds for the heat trace were shown in [6].

(3) Bañuelos, Kulczycki and Siudeja [3] have generalized Brown's results for the heat kernel to the case of the fractional Laplacian. Similarly, [12] generalizes the results from [10] for eigenvalue sums to the case of the fractional Laplacian. Combining these techniques one can probably extend the results in the present paper to the case of the fractional Laplacian.

Structure of the paper. We begin by introducing some notation, recalling the machinery developed in [10, 11] and proving some corollaries thereof. This is done in Section 2. In Section 3 we adapt the geometric constructions of [8] to the problem considered here. Section 4 is dedicated to the proof of Theorem 1.1 using the tools developed in Sections 2 and 3. We end the paper with the proof of Theorem 1.2 in Section 5.

2. Notation and Preliminaries

Throughout the paper we let $\operatorname{dist}(\,\cdot\,,\cdot\,)$ denote the distance between two sets in $\mathbb{R}^d$ (possibly singletons), that is,

$$\operatorname{dist}(A, B) = \inf_{x\in A,\, y\in B} |x-y|\,.$$

Given a Lipschitz set Ω define $\delta_\Omega(\,\cdot\,)$, the signed distance function of Ω, by

$$\delta_\Omega(x) = \operatorname{dist}(x, \Omega^c) - \operatorname{dist}(x, \Omega)\,.$$

Note that $\delta_\Omega(\,\cdot\,)$ and $\operatorname{dist}(\,\cdot\,, \partial\Omega)$ satisfy almost everywhere

$$|\nabla \delta_\Omega(x)| = 1\,, \qquad |\nabla \operatorname{dist}(x, \partial\Omega)| = 1\,. \tag{7}$$

Define also the inradius of $\Omega \subset \mathbb{R}^d$ by

$$r_{in}(\Omega) = \sup_{x\in\Omega} \operatorname{dist}(x, \Omega^c)\,.$$

We recall that for a Lipschitz domain $\Omega \subset \mathbb{R}^d$ the functions defined by

$$\vartheta_{inner}(\Omega, t) = \frac{|\{u\in\Omega : \operatorname{dist}(u, \partial\Omega) < t\}|}{t\mathcal{H}^{d-1}(\partial\Omega)} - 1\,,$$
$$\vartheta_{outer}(\Omega, t) = \frac{|\{u\in\Omega^c : \operatorname{dist}(u, \partial\Omega) < t\}|}{t\mathcal{H}^{d-1}(\partial\Omega)} - 1$$

are both $o(1)$ as $t \to 0^+$ [2]. In what follows we shall suppress Ω in the notation and let this dependence be understood implicitly. We also define

$$\overline{\vartheta}(t) = \frac{1}{2}\sup_{t_1, t_2 \le t}\big(|\vartheta_{inner}(t_1)| + |\vartheta_{outer}(t_2)|\big) \tag{8}$$

so that

$$\left|\frac{|\{u\in\mathbb{R}^d : \operatorname{dist}(u, \partial\Omega) < t\}|}{2t\mathcal{H}^{d-1}(\partial\Omega)} - 1\right| \le \overline{\vartheta}(t)\,. \tag{9}$$

The main contributions to the error term of Theorem 1.1 can be understood in terms of $\vartheta_{inner}(t), \vartheta_{outer}(t)$ and $\overline{\vartheta}(t)$.

In the following it will be convenient to introduce the operator

$$H_\Omega = -h^2\Delta_\Omega - 1 \qquad \text{in } L^2(\Omega)$$

with Dirichlet boundary conditions, depending on a parameter $h > 0$. Technically, H_Ω is defined as a self-adjoint operator in $L^2(\Omega)$ via the quadratic form $\int_\Omega (h^2|\nabla u|^2 - |u|^2)\,dx$ with form domain $H_0^1(\Omega)$. We have

$$\operatorname{Tr}(H_\Omega)_- = h^2 \sum_{\lambda_k < h^{-2}} (h^{-2} - \lambda_k) = h^2 \operatorname{Tr}(-\Delta_\Omega - h^{-2})_-\,,$$

and therefore the asymptotics in Theorem 1.1 as $\lambda \to \infty$ can be rephrased equivalently as asymptotics for $\operatorname{Tr}(H_\Omega)_-$ as $h \to 0^+$. Similarly, the universal bound in Theorem 1.2 can be rephrased equivalently as a universal bound for $\operatorname{Tr}(H_\Omega)_-$.

For $\phi \in C^\infty(\mathbb{R}^d)$, define $\phi H_\Omega \phi$ as a self-adjoint operator in $L^2(\Omega)$ via the quadratic form $\int_\Omega (h^2|\nabla(\phi u)|^2 - |\phi u|^2)\,dx$ with form domain $H_0^1(\Omega)$.

Let us recall three results from [10, 11] concerning localized traces of H_Ω.

Lemma 2.1 (Localized Berezin–Li–Yau inequality [10, Lemma 2.1])**.** *Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Then, for all $h > 0$,*

$$\operatorname{Tr}(\phi H_\Omega \phi)_- \leq L_d h^{-d} \int_\Omega \phi^2(x)\,dx\,.$$

Lemma 2.2 ([10, Proposition 1.2])**.** *Let $\phi \in C_0^\infty(\Omega)$ have support in a ball of radius $l > 0$ and satisfy*

$$\|\nabla\phi\|_{L^\infty} \leq Ml^{-1}\,.$$

Then, for all $h > 0$,

$$\left|\operatorname{Tr}(\phi H_\Omega \phi)_- - L_d h^{-d} \int_\Omega \phi^2(x)\,dx\right| \leq C l^{d-2} h^{-d+2}\,,$$

with a constant C depending only on M and d.

Lemma 2.3 ([10, Proposition 1.3], [11, Proposition 2.3])**.** *Let $\phi \in C_0^\infty(\mathbb{R}^d)$ have support in a ball of radius $l > 0$ and satisfy*

$$\|\nabla\phi\|_{L^\infty} \leq Ml^{-1}\,.$$

Assume that $\partial\Omega \cap \operatorname{supp}\phi$ can be represented as a graph $x_d = f(x')$ and that there is a point $(y', y_d) \in \partial\Omega \cap \operatorname{supp}\phi$ with $\nabla f(y') = 0$ and

$$|\nabla f(x')| \leq \omega(|x' - y'|) \qquad \text{for all } (x', x_d) \in \partial\Omega \cap \operatorname{supp}\phi\,,$$

where $\omega\colon [0,\infty) \to [0,\infty)$ is non-decreasing and $\lim_{\delta\to 0^+}\omega(\delta) = 0$. Then, if $\omega(l) \leq C_d$ and $0 < h \leq l$,

$$\left|\operatorname{Tr}(\phi H_\Omega \phi) - L_d h^{-d} \int_\Omega \phi^2(x)\,dx + \frac{L_{d-1}}{4} h^{-d+1} \int_{\partial\Omega} \phi^2(x)\,d\mathcal{H}^{d-1}(x)\right| \leq C \frac{l^d}{h^d}\left(\frac{h^2}{l^2} + \omega(l)\right),$$

where the constant C_d is universal and the constant C depends only on M and d.

Remark 2.4. This result appears in [10] in the special case $\omega(\delta) = C\delta^\alpha$. The case of a general function ω appears in [11], but for the Laplacian with Robin boundary conditions. The proof there, however, extends immediately to the case of Dirichlet boundary conditions. Moreover, a slightly stronger assumption on the parametrization is made in these papers, but only the above one is used, see [11, Equation (4.1)]. Also, the analysis in [10, 11] leads to an additional error term $\omega(l)^2 h/l$ in the parentheses on the right side, but since

$$\frac{\omega(l)^2 h}{l} \leq \frac{1}{2}\frac{h^2}{l^2} + \frac{1}{2}\omega(l)^4 \leq \frac{1}{2}\frac{h^2}{l^2} + \frac{C_d^3}{2}\omega(l)$$

this term is controlled by the other two terms in the parentheses. Finally, there are the following two minor changes. In [10, 11] it is stated that the constant C depends, in addition, on $\|\phi\|_{L^\infty}$ and Ω. However, since ϕ has support in a ball of radius l one easily finds $|\phi(x)| \leq l\|\nabla\phi\|_{L^\infty}$, so $\|\phi\|_{L^\infty} \leq M$, and an upper bound on $\|\phi\|_{L^\infty}$ was all that entered in the proof in [11]. Moreover, an inspection of the proof shows that the dependence on Ω enters only through the modulus of continuity ω and that, in fact, only $\omega(l) \leq C_d$ is needed.

Next, we recall a result of Solovej and Spitzer which provides a family of localization functions adapted to a given local length scale.

Lemma 2.5 ([32, Theorem 22]). *Let $\phi \in C_0^\infty(\mathbb{R}^d)$ with support in $\overline{B_1(0)}$ and $\|\phi\|_{L^2} = 1$ and let l be a bounded, positive Lipschitz function on $\mathbb{R}^d$ with Lipschitz constant $\|\nabla l\|_{L^\infty} < 1$. Let*

$$\phi_u(x) = \phi\Big(\frac{x-u}{l(u)}\Big)\sqrt{1 + \nabla l(u) \cdot \frac{x-u}{l(u)}}\,.$$

Then

$$\int_{\mathbb{R}^d} \phi_u(x)^2 l(u)^{-d}\, du = 1 \qquad \text{for all } x \in \mathbb{R}^d \tag{10}$$

and

$$\|\phi_u\|_{L^\infty} \leq \sqrt{2}\,\|\phi\|_{L^\infty} \quad \text{and} \quad \|\nabla\phi_u\|_{L^\infty} \leq Cl(u)^{-1}\|\nabla\phi\|_{L^\infty} \quad \text{for all } u \in \mathbb{R}^d\,, \tag{11}$$

where the constant C depends only on $(1 - \|\nabla l\|_{L^\infty})^{-1}$.

Remark 2.6. Strictly speaking, the functions ϕ_u are defined only for almost every $u \in \mathbb{R}^d$, namely, for those where $\nabla l(u)$ exists. Note that if $(x-u)/l(u) \in \operatorname{supp}\phi$, then $|\nabla l(u)\cdot(x-u)/l(u)| \leq \|\nabla l\|_{L^\infty} < 1$. Therefore the square root in the definition of ϕ_u is well-defined and $\phi_u \in C_0^\infty(\mathbb{R}^d)$.

Remark 2.7. The assumptions of Lemma 2.5 are weaker than those in [32]. However, the proof in [32] applies with almost no change, but for completeness we include it below. Moreover, the definition of ϕ_u in [32] reads

$$\phi_u(x) = l(u)^{d/2}\phi((x-u)/l(u))\sqrt{J(x,u)}\,,$$

where $J(x,u)$ is the absolute value of the Jacobi determinant of the map $u \mapsto (x-u)/l(u)$, that is,

$$J(x,u) = l(u)^{-d}\left|\det\left(1 + \nabla l(u) \otimes \frac{x-u}{l(u)}\right)\right|.$$

Computing the determinant one arrives at the above formula (which will be important for us later on).

Proof of Lemma 2.5. Without loss of generality we assume that $x = 0$. In order to prove (10) we shall show that the map $F\colon \mathbb{R}^d \to \mathbb{R}^d$ given by $F(u) = -u/l(u)$ is a bijection of $F^{-1}(B_1(0))$ onto $B_1(0)$. After this is established the desired equality follows by a change of variables since

$$l(u)^{-d}\Big(1 + \nabla l(u) \cdot \frac{x-u}{l(u)}\Big) = J(x,u),$$

where $J(x,u)$ is the absolute value of the Jacobi determinant of the map $u \mapsto (x-u)/l(u)$.

Fix $u \in \mathbb{R}^d$, since $|F(u)| \geq |u|/\|l\|_{L^\infty}$ and $F(0) = 0$ there exists a $t \in [-\|l\|_{L^\infty}, 0]$ such that $F(tu) = u$. Consequently F is surjective.

That the map is injective on $F^{-1}(B_1(0))$ can be seen as follows. Fix $u \neq 0$. We can write $F(tu) = -g(t)u$ where $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, indeed $g(t) = t/l(tu)$. Moreover, we claim that g is monotone increasing for all t such that $|F(tu)| = |g(t)||u| < \|\nabla l\|_{L^\infty}^{-1}$, and in particular for t such that $|F(tu)| = |g(t)||u| \leq 1$. For almost every t it holds that

$$g'(t) = l(tu)^{-1}[1 - tl(tu)^{-1}u \cdot \nabla l(tu)] \geq l(tu)^{-1}[1 - |g(t)||u|\|\nabla l\|_{L^\infty}] > 0,$$

which proves the claim. We conclude that F is a bijection from $F^{-1}(B_1(0))$ to $B_1(0)$.

Differentiating the formula for ϕ_u and using $\|\phi\|_{L^\infty} \leq \|\nabla\phi\|_{L^\infty}$ (see Remark 2.4) one immediately obtains (11). □

Lemma 2.8 (Localization)**.** *Let ϕ and l be as in Lemma 2.5. Then, for any $\varphi \in C^\infty(\mathbb{R}^d)$ and all $0 < h \leq M \min_{\mathrm{dist}(u,\Omega\,\cap\,\mathrm{supp}\,\varphi)\leq l(u)} l(u)$,*

$$\begin{aligned}\Big|\mathrm{Tr}(\varphi H_\Omega \varphi)_- - \int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u\varphi H_\Omega\varphi\phi_u)_- l(u)^{-d}\,du\Big|& \\ \leq C\|\varphi\|^2_{L^\infty(\Omega)} h^{-d+2} \int_{\mathrm{dist}(u,\Omega\,\cap\,\mathrm{supp}\,\varphi)\leq l(u)} l(u)^{-2}\,du\,,&\end{aligned} \tag{12}$$

where the constant depends only on $\|\nabla\phi\|_{L^\infty}$, $(1 - \|\nabla l\|_{L^\infty})^{-1}, M$ and d.

For $\varphi \equiv 1$ this is in essentially Proposition 1.1 [10]. Here we shall need the slightly more general statement above. However, the proof, which is given in Appendix A, is almost identical to that in [10].

Remark 2.9. In [10] the inequality corresponding to (12) is stated for all $h > 0$, however, the proof requires additionally an upper bound on $h/l(u)$. This does not affect the results in [10] because for an asymptotic result it suffices to apply the statement where this additional assumption is met. Nonetheless, in [10] the inequality is stated for a particular choice of l for which it can be extended to all $h > 0$, if one assumes that a parameter l_0 in their construction satisfies $\liminf_{h\to 0^+} l_0/h > 0$. This will be proved in Appendix A.

With these preparations at hand, we now show how the method of [10] can be used to compute a two-term asymptotic formula for cones.

Lemma 2.10 (Precise local asymptotics in cones)**.** *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ have support in a ball of radius $l > 0$ and satisfy*

$$\|\varphi\|_{L^\infty} \leq M\,. \tag{13}$$

Let $0 \leq \varepsilon \leq 1/2$ *and*

$$\Lambda_\varepsilon = \{x \in \mathbb{R}^d : x_d < \varepsilon |x|\}\,.$$

Then, for all $h > 0$,

$$\Big|\mathrm{Tr}(\varphi H_{\Lambda_\varepsilon}\varphi)_- - L_d h^{-d}\int_{\Lambda_\varepsilon}\varphi^2(x)\,dx + \frac{L_{d-1}}{4}h^{-d+1}\int_{\partial\Lambda_\varepsilon}\varphi^2(x)\,d\mathcal{H}^{d-1}(x)\Big| \leq C l^{d-4/3}h^{-d+4/3}\,,$$

and

$$\Big|\mathrm{Tr}(\varphi H_{\Lambda_\varepsilon^c}\varphi)_- - L_d h^{-d}\int_{\Lambda_\varepsilon^c}\varphi^2(x)\,dx + \frac{L_{d-1}}{4}h^{-d+1}\int_{\partial\Lambda_\varepsilon^c}\varphi^2(x)\,d\mathcal{H}^{d-1}(x)\Big| \leq C l^{d-4/3}h^{-d+4/3}\,,$$

where the constant C *depends only on* M *and* d *and, in particular, not on* ε.

The error $(l/h)^{d-4/3}$ is probably not sharp, but good enough for our purposes. After the proof we will explain that for $d = 2$, our proof actually yields the error $(l/h)^\gamma$ for any $\gamma > 0$.

Proof of Lemma 2.10. We only prove the first claim of the lemma, the second one follows analogously. The idea is to apply the arguments from [10, 11] to the operator $\varphi H_{\Lambda_\varepsilon}\varphi$ instead of H_{Λ_ε}.

Before we continue with the main part of the proof we show that the claimed inequality holds for $h \geq l$.

For all $h > 0$, Lemma 2.1 implies that

$$\begin{aligned}\Big|\mathrm{Tr}(\varphi H_{\Lambda_\varepsilon}\varphi)_- &- L_d h^{-d}\int_{\Lambda_\varepsilon}\varphi^2(x)\,dx + \frac{L_{d-1}}{4}h^{-d+1}\int_{\partial\Lambda_\varepsilon}\varphi^2(x)\,d\mathcal{H}^{d-1}(x)\Big|\\ &\leq 2L_d h^{-d}\int_{\Lambda_\varepsilon}\varphi^2(x)\,dx + \frac{L_{d-1}}{4}h^{-d+1}\int_{\partial\Lambda_\varepsilon}\varphi^2(x)\,d\mathcal{H}^{d-1}(x)\\ &\leq C(l^d h^{-d} + l^{d-1}h^{-d+1})\,.\end{aligned}$$

Here we used (13), $|\Lambda_\varepsilon \cap B_l| \leq C l^d$, and $\mathcal{H}^{d-1}(\partial\Lambda_\varepsilon \cap B_l) \leq C l^{d-1}$. The last inequality follows by noting that $\Lambda_\varepsilon^c \cap B_l$ is convex and the monotonicity of the measure of the perimeter of convex sets under inclusion.

Consequently the inequality claimed in the lemma holds for all $h \geq l$. Through the remainder of the proof we assume that $0 < h < l$.

Since Λ_ε is scale invariant, we may and will assume that $l = 1$.

Step 1: We derive a local C^1 modulus of continuity for $\partial\Lambda_\varepsilon$. We claim that for any $|u| \geq 4r$ and $B_r(u) \cap \partial\Lambda_\varepsilon \neq \emptyset$ we can choose a system of coordinates $(x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R}$ such that $\partial\Lambda_\varepsilon \cap B_r(u)$ can be parametrized as the graph $x_d = f(x')$ of a function f such that for some point in $\partial\Lambda_\varepsilon \cap B_r(u)$ with coordinates (y', y_d) and $\nabla f(y') = 0$ one has

$$|\nabla f(x')| \leq C_{d,\varepsilon}\frac{|x'-y'|}{|u|}\,, \tag{14}$$

where $C_{d,\varepsilon}$ is uniformly bounded for $0 \leq \varepsilon \leq 1/2$. (In fact, the constant here satisfies $C_{d,\varepsilon} = o_{\varepsilon\to 0^+}(1)$, but this will not be relevant for us. In $d = 2$ the boundary of Λ_ε consists of two rays and hence $C_{2,\varepsilon} = 0$.)

Let us prove (14). Pick $x_0 \in B_r(u) \cap \partial\Lambda_\varepsilon$. Then $B_r(u) \cap \partial\Lambda_\varepsilon \subset B_{2r}(x_0) \cap \partial\Lambda_\varepsilon$ and $0 \notin B_{2r}(x_0)$. After rescaling and rotating so that $x_0 = (1, 0, \dots, 0)$ and $\Lambda_\varepsilon \subset \{x \in \mathbb{R}^d : x_d \leq 0\}$ the above inclusions imply that it is sufficient to consider parametrizing $\partial\Lambda_\varepsilon$ as $x_d = f_0(x')$ in the ball $B_{2/3}(x_0)$. Clearly this is possible and f_0 is $C^{1,1}$-regular and thus, by the choice of coordinates, satisfies the estimate

$$|\nabla f_0(x')| \leq C_{d,\varepsilon}|x' - x_0'| \,, \quad x_0' = (1, 0, \dots, 0) \in \mathbb{R}^{d-1} \,,$$

where $C_{d,\varepsilon}$ is uniformly bounded for $0 \leq \varepsilon \leq 1/2$ and tends to zero as $\varepsilon \to 0^+$. After scaling and translating one obtains (14) since by assumption $|x_0| \geq \frac{3}{4}|u|$.

Step 2: We localize the problem. Fix a function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi = \overline{B_1(0)}$ and $\|\phi\|_{L^2} = 1$. With a parameter $l_0 \in (0, 1]$ depending on h to be determined, set

$$l(u) = \frac{1}{2}\min\bigl\{2, \max\{\operatorname{dist}(u, \Lambda_\varepsilon^c), 2l_0\}\bigr\} \,.$$

Note that $0 < l \leq 1$ and, by (7), $\|\nabla l\|_{L^\infty} \leq 1/2$, so Lemma 2.5 is applicable. Denote by ϕ_u the resulting family of functions from that lemma. Assume also that $h \leq l_0$ so that $h \leq l(u)$ for all $u \in \mathbb{R}^d$.

By Lemma 2.8, with $M = 1$, and a straightforward estimate of the integral remainder we have that

$$\left|\operatorname{Tr}(\varphi H_{\Lambda_\varepsilon}\varphi)_- - \int_{\mathbb{R}^d} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\, du\right| \leq C\|\varphi\|_{L^\infty}^2 l_0^{-1} h^{-d+2} \,. \tag{15}$$

Step 3: We split

$$\begin{aligned}\int_{\mathbb{R}^d} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\, du = &\int_{\Lambda^{(1)}} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\, du \\ &+ \int_{\Lambda^{(2)}} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\, du \,,\end{aligned} \tag{16}$$

where

$$\begin{aligned}\Lambda^{(1)} &= \{u \in \mathbb{R}^d :\ \emptyset \neq \operatorname{supp}\phi_u\varphi \subset \Lambda_\varepsilon\} \,,\\ \Lambda^{(2)} &= \{u \in \mathbb{R}^d :\ \operatorname{supp}\phi_u\varphi \cap \partial\Lambda_\varepsilon \neq \emptyset\} \,,\end{aligned}$$

and where we used the fact that $\operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- = 0$ when $\operatorname{supp}\phi_u\varphi \cap \Lambda_\varepsilon = \emptyset$. Since $\operatorname{supp}\varphi$ is contained in a ball of radius 1 and $\operatorname{supp}\phi_u$ is contained in a ball of radius $l(u) \leq 1$ the set $\Lambda^{(1)} \cup \Lambda^{(2)}$ is contained in a ball of radius 2. Moreover, it is easy to see that for all $u \in \Lambda^{(2)}$ one has $l(u) \geq \operatorname{dist}(u, \partial\Lambda_\varepsilon)$ and therefore $\operatorname{dist}(u, \partial\Lambda_\varepsilon) \leq l_0$ and $l(u) = l_0$.

Applying Lemma 2.2 to the first integral in (16) and using [10, Equation 8] (see also (35) below) yields

$$\begin{aligned}\int_{\Lambda^{(1)}} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\,du &= L_d h^{-d}\int_{\Lambda^{(1)}}\int_{\Lambda_\varepsilon}\phi_u^2(x)\varphi^2(x)l(u)^{-d}\,dx\,du\\ &\quad + O(h^{-d+2})\int_{\Lambda^{(1)}} l(u)^{-2}\,du \qquad (17)\\ &= L_d h^{-d}\int_{\Lambda^{(1)}}\int_{\Lambda_\varepsilon}\phi_u^2(x)\varphi^2(x)l(u)^{-d}\,dx\,du + l_0^{-1}O(h^{-d+2})\,.\end{aligned}$$

With a parameter $\delta > 0$ to be specified, we split the second integral of (16) further, depending on the distance of u from the vertex of Λ_ε,

$$\begin{aligned}\int_{\Lambda^{(2)}} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\,du &= \int_{\Lambda^{(2)}\setminus B_\delta} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\,du\\ &\quad + \int_{\Lambda^{(2)}\cap B_\delta} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\,du\,.\end{aligned} \qquad (18)$$

By Lemma 2.1 the second integral is small, that is,

$$\begin{aligned}\int_{\Lambda^{(2)}\cap B_\delta} \operatorname{Tr}(\phi_u\varphi H_{\Lambda_\varepsilon}\varphi\phi_u)_- l(u)^{-d}\,du &\le L_d h^{-d}\int_{\Lambda^{(2)}\cap B_\delta}\int_{\Lambda_\varepsilon}\phi_u^2(x)\varphi^2(x)l(u)^{-d}\,dx\,du\\ &\le C h^{-d}|\Lambda^{(2)}\cap B_\delta| \le C h^{-d}\delta^{d-1}l_0\,.\end{aligned} \qquad (19)$$

In the last inequality we used the fact that $\Lambda^{(2)}$ is contained in an l_0-neighborhood of $\partial\Lambda_\varepsilon$. For later purposes we also record that

$$\int_{\Lambda^{(2)}\cap B_\delta}\Big(\int_{\Lambda_\varepsilon}\phi_u^2(x)\varphi^2(x)\,dx + h\int_{\partial\Lambda_\varepsilon}\phi_u^2(x)\varphi^2(x)\,d\mathcal{H}^{d-1}(x)\Big)l(u)^{-d}du \le C\delta^{d-1}(l_0+h)\,, \quad (20)$$

where we used again $|\Lambda^{(2)}\cap B_\delta| \le C l_0\delta^{d-1}$.

To treat the remaining term of (18) we apply Lemma 2.3. Let $C_{d,\varepsilon}$ and C_d be the constants from Step 1 and Lemma 2.3, respectively, and let $\omega(r) = C_{d,\varepsilon}r/|u|$. Finally, set $A = \max\{C_{d,\varepsilon}/C_d, 4\}$.

We claim that, if $\delta \ge Al_0$, then $\omega(l(u)) \le C_d$ and for all $u \in \Lambda^{(2)}\setminus B_\delta$ one can parametrize $\partial\Lambda_\varepsilon \cap B_{l(u)}(u)$ as the graph of a function f and for a point $(y', y_d) \in \partial\Lambda_\varepsilon \cap B_{l(u)}(u)$ one has $\nabla f(y') = 0$ and $|\nabla f(x')| \le \omega(|x'-y'|)$ for all $x' \in \mathbb{R}^{d-1}$.

Indeed, for any $u \in \Lambda^{(2)} \setminus B_\delta$ one has $|u| \ge \delta \ge Al_0 = Al(u)$. Therefore, since $A \ge 4$, according to Step 1 such a parametrization is possible with the above choice of ω. In particular, $\omega(l(u)) = C_{d,\varepsilon}l(u)/|u| \le C_{d,\varepsilon}/A$. Since $A \ge C_{d,\varepsilon}/C_d$, the claimed inequality holds.

Since $l_0 \geq h$, we for all $u \in \Lambda^{(2)}$ have $l(u) = l_0 \geq h$ and therefore Lemma 2.3 yields

$$\begin{aligned}
&\int_{\Lambda^{(2)}\setminus B_\delta} \mathrm{Tr}(\phi_u \varphi H_{\Lambda_\varepsilon} \varphi \phi_u)_- l(u)^{-d}\, du \\
&\quad = L_d h^{-d} \int_{\Lambda^{(2)}\setminus B_\delta} \int_{\Lambda_\varepsilon} \phi_u^2(x) \varphi^2(x) l(u)^{-d}\, dx\, du \\
&\quad\quad - \frac{L_{d-1}}{4} h^{-d+1} \int_{\Lambda^{(2)}\setminus B_\delta} \int_{\partial\Lambda_\varepsilon} \phi_u^2(x)\varphi^2(x) l(u)^{-d}\, d\mathcal{H}^{d-1}(x)\, du \\
&\quad\quad + O(h^{-d}) \int_{\Lambda^{(2)}\setminus B_\delta} \left(\frac{h^2}{l(u)^2} + C_{d,\varepsilon} \frac{l(u)}{|u|} \right) du\, .
\end{aligned} \tag{21}$$

Combining (15), (16), (17), (18), (19), (20), (21) and using (10) we obtain

$$\mathrm{Tr}(\varphi H_{\Lambda_\varepsilon} \varphi)_- = L_d h^{-d} \int_{\Lambda_\varepsilon} \varphi^2(x)\, dx - \frac{L_{d-1}}{4} h^{-d+1} \int_{\partial\Lambda_\varepsilon} \varphi^2(x)\, d\mathcal{H}^{d-1}(x) + \mathcal{R}$$

with

$$|\mathcal{R}| \leq C h^{-d} \left(l_0^{-1} h^2 + \delta^{d-1}(l_0 + h) + \int_{\Lambda^{(2)}\setminus B_\delta} \left(\frac{h^2}{l(u)^2} + C_{d,\varepsilon} \frac{l(u)}{|u|} \right) du \right) . \tag{22}$$

Our final task in the proof is to choose l_0 and δ such that the right side here becomes $\leq C h^{-d+4/3}$. By [10, Equation 8], see also (34),

$$h^2 \int_{\Lambda^{(2)}\setminus B_\delta} l(u)^{-2}\, du \leq C l_0^{-1} h^2\, .$$

To bound the remaining term of the integral we consider two cases:

i. If $\Lambda^{(2)} \cap B_1 = \emptyset$, then

$$C_{d,\varepsilon} \int_{\Lambda^{(2)}\setminus B_\delta} \frac{l(u)}{|u|}\, du \leq C_{d,\varepsilon} \int_{\Lambda^{(2)}\setminus B_\delta} l(u)\, du \leq C C_{d,\varepsilon} l_0^2\, .$$

ii. If $\Lambda^{(2)} \cap B_1 \neq \emptyset$, then $\Lambda^{(2)} \subset B_5$ and

$$C_{d,\varepsilon} \int_{\Lambda^{(2)}\setminus B_\delta} \frac{l(u)}{|u|}\, du \leq C C_{d,\varepsilon} l_0^2 \int_\delta^5 \tau^{-1} \tau^{d-2}\, d\tau \leq C l_0^2 \times \begin{cases} 0 & \text{if } d = 2\, , \\ C_{3,\varepsilon}(1 + h \log(\delta^{-1})) & \text{if } d = 3\, , \\ C_{d,\varepsilon} & \text{if } d \geq 4\, . \end{cases}$$

In both cases we used the fact that $\Lambda^{(2)}$ is contained in an l_0-neighborhood of $\partial\Lambda_\varepsilon$.

In conclusion, the right side of (22) is bounded by

$$C h^{-d} \Big(l_0^{-1} h^2 + \delta^{d-1}(l_0 + h) + C_{d,\varepsilon} l_0^2 (1 + h \log(\delta^{-1})) \Big)\, , \tag{23}$$

where the log term appears only in $d = 3$. Setting $\delta = A l_0$ and $l_0 = h^{2/3}$, we obtain the claimed error bound. Note that $1 \geq l_0 \geq h$ for $0 < h \leq 1$, as required. □

Remark 2.11. In the two-dimensional case the above argument can be iterated to obtain Lemma 2.10 with an error term of order $l^\gamma h^{-\gamma}$ for any $\gamma > 0$. Indeed, if one has Lemma 2.10 with error term $l^{\gamma_0} h^{-\gamma_0}$ for some $\gamma_0 \in (0, 2]$, then one can replace the application of Lemma 2.1 in (19) by an application of this asymptotic expansion and one can avoid (20). Therefore (23) is replaced by $h^{-2}(l_0^{-1}h^2 + \delta h^{2-\gamma_0} l_0^{-1+\gamma_0})$. Choosing again $\delta = Al_0$ but now $l_0 = h^{\gamma_0/(1+\gamma_0)}$ yields a two-term expansion with error of order $l^{\gamma'} h^{-\gamma'}$ with $\gamma' = \frac{\gamma_0}{1+\gamma_0}$. Repeating this procedure the exponent γ can be made arbitrarily small. In higher dimensions the corresponding idea does not yield an improvement since the term $l_0^{-1}h^2 + C_{d,\varepsilon} l_0^2$ in (23) can be made no smaller than $h^{4/3}$.

3. Geometric constructions

In this section we adapt the geometric ideas used by Brown in [8] (see also [3]) to the setting considered here.

Definition 3.1. *Let $0 < \varepsilon \leq 1$ and $r > 0$. A point $p \subset \partial\Omega$ is called (ε, r)-good if the inner unit normal $\nu(p)$ exists and*

$$B_r(p) \cap \partial\Omega \subset \{x \in \mathbb{R}^d : |(x-p) \cdot \nu(p)| < \varepsilon |x-p|\} .$$

The set of all (ε, r)-good points of $\partial\Omega$ is denoted by $G_{\varepsilon,r}$.

In other words, p is (ε, r)-good if locally $\partial\Omega$ is contained in the complement of the two-sided circular cone with vertex p, symmetry axis $\nu(p)$, and opening angle $\sin^{-1}(\sqrt{1-\varepsilon^2}) = \cos^{-1}(\varepsilon)$ measured from the axis of symmetry.

Following [3, 8] we define a good subset of points near the boundary. In contrast to the constructions in [3, 8] this set will contain points both in Ω and in its complement Ω^c.

Definition 3.2. *Let*

$$\Gamma_{\varepsilon,r}(p) = \{x \in \mathbb{R}^d : |(x-p) \cdot \nu(p)| > \sqrt{1-\varepsilon^2}|x-p|\} \cap B_{r/2}(p)$$

and

$$\mathcal{G}_{\varepsilon,r} = \bigcup_{p \in G_{\varepsilon,r}} \Gamma_{\varepsilon,r}(p) .$$

We emphasize that $\Gamma_{\varepsilon,r}(p)$ differs from the corresponding set defined in [3, 8] in several ways. Here we avoid an additional degree of freedom by taking the union over *all* (ε, r)-good points instead of a subset of them, we consider two-sided cones instead of one-sided, and we also choose to truncate the cone at distance $r/2$ instead of r.

The two-sided cones appear since we, in contrast to [3, 8], do not work at a pointwise level but at the local length scale given by l. In particular, we have a non-trivial contribution to the trace from localizations centered at points $u \notin \Omega$ (see Lemma 2.8).

The reason for considering smaller cones is to ensure that if $u \in \mathcal{G}_{\varepsilon,r}$ then $\partial\Omega \cap B_{r'}(u)$ stays close to the hyperplane tangent to $\partial\Omega$ at p as long as $r' \leq r/2$. In particular, we shall make use of the following lemma.

Lemma 3.3. *Let $p \in \partial\Omega$ be (ε, r)-good with $0 < \varepsilon \le 1/2$. Then for any $u \in \Gamma_{\varepsilon,r}(p)$,*

$$|u - p| \le 2\,\mathrm{dist}(u, \partial\Omega)\,.$$

Proof of Lemma 3.3. Let $p' \in \partial\Omega$ satisfy $|u - p'| = \mathrm{dist}(u, \partial\Omega)$. Then, since $p \in \partial\Omega$,

$$|u - p'| = \mathrm{dist}(u, \partial\Omega) \le |u - p| < r/2$$

and so, in particular, $p' \in B_r(p)$. Let $\Lambda = \{y : |(y - p) \cdot \nu(p)| < \varepsilon|y - p|\}$. Then, since p is (ε, r)-good, $p' \in \partial\Omega \cap B_r(p)$ implies that $p' \in \Lambda$. Let $y \in \overline{\Lambda} \cap \overline{B_r(p)}$ satisfy $|u-y| = \mathrm{dist}(u, \Lambda \cap B_r(p))$. Then, since $p' \in \Lambda \cap B_r(p)$, $|u-y| \le |u-p'|$. By the choice of y and the construction of $\Gamma_{\varepsilon,r}(p)$ the points u, p, y form a right-angle triangle with the angle between the sides $u-p$ and $y-p$ larger than $\pi/2 - 2\sin^{-1}(\varepsilon)$. By elementary trigonometry it follows that, for $\varepsilon \in (0, 1/2]$,

$$|u - p'| \ge |u - y| \ge \sin(\pi/2 - 2\sin^{-1}(\varepsilon))|u - p| = (1 - 2\varepsilon^2)|u - p| \ge \frac{1}{2}|u - p|\,.$$

This completes the proof. □

The proof of the following result, which is omitted, is based on Rademacher's theorem on almost everywhere differentiability of Lipschitz functions.

Lemma 3.4 ([8, Section 4])**.** *For any $\varepsilon > 0$,*

$$\lim_{r\to 0^+} \mathcal{H}^{d-1}(\partial\Omega \setminus G_{\varepsilon,r}) = 0\,.$$

It follows that for any fixed $\varepsilon > 0$ we can find $r > 0$ small enough so that $G_{\varepsilon,r}$ is non-empty. Furthermore, defining for $\varepsilon > 0$

$$\mu_\Omega(\varepsilon, r) = \frac{\mathcal{H}^{d-1}(\partial\Omega \setminus G_{\varepsilon,r})}{\mathcal{H}^{d-1}(\partial\Omega)}\,, \tag{24}$$

there is an $r > 0$ so that $\mu(\varepsilon, r)$ is arbitrarily small. We shall often write simply μ and leave the dependence on Ω implicit. For the next lemma we recall that $\overline{\vartheta}$ was defined in (8).

Lemma 3.5 ([8, Proposition 1.3], [3, Lemma 2.7])**.** *Let $\varepsilon \in (0, 1]$ and $r > 0$. Then there exists an $s_0 = s_0(\varepsilon, r, \Omega) > 0$ such that for all $s \le s_0$,*

$$|\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \setminus \mathcal{G}_{\varepsilon,r}| \le 2s(\mu(\varepsilon, r) + \overline{\vartheta}(s) + \varepsilon^2)\mathcal{H}^{d-1}(\partial\Omega)\,. \tag{25}$$

Proof of Lemma 3.5. The proof follows closely those of Lemma 2.7 and Proposition 1.3 in [3] and [8], respectively. Write

$$\begin{aligned} |\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \setminus \mathcal{G}_{\varepsilon,r}| &= |\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\}| \\ &\quad - |\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \cap \mathcal{G}_{\varepsilon,r}|\,. \end{aligned} \tag{26}$$

The first term on the right side can be controlled using (9). To bound the second one, for some $\delta > 0$ to be determined later, choose $\nu_1, \dots, \nu_N \in \mathbb{S}^{d-1}$ and disjoint closed sets $F_1, \dots, F_N \subset G_{\varepsilon,r}$ such that $\mathcal{H}^{d-1}(G_{\varepsilon,r} \setminus \bigcup_{i=1}^N F_i) \le \delta\mathcal{H}^{d-1}(G_{\varepsilon,r})$ and $|\nu(p) - \nu_i| \le \varepsilon$ for all $p \in F_i$. Mimicking the proofs in [3, 8] one finds that $p + \rho\nu_i \in \Gamma_{\varepsilon,r}(p)$ for $p \in F_i$

and $-r/2 < \rho < r/2$ and that the map $(p,\rho) \mapsto p + \rho\nu_i$ is injective for $p \in F_i$ and $-r/2 < \rho < r/2$.

If s_0 is less than or equal to both $r/2$ and $\min_{i\neq j} \operatorname{dist}(F_i, F_j)/2$ one obtains by the area formula [9, Theorem 3.2.3] that, for $0 < s \leq s_0$,

$$
\begin{aligned}
|\{u \in \mathbb{R}^d : \operatorname{dist}(u, \partial\Omega) < s\} \cap \mathcal{G}_{\varepsilon,r}| &\geq \sum_{i=1}^N |\{p + \rho\nu_i : p \in F_i, -s < \rho < s\}| \\
&\geq (1-\varepsilon^2/2) \sum_{i=1}^N \int_{\{p+\rho\nu_i : p \in F_i, -s<\rho<s\}} \frac{dx}{\nu_i \cdot \nu(p)} \\
&= 2s(1-\varepsilon^2/2) \sum_{i=1}^N \mathcal{H}^{d-1}(F_i) \\
&\geq 2s(1-\varepsilon^2/2)(1-\delta)\mathcal{H}^{d-1}(G_{\varepsilon,r}) \\
&= 2s(1-\mu(\varepsilon,r))(1-\varepsilon^2/2)(1-\delta)\mathcal{H}^{d-1}(\partial\Omega)\,.
\end{aligned}
\tag{27}
$$

Combining (26), (27) and the definition of $\overline{\vartheta}$ yields

$$
|\{u \in \mathbb{R}^d : \operatorname{dist}(u,\partial\Omega) < s\} \setminus \mathcal{G}_{\varepsilon,r}| \leq 2s\mathcal{H}^{d-1}(\partial\Omega)(1-(1-\mu(\varepsilon,r))(1-\varepsilon^2/2)(1-\delta)+\overline{\vartheta}(s))\,.
$$

Choosing $\delta = \varepsilon^2/2$ and recalling that $\mu(\varepsilon, r) \leq 1$ completes the proof. $\square$

4. Asymptotics for Lipschitz domains

Our goal in this section is to prove the following

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with Lipschitz regular boundary. Then, as $h \to 0^+$,*

$$
\operatorname{Tr}(H_\Omega)_- = L_d|\Omega|h^{-d} - \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)h^{-d+1} + o(h^{-d+1})\,.
$$

Clearly, this is equivalent to Theorem 1.1. Our proof of Theorem 4.1 depends on three parameters

$$
\varepsilon_0 \in (0,4]\,, \qquad \varepsilon \in (0,1/2]\,, \qquad r > 0
$$

and we shall show that for each such choice of parameters there is an $h_0(\varepsilon_0,\varepsilon,r,\Omega) > 0$ such that for all $0 < h \leq h_0(\varepsilon_0,\varepsilon,r,\Omega)$ one has

$$
h^{d-1}\Big|\operatorname{Tr}(H_\Omega)_- - L_d|\Omega|h^{-d} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)h^{-d+1}\Big| \leq C\Big(\varepsilon_0^{1/3} + \frac{\varepsilon}{\varepsilon_0} + \frac{\overline{\vartheta}(l_0)}{\varepsilon_0} + \frac{\mu(\varepsilon,r)}{\varepsilon_0}\Big)\,, \tag{28}
$$

where C is a constant that depends in an explicit way on Ω. Here $\overline{\vartheta}(s)$ and $\mu(\varepsilon,r)$ are the functions from (8) and (24). Recalling that $\lim_{t\to 0}\overline{\vartheta}(t) = 0$ and $\lim_{r\to 0}\mu(\varepsilon,r) = 0$ for any fixed $\varepsilon > 0$ (see Lemma 3.4), Theorem 4.1 follows from (28) by letting h, r, ε and ε_0 tend to zero in that order.

There is nothing special about the assumption that $\varepsilon_0 \leq 4$. Any choice of upper bound is sufficient to complete the proof and would only result in a change of the constant C in (28). However, for our analysis in Section 5 allowing $\varepsilon_0 \in (0, 4]$ will be convenient.

We now give the details of our construction. We introduce a local length scale

$$l(u) = \frac{1}{2} \max\{\operatorname{dist}(u, \Omega^c), 2l_0\} \tag{29}$$

with a parameter $0 < l_0 \leq r_{in}(\Omega)/2$ that we will write as

$$l_0 = h/\varepsilon_0 \qquad \text{for } 0 < h \leq 2r_{in}(\Omega)\,.$$

Here $\varepsilon_0 \in (0, 4]$ is one of the parameters of our construction. We note in passing that the above definition of $l(u)$ is similar, but simpler than that in [10, 11] and has a natural scaling.

Note that $0 < l(u) \leq r_{in}(\Omega)/2$ and that, using (7), $\|\nabla l\|_{L^\infty} \leq 1/2$.

Fix a function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \phi = \overline{B_1(0)}$ and $\|\phi\|_{L^2} = 1$. Later on, it will also be important that ϕ is radially symmetric.

Lemma 2.5 now yields a family of functions $(\phi_u)_{u \in \mathbb{R}^d}$ such that $\operatorname{supp} \phi_u = \overline{B_{l(u)}(u)}$ and (10) and (11) are satisfied.

In what follows we will use the convention that C denotes a constant which may change from line to line but only depends on the dimension and the choice of ϕ. In particular, we emphasize that it is independent of Ω. Similarly, when we write $O(\,\cdot\,)$ the implicit constant is independent of Ω and all the parameters of the construction.

If $h \leq 2r_{in}(\Omega)$ then

$$\min_{\operatorname{dist}(u,\Omega) \leq l(u)} l(u) = h/\varepsilon_0 \geq h/4\,.$$

Thus, for $0 < h \leq 2r_{in}(\Omega)$ we can apply Lemma 2.8, with $M = 4$ and $\varphi \equiv 1$, and reduce our problem to studying the local contributions to the trace $\operatorname{Tr}(\phi_u H_\Omega \phi_u)_-$. (The fact that the integral on the right side of (12) is indeed negligible for small ε_0 will be proven below in (36).)

We now continue our construction and fix the parameters $\varepsilon \in (0, 1/2]$ and $r > 0$ and define the sets $\mathcal{G}_{\varepsilon,r}$ and $G_{\varepsilon,r}$ as in the previous section. According to Lemma 3.4 we may and will assume in the following that given ε, the parameter r is chosen so small that $G_{\varepsilon,r}$ is non-empty.

We divide the set of $u \in \mathbb{R}^d$ where $\operatorname{Tr}(\phi_u H_\Omega \phi_u)_-$ is non-zero into three parts,

$$\begin{aligned}
\Omega_* &= \{u \in \mathbb{R}^d : \operatorname{supp} \phi_u \subset \Omega\}\,, \\
\Omega_g &= \{u \in \mathcal{G}_{\varepsilon,r} : \operatorname{supp} \phi_u \cap \partial\Omega \neq \emptyset\}\,, \\
\Omega_b &= \{u \in \mathbb{R}^d \setminus \mathcal{G}_{\varepsilon,r} : \operatorname{supp} \phi_u \cap \partial\Omega \neq \emptyset\}\,.
\end{aligned} \tag{30}$$

Clearly these three sets are disjoint and $\mathrm{Tr}(\phi_u H_\Omega \phi_u)_- = 0$ for $u \notin \Omega_* \cup \Omega_g \cup \Omega_b$. Splitting the integral of Lemma 2.8 according to this partition we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du = &\int_{\Omega_*} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du\\ &+ \int_{\Omega_g} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du\\ &+ \int_{\Omega_b} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du\,.\end{aligned} \tag{31}$$

Let us pause for a moment and review the overall strategy of our proof. In Ω_* the effect of the boundary is not felt and a sufficiently precise asymptotic expansion follows from Lemma 2.2. By Lemma 3.5 the set Ω_b is small and its contribution to the trace is negligible. The set which is most difficult to analyse is Ω_g. Here the asymptotics in cones from Lemma 2.10 will play an important role.

4.1. **Some auxiliary estimates.** To control the error terms appearing in the proof we need to be able to control $l(u)$ on the sets in (30).

We begin with the following observation,

$$\Omega_g \cup \Omega_b = \{u \in \mathbb{R}^d : \ \mathrm{dist}(u, \partial\Omega) \le l_0\}\,. \tag{32}$$

Indeed, by definition of Ω_g and Ω_b and since $\operatorname{supp}\phi = \overline{B_1(0)}$, the set on the left equals $\{u \in \mathbb{R}^d : \ \mathrm{dist}(u, \partial\Omega) \le l(u)\}$. Therefore we need to prove that for any $u \in \mathbb{R}^d$, one has $\mathrm{dist}(u, \partial\Omega) \le l_0$ if and only if one has $\mathrm{dist}(u, \partial\Omega) \le l(u)$. This is trivial if $\mathrm{dist}(u, \Omega^c) \le 2l_0$, since then $l(u) = l_0$. On the other hand, if $\mathrm{dist}(u, \Omega^c) > 2l_0$, then $l(u) = (1/2)\,\mathrm{dist}(u, \Omega^c) = (1/2)\,\mathrm{dist}(u, \partial\Omega)$, and therefore neither of the two inequalities holds. This completes the proof of (32).

The equality (32) together with (9) implies that

$$|\Omega_g \cup \Omega_b| \le 2l_0 \mathcal{H}^{d-1}(\partial\Omega)(1 + \overline{\vartheta}(l_0))\,. \tag{33}$$

Note that it also follows from (32) that

$$l(u) = l_0 \qquad \text{if } u \in \Omega_g \cup \Omega_b\,.$$

Consequently, for any $\alpha \in \mathbb{R}$,

$$\int_{\Omega_g \cup \Omega_b} l(u)^\alpha\, du = l_0^\alpha |\Omega_g \cup \Omega_b| \le 2\mathcal{H}^{d-1}(\partial\Omega) l_0^{1+\alpha}(1 + \overline{\vartheta}(l_0))\,. \tag{34}$$

We now use (32) to bound integrals which will appear as error terms later on. We claim that

$$\int_{\Omega_*} l(u)^{-2}\, du \le C\mathcal{H}^{d-1}(\partial\Omega)\big[1 + \overline{\vartheta}(r_{in}(\Omega))\big] l_0^{-1}\,, \tag{35}$$

To prove this, we decompose

$$\int_{\Omega_*} l(u)^{-2}\, du = l_0^{-2} |\{u \in \Omega_* : \delta_\Omega(u) \le 2l_0\}| + 4 \int_{u \in \Omega_* : \delta_\Omega(u) > 2l_0} \delta_\Omega(u)^{-2}\, du\,.$$

Using (7) and the co-area formula and integrating by parts we find

$$\begin{aligned}\int_{u\in\Omega_*:\delta_\Omega(u)>2l_0} \delta_\Omega(u)^{-2}\,du &= \int_{2l_0}^{r_{in}(\Omega)} \mathcal{H}^{d-1}(\{u\in\Omega_*:\delta_\Omega(u)=t\})t^{-2}\,dt \\ &= 2\int_{2l_0}^{r_{in}(\Omega)} |\{u\in\Omega_*:\delta_\Omega(u)\le t\}|t^{-3}\,dt \\ &\qquad + |\Omega_*|r_{in}(\Omega)^{-2} - \frac{1}{4}|\{u\in\Omega_*:\delta_\Omega(u)\le 2l_0\}|l_0^{-2}\,,\end{aligned}$$

and therefore

$$\int_{\Omega_*} l(u)^{-2}\,du \le 8\int_{2l_0}^{r_{in}(\Omega)} |\{u\in\Omega:\delta_\Omega(u)\le t\}|t^{-3}\,dt + 4|\Omega|r_{in}(\Omega)^{-2}\,.$$

The second term on the right side can be bounded by

$$4|\Omega|r_{in}(\Omega)^{-2} \le 2|\Omega|r_{in}(\Omega)^{-1}l_0^{-1} \le 2\mathcal{H}^{d-1}(\partial\Omega)\big[1+2\overline{\vartheta}(r_{in}(\Omega))\big]l_0^{-1}\,.$$

In order to bound the first term, we use the definition of $\overline{\vartheta}$ and get

$$\begin{aligned}\int_{2l_0}^{r_{in}(\Omega)} |\{u\in\Omega:\delta_\Omega(u)\le t\}|t^{-3}\,dt &\le \mathcal{H}^{d-1}(\partial\Omega)\big[1+2\overline{\vartheta}(r_{in}(\Omega))\big]\int_{2l_0}^{r_{in}(\Omega)} t^{-2}\,dt \\ &\le \frac{1}{2}\mathcal{H}^{d-1}(\partial\Omega)\big[1+2\overline{\vartheta}(r_{in}(\Omega))\big]l_0^{-1}\,.\end{aligned}$$

This completes the proof of (35).

Next, we discuss the localization error coming from (12). We claim that

$$h^{-d+2}\int_{\mathrm{dist}(u,\Omega)\le l(u)} l(u)^{-2}\,du \le C\mathcal{H}^{d-1}(\partial\Omega)\big[1+\overline{\vartheta}(r_{in}(\Omega))\big]\varepsilon_0 h^{-d+1}\,. \tag{36}$$

Note that this term is negligible for the asymptotics if $\varepsilon_0 \ll 1$.

Indeed, taking into account (32) this follows from (34), (35) and the fact that $l_0 = h/\varepsilon_0$.

4.2. **Contribution from the bulk Ω_*.** For the first term on the right side of (31), Lemma 2.2 and (35) yield

$$\begin{aligned}\int_{\Omega_*} \mathrm{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\,du &= \int_{\Omega_*}\Big(L_d h^{-d}\int_\Omega \phi_u^2(x)\,dx + l(u)^{d-2}O(h^{-d+2})\Big)l(u)^{-d}\,du \\ &= L_d h^{-d}\int_{\Omega_*}\int_\Omega \phi_u^2(x)l(u)^{-d}\,dx\,du \\ &\quad + \mathcal{H}^{d-1}(\partial\Omega)\big[1+\overline{\vartheta}(r_{in}(\Omega))\big]l_0^{-1}O(h^{-d+2}) \\ &= L_d h^{-d}\int_{\Omega_*}\int_\Omega \phi_u^2(x)l(u)^{-d}\,dx\,du \\ &\quad + \varepsilon_0\mathcal{H}^{d-1}(\partial\Omega)\big[1+\overline{\vartheta}(r_{in}(\Omega))\big]O(h^{-d+1})\,.\end{aligned}$$

This is already the desired bound. Note that the second term on the right side is negligible for the asymptotics if $\varepsilon_0 \ll 1$.

4.3. **Contribution from the bad part of the boundary Ω_b.** For the third term on the right side of (31), Lemmas 2.1 and 3.5 yield

$$
\begin{aligned}
0 \leq \int_{\Omega_b} \operatorname{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du &\leq L_d h^{-d} \int_{\Omega_b}\int_\Omega \phi_u^2(x) l(u)^{-d}\, dx\, du \\
&\leq C h^{-d} |\Omega_b| \\
&\leq C\mathcal{H}^{d-1}(\partial\Omega) h^{-d+1}(\mu(\varepsilon, r) + \overline{\vartheta}(l_0) + \varepsilon^2)/\varepsilon_0\,,
\end{aligned}
\tag{37}
$$

Here we used $\Omega_b \subset \{u \in \mathbb{R}^d : \operatorname{dist}(u, \partial\Omega) \leq l_0\} \setminus \mathcal{G}_{\varepsilon,r}$ and assumed $l_0 \leq s_0$ where s_0 is the constant from Lemma 3.5. The latter condition holds for h small enough depending on ε_0, ε, r and Ω.

The bound (37) will be sufficient for us. Note that the term on the right side is negligible for the asymptotics if $(\mu(\varepsilon, r) + \overline{\vartheta}(l_0) + \varepsilon^2)/\varepsilon_0 \ll 1$.

4.4. **Contribution from the good part of the boundary Ω_g.** The term coming from Ω_g is more troublesome to deal with. It is the only term which contributes to the second term of the asymptotic expansion, and thus we need to understand its behavior in more detail.

Let $u \in \Omega_g$. Then by definition there is a $p(u) \in G_{\varepsilon,r}$ such that $u \in \Gamma_{\varepsilon,r}(p(u))$. We define two conical sets associated with u, namely,

$$
\begin{aligned}
\mathcal{I}_\varepsilon = \mathcal{I}_\varepsilon(u) &= \{x \in \mathbb{R}^d : (x - p(u)) \cdot \nu(p(u)) > \varepsilon |x - p(u)|\}\,, \\
\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(u) &= \{x \in \mathbb{R}^d : -(x - p(u)) \cdot \nu(p(u)) \geq \varepsilon |x - p(u)|\}^c\,.
\end{aligned}
$$

We note the inclusions $\mathcal{I}_\varepsilon \cap B_r(p) \subseteq \Omega \cap B_r(p) \subseteq \mathcal{U}_\varepsilon \cap B_r(p)$ and $\partial\Omega \cap B_r(p) \subset \mathcal{U}_\varepsilon \setminus \mathcal{I}_\varepsilon$. If h is small enough so that $l_0 \leq r/2$ (note that this condition on h depends only on ε_0 and r), then the fact that $l(u) = l_0$ implies that $B_{l(u)}(u) \subset B_r(p)$, and so

$$
\mathcal{I}_\varepsilon \cap B_{l(u)}(u) \subseteq \Omega \cap B_{l(u)}(u) \subseteq \mathcal{U}_\varepsilon \cap B_{l(u)}(u)\,. \tag{38}
$$

It is shown in [3] that there is a half-space $L^* = L^*(u)$ such that $p(u) \in \partial L^*$, $\operatorname{dist}(u, \partial L^*) = \operatorname{dist}(u, \partial\Omega)$ and $\mathcal{I}_\varepsilon \subset L^*(u) \subset \mathcal{U}_\varepsilon$. These inclusions together with (38) and domain monotonicity imply that

$$
\begin{aligned}
\operatorname{Tr}(\phi_u H_{\mathcal{I}_\varepsilon} \phi_u)_- &\leq \operatorname{Tr}(\phi_u H_\Omega \phi_u)_- \leq \operatorname{Tr}(\phi_u H_{\mathcal{U}_\varepsilon} \phi_u)_-\,, \\
\operatorname{Tr}(\phi_u H_{\mathcal{I}_\varepsilon} \phi_u)_- &\leq \operatorname{Tr}(\phi_u H_{L^*} \phi_u)_- \leq \operatorname{Tr}(\phi_u H_{\mathcal{U}_\varepsilon} \phi_u)_-\,.
\end{aligned}
$$

Since all the previous arguments hold for any $u \in \Omega_g$ we infer that

$$
\begin{aligned}
&\Big| \int_{\Omega_g} \operatorname{Tr}(\phi_u H_\Omega \phi_u)_- l(u)^{-d}\, du - \int_{\Omega_g} \operatorname{Tr}(\phi_u H_{L^*(u)} \phi_u)_- l(u)^{-d}\, du \Big| \\
&\qquad\qquad \leq \int_{\Omega_g} \big[\operatorname{Tr}(\phi_u H_{\mathcal{U}_\varepsilon(u)} \phi_u)_- - \operatorname{Tr}(\phi_u H_{\mathcal{I}_\varepsilon(u)} \phi_u)_-\big] l(u)^{-d}\, du\,.
\end{aligned}
\tag{39}
$$

A technical point here is that the choice of the point $p(u)$ and the half space $L^*(u)$ can be made so that it depends in a measurable way on u. The fact that this is possible can be seen by constructing the map $u \mapsto p(u)$ in the following manner. Take a countable dense

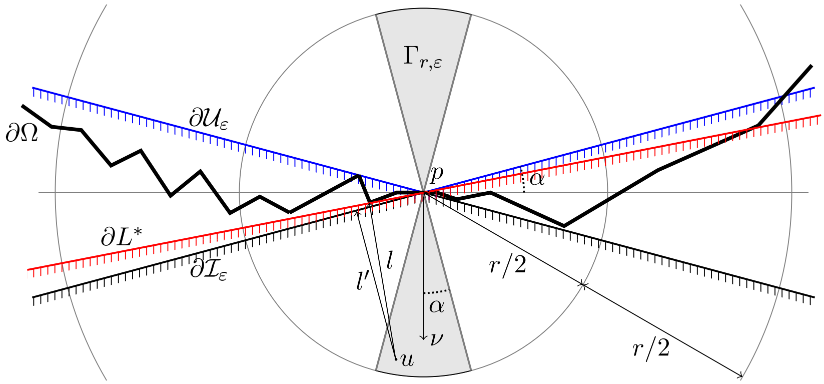

FIGURE 1. The different sets involved in the construction. Here $\alpha = \sin^{-1}(\varepsilon)$, $p = p(u)$, $\nu = \nu(p(u))$ and $l = \mathrm{dist}(u, \partial\Omega) = \mathrm{dist}(u, \partial L^*)$ and $l' = \mathrm{dist}(u, \partial\mathcal{I}_\varepsilon)$. The shaded two-sided truncated cone is the set $\Gamma_{r,\varepsilon}(p)$.

subset S in $G_{\varepsilon,r}$. The continuity of the map $p \mapsto \Gamma_{\varepsilon,r}(p)$ implies that $\mathcal{G}_{\varepsilon,r} = \cup_{p\in S}\Gamma_{\varepsilon,r}(p)$. Choose an ordering of S and define the $u \mapsto p(u)$ by mapping u to the point $p \in S$ which appears first in this ordering. The inverse image of any measurable subset of $\partial\Omega$ is then a countable union of intersections of the sets $\Gamma_{\varepsilon,r}$ which is measurable. The map $u \mapsto L^*(u)$ can be constructed in a similar manner.

We will argue that the second term on the left side of (39) contains the relevant terms in the asymptotics. In fact, by Lemma 2.3 in [10] (the case $\omega \equiv 0$ of Lemma 2.3 above but valid for all $h > 0$) it holds that

$$\mathrm{Tr}(\phi_u H_{L^*}\phi_u)_- = L_d h^{-d}\int_{L^*}\phi_u^2(x)\,dx - \frac{L_{d-1}}{4}h^{-d+1}\int_{\partial L^*}\phi_u^2(x)\,d\mathcal{H}^{d-1}(x) + l(u)^{d-2}O(h^{-d+2})\,.$$

Integrating these asymptotics we obtain

$$\begin{aligned}\int_{\Omega_g}\mathrm{Tr}(\phi_u H_{L^*(u)}\phi_u)_- l(u)^{-d}\,du &= L_d h^{-d}\int_{\Omega_g}\int_{L^*(u)}\phi_u^2(x)l(u)^{-d}\,dx\,du\\ &\quad - \frac{L_{d-1}}{4}h^{-d+1}\int_{\Omega_g}\int_{\partial L^*(u)}\phi_u^2(x)l(u)^{-d}\,d\mathcal{H}^{d-1}(x)\,du \\ &\quad + \int_{\Omega_g} l(u)^{-2}\,du\;O(h^{-d+2})\,.\end{aligned} \tag{40}$$

The first two terms on the right side are almost the terms that we are looking for, namely,

$$L_d h^{-d}\int_{\Omega_g}\int_{\Omega}\phi_u^2(x)l(u)^{-d}\,dx\,du - \frac{L_{d-1}}{4}h^{-d+1}\mathcal{H}^{d-1}(\partial\Omega)\,. \tag{41}$$

Note that in the first term on the right side of (40) we want to replace the domain $L^*(u)$ of the u-integration by Ω. Similarly, in the second term we essentially want to replace $\partial L^*(u)$

by $\partial\Omega$ (although eventually we will argue slightly differently). The last term on the right side of (40) is controlled by (34).

Thus, in the remainder of this subsection we need to do two things, namely first to control the error between the right side of (40) and (41), and second to bound the term on the right side of (39).

4.4.1. *The volume terms.* First we show that the difference between the first term on the right side of (40) and the first term in (41) is small. We bound

$$\begin{aligned}\int_{\Omega_g}\Big|\int_\Omega \phi_u^2(x)\,dx-\int_{L^*(u)}\phi_u^2(x)\,dx\Big|l(u)^{-d}\,du &\le \int_{\Omega_g}\int_{\Omega\Delta L^*(u)}\phi_u^2(x)l(u)^{-d}\,dx\,du\\ &\le \int_{\Omega_g}\int_{\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p)}\phi_u^2(x)l(u)^{-d}\,dx\,du \qquad (42)\\ &\le C\int_{\Omega_g}|(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap\operatorname{supp}\phi_u|l(u)^{-d}\,du\,.\end{aligned}$$

For $u\in\Omega_g$ we have $l(u)\ge \operatorname{dist}(u,\partial\Omega)$. By Lemma 3.3 we find $|u-p(u)|\le 2l(u)$ and hence

$$(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap B_{l(u)}(u)\subset(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap B_{3l(u)}(p(u))\,, \qquad (43)$$

which in turn implies that

$$|(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap B_{l(u)}(u)|\le|(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap B_{3l(u)}(p(u))|\le C\varepsilon l(u)^d\,.$$

Inserting this bound into (42) and recalling (33) yields

$$\begin{aligned}h^{-d}\int_{\Omega_g}|(\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p))\cap\operatorname{supp}\phi_u|l(u)^{-d}\,du &\le Ch^{-d}\varepsilon|\Omega_g|\\ &\le Ch^{-d}\varepsilon l_0\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\\ &= C\varepsilon\varepsilon_0^{-1}h^{-d+1}\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\,.\end{aligned}$$

Note that this term is negligible for the asymptotics if $\varepsilon\varepsilon_0^{-1}\ll 1$.

4.4.2. *The boundary terms.* Next, we consider the difference between the second term on the right side of (40) and the second term in (41). We shall show that

$$\Big|\int_{\Omega_g}\int_{\partial L^*(u)}\phi_u^2(x)l(u)^{-d}d\mathcal{H}^{d-1}(x)\,du-\mathcal{H}^{d-1}(\partial\Omega)\Big|\le C\mathcal{H}^{d-1}(\partial\Omega)\big(\mu(\varepsilon,r)+\overline{\vartheta}(l_0)+\varepsilon^2\big)\,. \qquad (44)$$

Note that the right side is negligible for the asymptotics if $\mu(\varepsilon,r)+\overline{\vartheta}(l_0)+\varepsilon^2\ll 1$. This is a weaker requirement than the one we met in (37).

Let $u\in\Omega_g$. We know from (32) that $l(u)=l_0$ and therefore $\phi_u(x)=\phi((x-u)/l_0)$.

We define

$$f(x_d)=\int_{\mathbb{R}^{d-1}}\phi(x',x_d)^2\,dx'\,.$$

Let $y \in \partial L^*(u)$ such that $|u-y| = \mathrm{dist}(u, \partial L^*(u)) = \mathrm{dist}(u, \partial\Omega)$. Then $\partial L^*(u) = \{x \in \mathbb{R}^d : (x-y)\cdot(u-y) = 0\}$ and

$$\int_{\partial L^*(u)} \phi_u(x)^2 \, d\mathcal{H}^{d-1}(x) = \int_{\partial L^*(u)} \phi\Big(\frac{x-y}{l_0} - \frac{u-y}{l_0}\Big)^2 \, d\mathcal{H}^{d-1}(x)$$
$$= l_0^{d-1} f(|u-y|/l_0)\,.$$

The last equality follows by scaling and from the fact that ϕ is radial. Since f is even, we can write

$$f(|u-y|/l_0) = f(\delta_\Omega(u)/l_0)\,.$$

This proves that

$$\int_{\Omega_g}\int_{\partial L^*(u)} \phi_u^2(x) l(u)^{-d} \, d\mathcal{H}^{d-1}(x)\, du = l_0^{-1} \int_{\Omega_g} f(\delta_\Omega(u)/l_0)\, du\,.$$

Next, we show that, up to a controllable error, the set Ω_g on the right side can be replaced by $\mathbb{R}^d$. Indeed, we have

$$\begin{aligned} 0 &\le l_0^{-1}\int_{\Omega_b} f(\delta_\Omega(u)/l_0)\, du \le l_0^{-1}\|f\|_{L^\infty}|\Omega_b| \\ &\le C\mathcal{H}^{d-1}(\partial\Omega)\big(\mu(\varepsilon, r) + \overline{\vartheta}(l_0) + \varepsilon^2\big)\,, \end{aligned} \tag{45}$$

where we used the same bound as in (37). Moreover, since ϕ has support in $\overline{B_1(0)}$, f has support in $[-1,1]$ and therefore (32) implies that $f(\delta_\Omega(u)/l_0) = 0$ for $u \notin \Omega_g \cup \Omega_b$.

Thus, we are left with analysing

$$l_0^{-1}\int_{\mathbb{R}^d} f(\delta_\Omega(u)/l_0)\, du = l_0^{-1}\int_{\mathbb{R}} f(t/l_0)\mathcal{H}^{d-1}(\{u \in \mathbb{R}^d : \delta_\Omega(u) = t\})\, dt\,.$$

The identity here comes again from the co-area formula together with (7).

The idea in the following is that $l_0^{-1} f(t/l_0)$ is an approximate delta function at $t = 0$. Note that

$$\int_{\mathbb{R}} f(x_d)\, dx_d = \|\phi\|_{L^2}^2 = 1\,.$$

The following argument is a quantitative, 'two-sided' version of a special case of [8, Proposition 1.1]. To justify the replacement of $l_0^{-1} f(t/l_0)$ by a delta function write

$$\begin{aligned} &l_0^{-1}\int_0^\infty f(t/l_0)\mathcal{H}^{d-1}(\{u \in \mathbb{R}^d : \delta_\Omega(u) = t\})\, dt - (1/2)\mathcal{H}^{d-1}(\partial\Omega) \\ &= l_0^{-1}\int_0^\infty f(t/l_0)\frac{d}{dt}\Big(|\{u \in \Omega : \delta_\Omega(u) \le t\}| - \mathcal{H}^{d-1}(\partial\Omega)t\Big) dt \\ &= l_0^{-2}\int_0^\infty f'(t/l_0)\Big(|\{u \in \Omega : \delta_\Omega(u) \le t\}| - \mathcal{H}^{d-1}(\partial\Omega)t\Big) dt \\ &= l_0^{-2}\mathcal{H}^{d-1}(\partial\Omega)\int_0^\infty f'(t/l_0)\, t\, \vartheta_{inner}(t)\, dt\,. \end{aligned}$$

This, together with a similar formula for $t<0$ and the fact that f is supported in $[-1,1]$, implies that

$$\begin{aligned}
&\left| l_0^{-1} \int_{\mathbb{R}} f(t/l_0)\mathcal{H}^{d-1}(\{u \in \mathbb{R}^d : \ \delta_\Omega(u) = t\})\, dt - \mathcal{H}^{d-1}(\partial\Omega) \right| \\
&\quad \leq 2 l_0^{-2} \mathcal{H}^{d-1}(\partial\Omega) \overline{\vartheta}(l_0) \int_0^\infty |f'(t/l_0)|\, t\, dt \\
&\quad = 2\mathcal{H}^{d-1}(\partial\Omega)\overline{\vartheta}(l_0) \int_0^\infty |f'(x_d)|\, x_d\, dx_d\, .
\end{aligned}$$

This completes the proof of (44).

4.4.3. *Estimating the error from* (39). To complete the proof, it remains to control the error made in our local approximation of $B_{l(u)}(u)\cap\Omega$ by $B_{l(u)}(u)\cap L^*(u)$, that is, the right side of (39). We shall show that

$$\begin{aligned}
&\int_{\Omega_g} \big[\mathrm{Tr}(\phi_u H_{\mathcal{U}_\varepsilon(u)}\phi_u)_- - \mathrm{Tr}(\phi_u H_{\mathcal{I}_\varepsilon(u)}\phi_u)_-\big] l(u)^{-d}\, du \\
&\qquad \leq C\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\big(\varepsilon\varepsilon_0^{-1} + \varepsilon_0^{1/3}\big)h^{-d+1}\, .
\end{aligned}$$

Note that in order to show that this term does not interfere with the asymptotics we need to make $\varepsilon\varepsilon_0^{-1}+\varepsilon_0^{1/3}$ small.

Plugging in the asymptotics of Lemma 2.10 we find that

$$\begin{aligned}
&\int_{\Omega_g} \big[\mathrm{Tr}(\phi_u H_{\mathcal{U}_\varepsilon(u)}\phi_u)_- - \mathrm{Tr}(\phi_u H_{\mathcal{I}_\varepsilon(u)}\phi_u)_-\big] l(u)^{-d}\, du \\
&\qquad \leq L_d h^{-d} \int_{\Omega_g}\int_{\mathcal{U}_\varepsilon(p)\setminus\mathcal{I}_\varepsilon(p)} \phi_u^2(x) l(u)^{-d}\, dx\, du \\
&\qquad\quad - \frac{L_{d-1}}{4} h^{-d+1} \int_{\Omega_g}\bigg(\int_{\partial\mathcal{U}_\varepsilon(p)} \phi_u^2(x)\, d\mathcal{H}^{d-1}(x) - \int_{\partial\mathcal{I}_\varepsilon(p)} \phi_u^2(x)\, d\mathcal{H}^{d-1}(x)\bigg) l(u)^{-d}\, du \\
&\qquad\quad + Ch^{-d+4/3}\int_{\Omega_g} l(u)^{-4/3}\, du\, .
\end{aligned}$$

The first term can be handled as in (42) and is thus $\leq C\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\varepsilon\varepsilon_0^{-1}h^{-d+1}$. The third term is $\leq C\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\varepsilon_0^{1/3}h^{-d+1}$ by (34) and the choice of l_0.

In order to bound the second term, let H denote the hyperplane through $p(u)$ orthogonal to $\nu(p(u))$. Then the map $s\colon \mathbb{R}^{d-1}\to\mathbb{R}$, $x'\mapsto \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}|x'|$, parametrizes $\partial\mathcal{U}_\varepsilon$ and $\partial\mathcal{I}_\varepsilon$ as graphs over H. In coordinates chosen so that $p(u)=0$ and $H=\{(x',0): \ x'\in\mathbb{R}^{d-1}\}$, we find that

$$\begin{aligned}
\Big|\int_{\partial\mathcal{U}_\varepsilon} \phi_u^2(x)\,d\mathcal{H}^{d-1}(x) - \int_{\partial\mathcal{I}_\varepsilon} \phi_u^2(x)\,d\mathcal{H}^{d-1}(x)\Big| \\
&\le \int_{\mathbb{R}^{d-1}} |\phi_u^2(x', s(x')) - \phi_u^2(x', -s(x'))|\sqrt{1+|\nabla s|^2}dx' \\
&\le \frac{4\varepsilon}{1-\varepsilon^2}\|\phi_u\|_{L^\infty}\|\nabla\phi_u\|_{L^\infty}\int_{B_{3l(u)}} |x'|\,dx' \\
&\le \frac{C\varepsilon}{\sqrt{1-\varepsilon^2}} l(u)^{d-1}\,,
\end{aligned}$$

where we used $|x'| \le 3l(u)$ in $\operatorname{supp}\phi_u$, see (43). Combined with (33) we find that the error coming from the second term of (34) is $\le C\mathcal{H}^{d-1}(\partial\Omega)(1+\overline{\vartheta}(l_0))\varepsilon h^{-d+1}$.

4.5. **Gathering the error terms.** The proof of Theorem 4.1 can now be completed by combining the contributions from $\Omega_*, \Omega_b, \Omega_g$ and estimating the localization error from Lemma 2.8. Note that (10) implies that

$$\int_{\Omega_*}\int_\Omega \phi_u^2(x)l(u)^{-d}\,dx\,du + \int_{\Omega_g}\int_\Omega \phi_u^2(x)l(u)^{-d}\,dx\,du + \int_{\Omega_b}\int_\Omega \phi_u^2(x)l(u)^{-d}\,dx\,du = |\Omega|\,.$$

For all $0 < h \le 2r_{in}(\Omega)$, $r > 0$, $\varepsilon \in (0, 1/2]$ and $\varepsilon_0 \in (0, 4]$ satisfying

$$h/\varepsilon_0 = l_0 \le \min\{r/2, s_0, r_{in}(\Omega)/2\}$$

(with $s_0 = s_0(\varepsilon, r, \Omega)$ given by Lemma 3.5) we can conclude that

$$\begin{aligned}
h^{-d+1}&\Big|\mathrm{Tr}(H_\Omega)_- - L_d|\Omega|h^{-d} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)h^{-d+1}\Big| \\
&\le C\mathcal{H}^{d-1}(\partial\Omega)\Big[\varepsilon_0\big[1+\overline{\vartheta}(r_{in}(\Omega))\big] + \frac{\mu(\varepsilon,r)+\overline{\vartheta}(l_0)}{\varepsilon_0} + \big(\varepsilon_0^{-1}\varepsilon + \varepsilon_0^{1/3}\big)\big[1+\overline{\vartheta}(l_0)\big]\Big]\,,
\end{aligned} \tag{46}$$

where the constant C depends only on the dimension. (Here we have simplified some terms using the fact that $\varepsilon \le 1/2$ and $\varepsilon_0 \le 4$.) This proves (28) and therefore concludes the proof of Theorem 4.1. □

5. Uniform asymptotics for convex sets

Our goal in this section is to prove the following

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a convex bounded open set. Then, for all $h > 0$,*

$$h^{d-1}\Big|\mathrm{Tr}(H_\Omega)_- - L_d|\Omega|h^{-d} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)h^{-d+1}\Big| \le C\mathcal{H}^{d-1}(\partial\Omega)\Big(\tfrac{h}{r_{in}(\Omega)}\Big)^{1/11}\,,$$

where the constant C depends only on the dimension.

Clearly, this is equivalent to Theorem 1.2. To prove Theorem 5.1 we follow the same strategy as in the proof of Theorem 4.1. The geometry of Ω enters into the final inequality (46) in that proof via the three quantities $\overline{\vartheta}(l_0)$, $\mu(\varepsilon, r)$ and $s_0(\varepsilon, r, \Omega)$ (the latter as a constraint on the size of h).

Our first goal in this section is to show that $\overline{\vartheta}(\Omega,t)$ can be bounded for convex Ω through $t/r_{in}(\Omega)$ only. This makes the geometric dependence of the term $\overline{\vartheta}(l_0)$ in (46) explicit.

It is not so easy to bound $\mu(\varepsilon,r)$ and $s_0(\varepsilon,r,\Omega)$ explicitly, even for convex sets. Our second goal in this section is therefore to prove a replacement of Lemma 3.5 for convex sets where the geometry enters only through $r_{in}(\Omega)$ and $\mathcal{H}^{d-1}(\partial\Omega)$.

Having achieved these two goals, a straightforward modification of the proof of Theorem 4.1 will prove Theorem 5.1.

Throughout this section we assume that $\Omega\subset\mathbb{R}^d$ is a convex open set. The arguments that follow are based on ideas related to the notion of inner parallel sets. The inner parallel set of Ω at distance t is defined to be

$$\Omega_t=\{u\in\Omega:\operatorname{dist}(u,\Omega^c)>t\}\,. \tag{47}$$

By [22, Theorem 1.2] and monotonicity of the measure of the perimeter of convex bodies under inclusions we know that

$$\mathcal{H}^{d-1}(\partial\Omega)\Big(1-\frac{t}{r_{in}(\Omega)}\Big)_+^{d-1}\le\mathcal{H}^{d-1}(\partial\Omega_t)\le\mathcal{H}^{d-1}(\partial\Omega)\qquad\text{for all } t\ge 0\,. \tag{48}$$

Our first application of (48) will be to show that, as claimed above, one has two-sided bounds for $r_{in}(\Omega)$ in terms of $|\Omega|$ and $\mathcal{H}^{d-1}(\partial\Omega)$. Indeed, by the co-area formula and (7) one has

$$|\Omega|=\int_0^{r_{in}(\Omega)}\mathcal{H}^{d-1}(\partial\Omega_s)\,ds\,.$$

Applying (48) and integrating we find that

$$\frac{|\Omega|}{\mathcal{H}^{d-1}(\partial\Omega)}\le r_{in}(\Omega)\le\frac{d|\Omega|}{\mathcal{H}^{d-1}(\partial\Omega)}\,. \tag{49}$$

Remark 5.2. It might be worth noting that both bounds in (49) cannot be improved. In the upper bound equality is achieved if Ω is a ball and, more generally, if and only if Ω is a form body (see [22, 29]). In the lower bound equality is asymptotically achieved by $(0,L)^{d-1}\times(0,1)$ in the limit $L\to\infty$.

The following lemma achieves the first goal mentioned at the beginning of this section.

Lemma 5.3. *Let $\Omega\subset\mathbb{R}^d$ be a convex open set. Then for all $0\le t\le r_{in}(\Omega)$,*

$$|\vartheta_{inner}(\Omega,t)|\le C\frac{t}{r_{in}(\Omega)}\,,\quad|\vartheta_{outer}(\Omega,t)|\le C\frac{t}{r_{in}(\Omega)}\,,\quad\overline{\vartheta}(\Omega,t)\le C\frac{t}{r_{in}(\Omega)}\,, \tag{50}$$

where the constants depend only on the dimension.

Proof of Lemma 5.3. We first bound the measure of $\{u\in\Omega:\operatorname{dist}(u,\Omega^c)<t\}$ from both above and below. Using the co-area formula and (7) in the same manner as above we have that, for $0\le t\le r_{in}(\Omega)$,

$$|\{u\in\Omega:\operatorname{dist}(u,\Omega^c)<t\}|=\int_0^t\mathcal{H}^{d-1}(\partial\Omega_s)\,ds\,.$$

By the upper bound in (48) it follows that, for $t \geq 0$,

$$|\{u \in \Omega : \operatorname{dist}(u, \Omega^c) < t\}| \leq t\mathcal{H}^{d-1}(\partial\Omega)\,.$$

Correspondingly, the lower bound in (48) implies that, for $0 \leq t \leq r_{in}(\Omega)$,

$$\begin{aligned}
|\{u \in \Omega : \operatorname{dist}(u, \Omega^c) < t\}| &= \int_0^t \mathcal{H}^{d-1}(\partial\Omega_s)\,ds \\
&\geq \mathcal{H}^{d-1}(\partial\Omega) \int_0^t \Big(1 - \frac{s}{r_{in}(\Omega)}\Big)^{d-1} ds \\
&= \frac{\mathcal{H}^{d-1}(\partial\Omega) r_{in}(\Omega)}{d}\Big(1 - \Big(1 - \frac{t}{r_{in}(\Omega)}\Big)^d\Big) \\
&\geq t\mathcal{H}^{d-1}(\partial\Omega)\Big(1 - \frac{d-1}{2r_{in}(\Omega)}t\Big)\,.
\end{aligned}$$

Consequently we find that

$$-\frac{d-1}{2r_{in}(\Omega)}\,t \leq \vartheta_{inner}(t) \leq 0\,.$$

To obtain the corresponding bounds for the measure of $\{x \in \Omega^c : \operatorname{dist}(x, \Omega) < t\}$ we first note that $\{u \in \mathbb{R}^d : \operatorname{dist}(u, \Omega) < t\}$ is convex and its inner parallel set at distance t is Ω. By applying (48) to this set and using $r_{in}(\{u \in \mathbb{R}^d : \operatorname{dist}(u, \Omega) < t\}) = r_{in}(\Omega) + t$ we find that

$$\begin{aligned}
\mathcal{H}^{d-1}(\{u \in \mathbb{R}^d : &\operatorname{dist}(u, \Omega) = t\})\Big(\frac{r_{in}(\Omega)}{r_{in}(\Omega) + t}\Big)^{d-1} \\
&\leq \mathcal{H}^{d-1}(\partial\Omega) \leq \mathcal{H}^{d-1}(\{u \in \mathbb{R}^d : \operatorname{dist}(u, \Omega) = t\})\,.
\end{aligned}$$

Rearranging and arguing as before one finds

$$t\mathcal{H}^{d-1}(\partial\Omega) \leq |\{u \in \mathbb{R}^d : \operatorname{dist}(u, \Omega) < t\}| \leq t\mathcal{H}^{d-1}(\partial\Omega)\Big(1 + \frac{2^d - d - 1}{d\,r_{in}(\Omega)}t\Big)\,,$$

and hence

$$0 \leq \vartheta_{outer}(t) \leq \frac{2^d - d - 1}{d\,r_{in}(\Omega)}t\,.$$

By combining the bounds for ϑ_{inner} and ϑ_{outer} one obtains the third inequality in (50). This completes the proof of the lemma. □

The following lemma achieves the second goal mentioned at the beginning of this section. Note that this is similar to (25) but without involving $\mu(\varepsilon, r)$ or $\overline{\vartheta}$ and with an explicit value for s_0.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ be a convex open set. Then, for all $\varepsilon \in (0, 1]$, $r \in (0, \varepsilon r_{in}(\Omega))$ and $s \in (0, r/2]$,*

$$|\{u \in \mathbb{R}^d : \operatorname{dist}(u, \partial\Omega) < s\} \setminus \mathcal{G}_{\varepsilon,r}| \leq C\mathcal{H}^{d-1}(\partial\Omega)\frac{sr}{\varepsilon r_{in}(\Omega)}\,,$$

where C depends only on the dimension.

Proof of Lemma 5.4. We divide the proof into three steps.

Step 1: We define a set $G \subseteq \partial\Omega$.

We recall that Ω_t is defined in (47). We denote by $\mathrm{reg}(\partial\Omega_t)$ the set of points $x \in \partial\Omega_t$ for which the inner unit normal $\nu_t(x)$ exists. We consider the natural normal-map defined for $t \in [0, r_{in}(\Omega))$ by

$$f_t\colon \mathrm{reg}(\partial\Omega_t) \times \mathbb{R}_+ \to \mathbb{R}^d\,,\ (x, s) \mapsto x - s\nu_t(x)\,.$$

We observe that $f_t(\mathrm{reg}(\partial\Omega_t), s) \subseteq \mathrm{reg}(\partial\Omega_{t-s})$ for $0 < s \leq t$ and, in particular, that $f_t(\mathrm{reg}(\partial\Omega_t), t) \subseteq \mathrm{reg}(\partial\Omega)$. We also note that for all $s \in [0, t]$ the inwards pointing normal to $\partial\Omega_{t-s}$ at $f_t(x, t-s)$ is equal to the normal at x, $\nu_t(x)$. It follows that the image of the map $f_t(x, \cdot\,)\colon [0,\infty) \to \mathbb{R}^d$ is a ray starting at x and passing orthogonally through $\partial\Omega$ at the point $f_t(x,t)$. If $f_t(x,t)$ is (ε, r)-good this ray forms the axis of symmetry for the cone $\Gamma_{\varepsilon,r}(f_t(x,t))$. After these preparations, we now set

$$G = f_{r/\varepsilon}(\mathrm{reg}(\partial\Omega_{r/\varepsilon}), r/\varepsilon)\,.$$

Step 2: We show that for $\varepsilon \in (0,1)$ and $r \in (0, \varepsilon r_{in}(\Omega))$ every $p \in G$ is (ε, r)-good.

Note that we only need to check the (ε, r)-condition in the inwards direction, since for any $y \in \mathrm{reg}(\partial\Omega)$ the boundary $\partial\Omega$ is contained in the half-space $\{u \in \mathbb{R}^d : (u-y)\cdot\nu(y) \geq 0\}$.

The main idea behind the construction of G is based on the observation that if a point $y \in \mathrm{reg}(\partial\Omega)$ fails to be (ε, r)-good then it cannot be in the image of f_t for suitably chosen t, see Figure 2.

Assume that $y \in \mathrm{reg}(\partial\Omega)$ fails to be (ε, r)-good. If there is a point of $\mathrm{reg}(\partial\Omega_t)$ which is mapped to $y \in \mathrm{reg}(\partial\Omega)$ under the normal map f_t it must be the point $y + t\nu(y)$. However, since y is not (ε, r)-good there is a point $y' \in \Omega^c$ such that $|y'-y| = r$ and $(y'-y)\cdot\nu(y) = \varepsilon r$. By elementary trigonometry we find that if $t > \frac{r}{2\varepsilon}$ then $|y + t\nu(y) - y'| < t$, and therefore $y + t\nu(t)$ does not belong to $\partial\Omega_t$ implying that $y \notin f_t(\mathrm{reg}(\partial\Omega_t), t)$. This proves that any $p \in G = f_{r/\varepsilon}(\mathrm{reg}(\partial\Omega_{r/\varepsilon}), r/\varepsilon)$ is an (ε, r)-good point of $\partial\Omega$.

Step 3: We now prove the inequality in the lemma.

We observe that for any fixed $t > 0$ and all $s \geq 0$ the map $f_t(\,\cdot\,, s)$ is injective, and by convexity $\mathcal{H}^{d-1}(f_t(\mathrm{reg}(\partial\Omega_t), s))$ is an increasing functions of s. Note also that $\mathcal{H}^{d-1}(\mathrm{reg}(\partial\Omega_t)) = \mathcal{H}^{d-1}(\partial\Omega_t)$ since $\mathcal{H}^{d-1}$-a.e. point of the boundary of a d-dimensional convex set is regular (see [29]).

Lemma 5.3 implies that

$$\begin{aligned}|\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \setminus \mathcal{G}_{\varepsilon,r}| \leq 2s\mathcal{H}^{d-1}(\partial\Omega)(1 + Cs/r_{in}(\Omega))\\ - |\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \cap \mathcal{G}_{\varepsilon,r}|\,.\end{aligned}$$

Therefore using $s \leq r/2 \leq r/(2\varepsilon)$ we see that the claimed inequality will follow from

$$|\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \cap \mathcal{G}_{\varepsilon,r}| \geq 2s\mathcal{H}^{d-1}(\partial\Omega)\Big(1 - \frac{Cr}{\varepsilon r_{in}(\Omega)}\Big)\,, \quad \forall s \leq r/2\,.$$

Since every $p \in G$ is (ε, r)-good

$$f_{r/\varepsilon}(\mathrm{reg}(\partial\Omega_{r/\varepsilon}), r/\varepsilon + s') \subset \mathcal{G}_{\varepsilon,r}\,, \quad \forall s' \in (-r/2, r/2)\,.$$

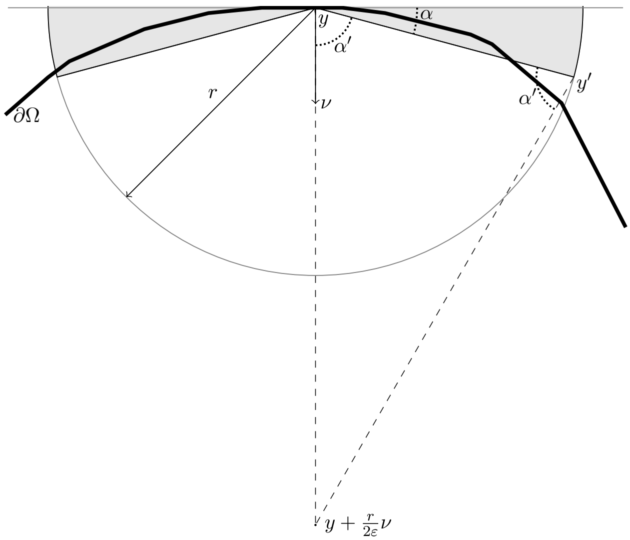

FIGURE 2. A 2-dimensional cross-section of a neighborhood of y illustrating the idea behind the construction of G. Here $\alpha = \sin^{-1}(\varepsilon)$ and $\alpha' = \pi/2 - \alpha$.

Therefore, using again the co-area formula, (7), (48) and the fact that $\mathcal{H}^{d-1}(f_t(\partial\Omega_t, s))$ is increasing in s,

$$\begin{aligned}
|\{u \in \mathbb{R}^d : \mathrm{dist}(u, \partial\Omega) < s\} \cap \mathcal{G}_{\varepsilon,r}| &\geq \int_{-s}^{s} \mathcal{H}^{d-1}(f_{r/\varepsilon}(\mathrm{reg}(\partial\Omega_{r/\varepsilon}), r/\varepsilon + s')\, ds' \\
&\geq 2s\mathcal{H}^{d-1}(f_{r/\varepsilon}(\mathrm{reg}(\partial\Omega_{r/\varepsilon}), r/\varepsilon - s) \\
&\geq 2s\mathcal{H}^{d-1}(\partial\Omega_{r/\varepsilon}) \\
&\geq 2s\mathcal{H}^{d-1}(\partial\Omega)\Big(1 - \frac{r}{\varepsilon r_{in}(\Omega)}\Big)^{d-1} \\
&\geq 2s\mathcal{H}^{d-1}(\partial\Omega)\Big(1 - \frac{(d-1)r}{\varepsilon r_{in}(\Omega)}\Big)\,.
\end{aligned}$$

This completes the proof of Lemma 5.4. □

Remark 5.5. The points in the set G in the previous proof are a lot better than (ε, r)-good. The proof shows essentially that for any $p \in G$ the principal curvatures of $\partial\Omega$ are bounded from above by $\sim \varepsilon r^{-1}$. That this set is large for r small enough follows from Aleksandrov's theorem on a.e. twice differentiability of convex functions.

As explained at the beginning of this subsection, proving Theorem 5.1 is now simply a matter of bounding all the relevant error terms in the derivation of the asymptotic expansion.

Proof of Theorem 5.1. We repeat the proof of Theorem 4.1 but in (37) and (45), where we used Lemma 3.5, we simply keep the term $|\Omega_b|$. In this way we find

$$\begin{aligned} &h^{-d+1}\Big|\mathrm{Tr}(H_\Omega)_- - L_d|\Omega|h^{-d} + \frac{L_{d-1}}{4}\mathcal{H}^{d-1}(\partial\Omega)h^{-d+1}\Big| \\ &\leq C\mathcal{H}^{d-1}(\partial\Omega)\Big[\varepsilon_0\big[1+\overline{\vartheta}(r_{in}(\Omega))\big] + \frac{|\Omega_b|}{h\mathcal{H}^{d-1}(\partial\Omega)} + \overline{\vartheta}(l_0) + \big(\varepsilon_0^{-1}\varepsilon + \varepsilon_0^{1/3}\big)\big[1+\overline{\vartheta}(l_0)\big]\Big]\,, \end{aligned}$$

where we again require $0 < h < 2r_{in}(\Omega)$, $r > 0$, $\varepsilon \in (0, 1/2]$ and $\varepsilon_0 \in (0, 4]$ to be chosen so that

$$h/\varepsilon_0 \leq \min\big\{r/2, r_{in}(\Omega)/2\big\}\,.$$

We now use the convexity of Ω to bound the terms which still depend on the geometry. By (49) we have

$$\overline{\vartheta}(r_{in}(\Omega)) \leq C \qquad \text{and} \qquad \overline{\vartheta}(l_0) \leq C\frac{l_0}{r_{in}(\Omega)}\,.$$

Furthermore, if $r \leq \varepsilon r_{in}(\Omega)$ and $l_0 \leq r/2$, then Lemma 5.4 implies that

$$|\Omega_b| \leq C\mathcal{H}^{d-1}(\partial\Omega)\frac{l_0 r}{\varepsilon r_{in}(\Omega)} = C\mathcal{H}^{d-1}(\partial\Omega)\frac{hr}{\varepsilon\varepsilon_0 r_{in}(\Omega)}\,.$$

Therefore, the error term above is bounded by

$$C\mathcal{H}^{d-1}(\partial\Omega)\Big[\frac{r}{\varepsilon\varepsilon_0 r_{in}(\Omega)} + \varepsilon_0^{-1}\varepsilon + \varepsilon_0^{1/3}\Big].$$

(Here we have dropped a term $h/(\varepsilon_0 r_{in}(\Omega))$ coming from the bound on $\overline{\vartheta}(l_0)$, since $h \leq \varepsilon_0 r \leq \varepsilon_0\varepsilon r_{in}(\Omega)$, so this term is $\leq \varepsilon$ and therefore also $\leq 4\,\varepsilon_0^{-1}\varepsilon$.) The above bound is valid provided the parameters satisfy

$$h \leq \varepsilon_0\, r/2 \qquad \text{and} \qquad r \leq \varepsilon\, r_{in}(\Omega)\,.$$

It remains to choose the parameters. We first assume that $s = h/r_{in}(\Omega) \leq 1$. Optimizing successively over r, ε and ε_0 in that order and adjusting the constants we arrive at the choices

$$r = (1/2)r_{in}(\Omega)\, s^{8/11}\,, \qquad \varepsilon = (1/2)\, s^{4/11}\,, \qquad \varepsilon_0 = 4\, s^{3/11}\,.$$

Clearly all constraints are satisfied and the final error is

$$C\mathcal{H}^{d-1}(\partial\Omega)\, s^{1/11} = C\mathcal{H}^{d-1}(\partial\Omega)(h/r_{in}(\Omega))^{1/11}\,.$$

This is the claimed bound for $h \leq r_{in}(\Omega)$.

Finally, for any convex $\Omega \subset \mathbb{R}^d$ the first eigenvalue of $-\Delta_\Omega$ satisfies $\lambda_1(\Omega) \geq \frac{\pi^2}{4r_{in}(\Omega)^2}$ [17, 27]. Hence $\mathrm{Tr}(H_\Omega)_- = 0$ for all $h \geq (2/\pi)r_{in}(\Omega)$ and, in particular, for $h \geq r_{in}(\Omega)$. Combining this observation with the fact that $\frac{|\Omega|}{r_{in}(\Omega)} \leq \mathcal{H}^{d-1}(\partial\Omega)$ (see (49)) the claimed bound holds also for any $h \geq r_{in}(\Omega)$, which completes the proof. $\square$

Appendix A. Proof of Lemma 2.8

What remains to conclude our analysis is to prove Lemma 2.8. As mentioned earlier the proof follows the same strategy as the proof of Proposition 1.1 in [10].

Proof of Lemma 2.8. Set

$$\gamma = \int_{\mathbb{R}^d} \phi_u (\phi_u \varphi H_\Omega \varphi \phi_u)^0_- \phi_u l(u)^{-d}\, du\,.$$

Clearly $\gamma \geq 0$ and by (10) $\gamma \leq 1$. Since the range of γ is a subset of $H^1_0(\Omega)$ the variational principle tells us that

$$\operatorname{Tr}(\varphi H_\Omega \varphi)_- \geq -\operatorname{Tr}(\gamma \varphi H_\Omega \varphi) = \int_{\mathbb{R}^d} \operatorname{Tr}(\phi_u \varphi H_\Omega \varphi \phi_u)_- l(u)^{-d}\, du\,.$$

This completes the proof of one side of the inequality.

To complete the proof we use the following version of the IMS-localization formula, for $f \in H^1_0(\Omega)$,

$$\frac{1}{2}(f, \phi_u^2 \varphi(-\Delta)\varphi f) + \frac{1}{2}(f, \varphi(-\Delta)(\phi_u^2 \varphi f)) = (f, \phi_u \varphi(-\Delta)\varphi\phi_u f) - (\varphi f, \varphi f (\nabla \phi_u)^2)\,.$$

By (10) this yields that

$$(f, \varphi(-\Delta)\varphi f) = \int_{\mathbb{R}^d} \big((f, \phi_u\varphi(-\Delta)\varphi\phi_u f) - (\varphi f, \varphi f(\nabla\phi_u)^2)\big) l(u)^{-d}\, du\,. \tag{51}$$

Using the properties of l and ϕ_u in Lemma 2.5 one can show, see the proof of [32, eq. (68)], that

$$\int_{\mathbb{R}^d} (\nabla \phi_u)^2(x) l(u)^{-d}\, du \leq C \int_{\mathbb{R}^d} \phi_u^2(x) l(u)^{-d-2}\, du\,.$$

When combined with (51) we find that

$$\operatorname{Tr}(\varphi H_\Omega \varphi)_- \leq \int_{\operatorname{dist}(u, \Omega \cap \operatorname{supp}\varphi) \leq l(u)} \operatorname{Tr}(\phi_u \varphi (H_\Omega - Ch^2 l(u)^{-2})\varphi\phi_u)_- l(u)^{-d}\, du\,. \tag{52}$$

Let $0 < \rho_u \leq 1$ be an additional parameter to be chosen later. By the variational principle

$$\begin{aligned}
&\operatorname{Tr}(\phi_u\varphi(H_\Omega - Ch^2 l(u)^{-2})\varphi\phi_u)_- \\
&\quad \leq \operatorname{Tr}(\phi_u \varphi H_\Omega \varphi \phi_u)_- + \operatorname{Tr}(\phi_u\varphi(-\rho_u h^2 \Delta_\Omega - \rho_u - Ch^2 l(u)^{-2})\varphi\phi_u)_- \\
&\quad \leq \operatorname{Tr}(\phi_u \varphi H_\Omega \varphi \phi_u)_- + L_d(\rho_u + Ch^2 l(u)^{-2})^{1+d/2} \rho_u^{-d/2} h^{-d} \int_\Omega \phi_u^2(x)\varphi(x)^2\, dx\,,
\end{aligned}$$

where we in the last step used Lemma 2.1.

Setting $\rho_u = h^2 l(u)^{-2}/M^2$, which by assumption is bounded by 1, we conclude that

$$\begin{aligned}
&\operatorname{Tr}(\phi_u\varphi(H_\Omega - Ch^2 l(u)^{-2})\varphi\phi_u)_- \\
&\quad \leq \operatorname{Tr}(\phi_u \varphi H_\Omega \varphi\phi_u)_- + L_d M^{-2}(1 + CM^2)^{1+d/2} h^{-d+2} l(u)^{-2} \int_\Omega \phi_u^2(x)\varphi(x)^2\, dx\,.
\end{aligned} \tag{53}$$

Since $\|\phi_u\|_{L^\infty} \le C$ and $|\operatorname{supp}\phi_u| \le Cl(u)^d$ it holds that

$$\begin{aligned}\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)} &\int_\Omega \phi_u(x)^2\varphi(x)^2 l(u)^{-d-2}\,dx\,du \\ &\le \|\varphi\|^2_{L^\infty(\Omega)}\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)}\int_\Omega \phi_u(x)^2 l(u)^{-d-2}\,dx\,du \\ &\le C\|\varphi\|^2_{L^\infty(\Omega)}\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)} l(u)^{-2}\,du\,.\end{aligned} \tag{54}$$

Combining (52), (53) and (54) completes the proof of the lemma. □

We now move on to proving that the inequality of Proposition 1.1 in [10] can be extended to all $h>0$. We also show that the same construction allows us to prove the analogous statement for the length scale used in the proof of Theorem 1.1.

We begin with a function l as in Lemma 2.5 and any constant $S>0$. Assuming that $h \ge S\max_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)} l(u)$ then by Lemma 2.1 and (10)

$$\begin{aligned}\Big|\operatorname{Tr}(\varphi H_\Omega\varphi)_- &- \int_{\mathbb{R}^d}\operatorname{Tr}(\phi_u\varphi H_\Omega\varphi\phi_u)_- l(u)^{-d}\,du\Big| \\ &\le h^{-d}L_d\int_\Omega \varphi^2(x)\,dx + h^{-d}L_d\int_{\mathbb{R}^d}\int_\Omega \varphi^2(x)\phi_u^2(x)l(u)^{-d}\,dx\,du \\ &= h^{-d}2L_d\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)}\int_\Omega \varphi^2(x)\phi_u^2(x)l(u)^{-d}\,dx\,du \\ &\le h^{-d}C\|\varphi\|^2_{L^\infty(\Omega)}\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)} du \\ &\le h^{-d+2}C\|\varphi\|^2_{L^\infty(\Omega)}S^{-2}\int_{\operatorname{dist}(u,\Omega\cap\operatorname{supp}\varphi)\le l(u)} l(u)^{-2}\,du\,.\end{aligned} \tag{55}$$

Here we used that $\int_\Omega \varphi^2(x)\phi_u(x)^2\,dx \le \|\varphi\|^2_{L^\infty}Cl(u)^d$ to obtain an estimate which matches that of Lemma 2.8.

Assume now that we are given a length scale l depending on a parameter l_0, which itself depends on h in such a way that there are constants $\delta,\mu>0$ such that for $h\le\delta$ one has $l_0\ge\mu h$.

We first consider the length scale used in [10]:

$$l(u) = \frac{1}{2}\big(1+(\operatorname{dist}(u,\Omega^c)+l_0^2)^{-1/2}\big)^{-1}, \quad \text{with } 0<l_0\le 1\,.$$

We have that

$$\min_{\operatorname{dist}(u,\Omega)\le l(u)} l(u) = \frac{l_0}{2+2l_0}\,,$$
$$\max_{\operatorname{dist}(u,\Omega)\le l(u)} l(u) \le 1/2\,.$$

If $h \leq \delta$ and we set $M = \frac{2+\mu\delta}{\mu}$ then

$$M \min_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) = \frac{2+2\mu\delta}{\mu}\frac{l_0}{2+2l_0} \geq \frac{2+2\mu\delta}{\mu}\frac{\mu h}{2+2\mu h} \geq h\,.$$

Therefore, we can in the regime $h \leq \delta$ apply Lemma 2.8 with M as above. On the other hand, if $h > \delta$ then with $S = 2\delta$ we have

$$S \max_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) \leq 2\delta/2 < h\,.$$

Thus if $h > \delta$ we can apply (55) with $S = 2\delta$. In conclusion, with the choices of l and l_0 made in [10] the claimed inequality is valid for all $h > 0$.

Similarly, for the length scale (29) used in the proof of Theorem 1.1 we have

$$\min_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) = l_0\,,$$
$$\max_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) \leq r_{in}(\Omega)/2\,.$$

Setting $M = 1/\mu$ and $S = 2\delta/r_{in}(\Omega)$ we find

$$M \min_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) = l_0/\mu \geq h\,, \qquad \text{for } h \leq \delta\,,$$
$$S \max_{\operatorname{dist}(u,\Omega)\leq l(u)} l(u) \leq \delta < h\,, \qquad \text{for } h > \delta\,,$$

and we can conclude in the same manner as above.

References

1. M. Aizenman and E. H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429.
2. L. Ambrosio, A. Colesanti, and E. Villa, *Outer Minkowski content for some classes of closed sets*, Math. Ann. **342** (2008), no. 4, 727–748.
3. R. Bañuelos, T. Kulczycki, and B. Siudeja, *On the trace of symmetric stable processes on Lipschitz domains*, J. Funct. Anal. **257** (2009), no. 10, 3329–3352.
4. F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167.
5. M. van den Berg, *A uniform bound on trace$(e^{t\Delta})$ for convex regions in $\mathbb{R}^n$ with smooth boundaries*, Comm. Math. Phys. **92** (1984), no. 4, 525–530.
6. M. van den Berg, *On the asymptotics of the heat equation and bounds on traces associated with the Dirichlet Laplacian*, J. Funct. Anal. **71** (1987), no. 2, 279–293.
7. M. Bronstein and V. Ivrii, *Sharp spectral asymptotics for operators with irregular coefficients. I. Pushing the limits.* Comm. Partial Differential Equations **28** (2003), no. 1–2, 83–102.
8. R. M. Brown, *The trace of the heat kernel in Lipschitz domains*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 889–900.
9. H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, New York, 1969.

10. R. L. Frank and L. Geisinger, *Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain*, Mathematical results in quantum physics, World Sci. Publ., Hackensack, NJ, 2011, pp. 138–147.
11. R. L. Frank and L. Geisinger, *Semi-classical analysis of the Laplace operator with Robin boundary conditions*, Bull. Math. Sci. **2** (2012), no. 2, 281–319.
12. R. L. Frank and L. Geisinger, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*, J. Reine Angew. Math. **712** (2016), 1–37.
13. L. Geisinger, A. Laptev, and T. Weidl, *Geometrical Versions of improved Berezin-Li-Yau Inequalities*, Journal of Spectral Theory **1** (2011), 87–109.
14. L. Geisinger and T. Weidl, *Universal bounds for traces of the Dirichlet Laplace operator*, J. Lond. Math. Soc. **82** (2010), no. 2, 395–419.
15. E. M. Harrell II and J. Stubbe, *Two-term asymptotically sharp estimates for eigenvalue means of the Laplacian*, Journal of Spectral Theory **8** (2018), 1529–1550.
16. E. M. Harrell II, L. Provenzano, and J. Stubbe, *Complementary asymptotically sharp estimates for eigenvalue means of Laplacians*, `arXiv:1806.10366 [math.SP]` (2018).
17. J. Hersch, *Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum*, Z. Angew. Math. Phys. **11** (1960), 387–413.
18. V. Ivrii, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34.
19. V. Ivrii, *Sharp spectral asymptotics for operators with irregular coefficients*, Internat. Math. Res. Notices **22** (2000), 1155–1166.
20. V. Ivrii, *Sharp spectral asymptotics for operators with irregular coefficients. II. Domains with boundaries and degenerations*. Comm. Partial Differential Equations **28** (2003), no. 1–2, 103–128.
21. H. Kovařík, S. Vugalter, and T. Weidl, *Two-dimensional Berezin–Li–Yau inequalities with a correction term*, Comm. Math. Phys. **287** (2009), no. 3, 959–981.
22. S. Larson, *A bound for the perimeter of inner parallel bodies*, J. Funct. Anal. **271** (2016), no. 3, 610–619.
23. S. Larson, *On the remainder term of the Berezin inequality on a convex domain*, Proc. Amer. Math. Soc., **145** (2017), no. 5, 2167–2181.
24. S. Larson, *Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains*, Journal of Spectral Theory (to appear).
25. P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, 309–318.
26. A. D. Melás, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 631–636.
27. M. H. Protter, *A lower bound for the fundamental frequency of a convex region*, Proc. Amer. Math. Soc. **81** (1981), no. 1, 65–70.
28. G. V. Rozenblum, *On the eigenvalues of the first boundary value problem in unbounded domains*. Math. USSR Sb. **18** (1972), no. 2, 235–248.
29. R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Second Expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge Univ. Press, Cambridge, 2014.

30. R. Seeley, *A sharp asymptotic remainder estimate for the eigenvalues of the Laplacian in a domain of* $\mathbb{R}^3$. Adv. in Math. **29** (1978), no. 2, 244–269.
31. R. Seeley, *An estimate near the boundary for the spectral function of the Laplace operator.* Amer. J. Math. **102** (1980), no. 5, 869–902.
32. J. P. Solovej and W. L. Spitzer, *A new coherent states approach to semiclassics which gives Scott's correction*, Comm. Math. Phys. **241** (2003), no. 2-3, 383–420.
33. D. Vassiliev, *Two-term asymptotics of the spectrum of a boundary value problem under an interior reflection of general form.* Functional Anal. Appl. **18** (1984), 267–277.
34. D. Vassiliev, *Two-term asymptotics of the spectrum of a boundary value problem in the case of a piecewise smooth boundary.* Soviet Math. Dokl. **33** (1986), no. 1, 227–230.
35. T. Weidl, *Improved Berezin–Li–Yau inequalities with a remainder term*, Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, pp. 253–263, 2008.
36. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen* (*mit einer Anwendung auf die Theorie der Hohlraumstrahlung*), Math. Ann. **71** (1912), no. 4, 441–479.
37. H. Weyl, *Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze*, J. Reine Angew. Math. **143** (1913), 177–202.

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Paper F

Maximizing Riesz means of anisotropic harmonic oscillators
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MAXIMIZING RIESZ MEANS OF ANISOTROPIC HARMONIC OSCILLATORS

SIMON LARSON

Abstract. We consider problems related to the asymptotic minimization of eigenvalues of anisotropic harmonic oscillators in the plane. In particular we study Riesz means of the eigenvalues and the trace of the corresponding heat kernels. The eigenvalue minimization problem can be reformulated as a lattice point problem where one wishes to maximize the number of points of $(\mathbb{N}-\frac{1}{2})\times(\mathbb{N}-\frac{1}{2})$ inside triangles with vertices $(0,0),(0,\lambda\sqrt{\beta})$ and $(\lambda/\sqrt{\beta},0)$ with respect to $\beta>0$, for fixed $\lambda\geq 0$. This lattice point formulation of the problem naturally leads to a family of generalized problems where one instead considers the shifted lattice $(\mathbb{N}+\sigma)\times(\mathbb{N}+\tau)$, for $\sigma,\tau>-1$. We show that the nature of these problems are rather different depending on the shift parameters, and in particular that the problem corresponding to harmonic oscillators, $\sigma=\tau=-\frac{1}{2}$, is a critical case.

1. Introduction and main result

For $\beta>0$, let L_β denote the self-adjoint operator on $L^2(\mathbb{R}^2)$ acting as

$$-\Delta+\beta x^2+\beta^{-1}y^2,$$

which we will refer to as the *anisotropic harmonic oscillator*. For any $\beta>0$ the spectrum of L_β is positive and purely discrete, consisting of an infinite number of eigenvalues. Let $\{\lambda_k(\beta)\}_{k\in\mathbb{N}}$ denote the eigenvalues of L_β numbered in increasing order and each repeated according to its multiplicity. Here and in what follows we use the convention that $\mathbb{N}=\{1,2,\ldots\}$. It is well known that the eigenvalues have a one-to-one correspondence with $\mathbb{N}^2$, explicitly given by

$$(k_1,k_2)\mapsto 2(k_1-1/2)\sqrt{\beta}+2(k_2-1/2)/\sqrt{\beta}=\lambda_{(k_1,k_2)}(\beta). \tag{1}$$

In this paper we consider a number of problems related to the following question: Given $k\in\mathbb{N}$ for what values of β is the minimum

$$\min\{\lambda_k(\beta):\beta>0\}$$

realized? In particular we are interested in how the set of minimizing β behaves as k tends to infinity. Similar questions concerning minimizing or maximizing functions of the spectrum of differential operators has in recent years seen large interest, see for instance [12] and references therein.

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1.1. **Minimizing eigenvalues and counting lattice points.** The problem of minimizing the k-th eigenvalue among the operators L_β can be reformulated as finding the β for which the *eigenvalue counting function*,

$$N(\beta, \lambda) := \#\{j \in \mathbb{N} : \lambda_j(\beta) \leq \lambda\}, \tag{2}$$

is first to reach k. Hence, if one understands the maximization problem

$$\max\{N(\beta, \lambda) : \beta > 0\} \tag{3}$$

for all $\lambda \geq 0$, then one also understands the problem of minimizing $\lambda_k(\beta)$ for any $k \in \mathbb{N}$.

Due to the form of the eigenvalues of L_β this maximization problem can be reformulated as a geometric lattice point problem: Given $\lambda \geq 0$ find the triangle, amongst those given by the vertices $(0,0)$, $(\lambda/\sqrt{\beta}, 0)$ and $(0, \sqrt{\beta}\lambda)$, which contains the greatest number of points of the lattice $(\mathbb{N} - \frac{1}{2}) \times (\mathbb{N} - \frac{1}{2})$. (We have here rescaled the problem to avoid the factor 2 appearing in the explicit form of the eigenvalues (1).)

In a similar manner the problem of minimizing eigenvalues of the Dirichlet Laplacian among cuboids of unit measure, i.e. domains of the form $Q = (0, a_1) \times \ldots \times (0, a_d) \subset \mathbb{R}^d$ with $\prod_{i=1}^d a_i = 1$, can be recast as finding which ellipsoid centered at the origin and of fixed volume contains the largest number of positive integer lattice points. In [2] Antunes and Freitas used this idea to show that if Q_k is a sequence of unit area rectangles such that Q_k minimizes λ_k then Q_k converges to the square as k tends to infinity. In [4] a similar result was proven for the case of the Neumann Laplacian. The result of Antunes and Freitas was generalized to the three-dimensional case in [5], and to arbitrary dimension in [7] where also the corresponding Neumann result was proven to hold in any dimension.

Generalizing the work of Antunes and Freitas from the viewpoint of lattice point problems, Laugesen and Liu [16] studied the following problem: Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing concave function with $f(0) = 1$ and $f(1) = 0$. Define, for $s, r > 0$, the function

$$N(s, r) := \#\{(k_1, k_2) \in \mathbb{N}^2 : k_2 \leq rsf(k_1 s/r)\}. \tag{4}$$

This function counts the number of integer lattice points under the graph of f after it has been compressed in the x-direction by a factor s, stretched in the y-direction by the same factor, and scaled by a factor r. What happens to the set of maximizers, $\operatorname{argmax}_{s>0} N(s, r)$, as r (the area under the rescaled graph) tends to infinity? For a large family of functions f they prove that the maximizing set of s tends to 1. The corresponding problem with concave curves replaced by convex ones was treated in [3]. More recently Laugesen and Liu [17] have studied the case of both concave and convex curves where they also allow for shifting the lattice, i.e. replacing $\mathbb{N}^2$ by $(\mathbb{N} + \sigma) \times (\mathbb{N} + \tau)$. For work on similar problems in higher dimensions see also [8, 18].

However, the results of [3, 16, 17] all require that the graph of the function f has non-vanishing curvature. In particular, the case of $f(x) = 1 - x$ is not covered, which is precisely the problem of interest here. That the case of vanishing curvature is excluded from the results of [3, 16, 17, 18] is no accident, and also more classical problems in lattice point theory are less well understood in this setting [13, 20]. In fact it was conjectured in [16] that the problem with $f(x) = 1 - x$ fails to have an asymptotic maximizer (see also [17] for the shifted case), and that instead the sequence of maximizing values of s has an infinite number of limit points. In [19] Marshall and Steinerberger prove the conjecture in the case of the non-shifted lattice $\mathbb{N}^2$.

1.2. **Maximizing Riesz means.** In what follows we will consider a family of problems closely related to the maximization problem in (3). The main problem that we are interested in is the behavior of β which maximizes the function

$$R^\gamma_{\sigma,\tau}(\beta,\lambda) := \sum_{k\in\mathbb{N}^2} (\lambda - (k_1+\sigma)\sqrt{\beta} - (k_2+\tau)/\sqrt{\beta})^\gamma_+, \tag{5}$$

for $\gamma > 0$ and $\sigma, \tau > -1$, as λ tends to infinity (if $\sigma = \tau$ we will write simply R^γ_σ). Here and in what follows $x_\pm := (|x| \pm x)/2$.

Setting $\gamma = 0$ and interpreting the sum appropriately, (5) reduces to the function

$$N_{\sigma,\tau}(\beta,\lambda) := \#\{(k_1,k_2) \in (\mathbb{N}+\sigma)\times(\mathbb{N}+\tau) : k_1\sqrt{\beta} + k_2/\sqrt{\beta} \le \lambda\}. \tag{6}$$

If $\sigma = \tau = 0$ then (6) corresponds to the case considered in [16, 19]. If $\sigma = \tau = -1/2$ then (6) is the eigenvalue counting function (2) evaluated at 2λ. Similarly, for $\gamma > 0$, $R^\gamma_{-1/2}(\beta,\lambda) = \mathrm{Tr}(L_\beta - 2\lambda)^\gamma_-$ is the *Riesz mean* of order γ of L_β. Here we will adopt this name also for other σ and τ.

Taking $\gamma > 0$ (instead of $\gamma = 0$ as in the original problem) leads to a regularization of the problem and will allow us to use certain tools that are effectively excluded in the case of the counting function. Using the Aizenman–Lieb Identity [1] the regularizing effect of increasing γ becomes clear as it allows one to write $R^\gamma_{\sigma,\tau}$ as a weighted mean of lower order Riesz means: for $\gamma_2 > \gamma_1 \ge 0$ and $\lambda \ge 0$,

$$R^{\gamma_2}_{\sigma,\tau}(\lambda) = B(1+\gamma_1, \gamma_2-\gamma_1)^{-1} \int_0^\infty \eta^{-1+\gamma_2-\gamma_1} R^{\gamma_1}_{\sigma,\tau}(\lambda-\eta)\, d\eta, \tag{7}$$

where B denotes the Euler Beta function, and we as above interpret $R^0_{\sigma,\tau}$ as $N_{\sigma,\tau}$. This identity follows from linearity and the fact that

$$\int_0^\infty \tau^{-1+\gamma_2-\gamma_2}(\tau-a)^{\gamma_1}_+\, d\tau = a^{\gamma_2}_+ B(1+\gamma_1, \gamma_2-\gamma_1).$$

We will also consider a further regularized problem which in the harmonic oscillator case corresponds to the *trace of the heat kernel* of L_β, that is $\mathrm{Tr}(e^{-tL_\beta})$. For general shift parameters σ, τ we define

$$H_{\sigma,\tau}(\beta,t) := \sum_{k\in\mathbb{N}^2} e^{-t((k_1+\sigma)\sqrt{\beta}+(k_2+\tau)/\sqrt{\beta})}. \tag{8}$$

The problem of asymptotically maximizing this function in β as $t \to 0^+$ can in a certain sense be seen as a limiting version of the Riesz mean problems with λ and γ going to infinity simultaneously. A further connection to the Riesz means can be found by noticing that $H_{\sigma,\tau}$ can be written using the Laplace transform of the $R^\gamma_{\sigma,\tau}$:

$$H_{\sigma,\tau}(\beta,t) = \frac{t^{1+\gamma}}{\Gamma(1+\gamma)} \int_0^\infty R^\gamma_{\sigma,\tau}(\beta,\lambda) e^{-\lambda t}\, d\lambda. \tag{9}$$

This connection via the Laplace transform of $R^\gamma_{\sigma,\tau}$ and $H_{\sigma,\tau}$, combined with the fact that $H_{\sigma,\beta}$ can be explicitly computed, will be of importance when we study the behavior of the Riesz means for large λ (following [10, 11]). The main motivation for including the study of the heat kernel problem here is that it is easier to understand than the Riesz mean problem, and can thus serve as a guide to what we might expect when studying $R^\gamma_{\sigma,\tau}$.

1.3. **Main results and conjectures.** Throughout the paper $\beta^\gamma_{\sigma,\tau}(\lambda)$, for $\lambda \geq 0$, will denote a β which maximizes $R^\gamma_{\sigma,\tau}(\,\cdot\,,\lambda)$, that is, satisfies

$$R^\gamma_{\sigma,\tau}(\beta^\gamma_{\sigma,\tau}(\lambda),\lambda) = \max\{R^\gamma_{\sigma,\tau}(\beta,\lambda) : \beta > 0\}.$$

As such a maximizer is not necessarily unique we emphasize that when we make a claim concerning $\beta^\gamma_{\sigma,\tau}(\lambda)$ we mean that this holds for *all* maximizers. Similarly we let $\beta^H_{\sigma,\tau}(t)$, with $t > 0$, denote a maximizer of $H_{\sigma,\tau}(\,\cdot\,,t)$, i.e. such that

$$H_{\sigma,\tau}(\beta^H_{\sigma,\tau}(t),t) = \max\{H_{\sigma,\tau}(\beta,t) : \beta > 0\}. \tag{10}$$

We first turn to what we are able to prove for $\beta^H_{\sigma,\tau}(t)$. The problem is made easier due to the fact that the sum (8) can be explicitly computed:

$$H_{\sigma,\tau}(\beta,t) = \sum_{k\in\mathbb{N}^2} e^{-t((k_1+\sigma)\sqrt{\beta}+(k_2+\tau)/\sqrt{\beta})} = \frac{e^{-t(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)}. \tag{11}$$

The question of maximizing with respect to β is thus reduced to an explicit optimization problem in one variable. However, the behavior of this function depends strongly on the parameters t, σ, τ and carrying out the maximization explicitly is difficult.

For $\beta^H_{\sigma,\tau}(t)$ there are two asymptotic regions that we wish to study: when $t \to 0^+$ and when $t \to \infty$. The asymptotic problem $t \to 0^+$ is most closely related to that studied for the Riesz means as more and more of the lattice points (eigenvalues) become relevant as t becomes smaller, while if t goes to ∞ the main contribution comes from the lattice points which are closest to the origin. Our first theorem tells us that we can determine the behavior of $\beta^H_{\sigma,\tau}(t)$ in both limits.

Theorem 1.1. *For each $t > 0$ and $\sigma, \tau > -1$ there exists a maximizing value $\beta^H_{\sigma,\tau}(t)$ satisfying* (10)*. If $\max\{\sigma,\tau\} \geq -1/2$ then the maximizer is unique for each $t > 0$, moreover, if $\sigma = \tau \geq -1/2$ then $\beta^H_\sigma(t) = 1$.*

Furthermore, for all $\sigma, \tau > -1$, it holds that

$$\lim_{t\to\infty} \beta^H_{\sigma,\tau}(t) = \frac{1+\tau}{1+\sigma},$$

similarly, for all $\sigma, \tau > -1/2$,

$$\lim_{t\to 0^+} \beta^H_{\sigma,\tau}(t) = \frac{1+2\tau}{1+2\sigma}.$$

For all values of $\sigma, \tau > -1$ not covered above, any sequence of maximizers degenerates, i.e. $\beta^H_{\sigma,\tau}(t)$ tends to 0 or ∞ as $t \to 0^+$.

Remark 1.2. One should note that the asymptotic maximizer in the limit $t \to \infty$ is precisely the β which minimizes the area of the first triangle containing any lattice points at all. In the limit $t \to 0^+$ we find the same limit as Laugesen–Liu [17] found for the counting function (4). This limit corresponds to balancing the area of the region below the bounding curve (in our case a line) to the left of the first column of lattice points, with that of the region below the bounding curve and below the first row of lattice points (see [17, Figure 1]).

In the same direction we prove the following for Riesz means:

Theorem 1.3. *For all $\gamma > 0$ and $\sigma, \tau > -1/2$ it holds that*

$$\lim_{\lambda \to \infty} \beta^{\gamma}_{\sigma,\tau}(\lambda) = \frac{1+2\tau}{1+2\sigma}.$$

That is, for all shifts $\sigma, \tau > -1/2$ any sequence of maximizers, with $\lambda \to \infty$, for positive order Riesz means has a unique limit. Thus the behavior observed in [16] and studied in [19] for the counting function with $\sigma = \tau = 0$ effectively vanishes as soon as we consider the regularized problem of Riesz means with $\gamma > 0$.

In the case of the harmonic oscillators, $\sigma = \tau = -1/2$, we find a unique limit first when $\gamma > 1$. Specifically we prove that:

Theorem 1.4. *For all $\gamma > 1$ it holds that*

$$\lim_{\lambda \to \infty} \beta^{\gamma}_{-1/2}(\lambda) = 1.$$

We do not believe that the failure to prove the corresponding result for smaller γ is a result of our methods, but that in these cases the behavior of the maximizers resembles that in [19]. In fact, for the cases that are not covered by the above we conjecture the following, which extends the conjecture of Laugesen and Liu [16]:

Conjecture 1.5. *The conjecture is split into two parts:*

(i) For all $\sigma, \tau > -1/2$ the set

$$\bigcap_{\lambda > 0} \overline{\bigcup_{\lambda' > \lambda} \operatorname*{argmax}_{\beta > 0} N_{\sigma,\tau}(\beta, \lambda')}$$

is infinite.

(ii) For all $0 \leq \gamma \leq 1$ the set

$$\bigcap_{\lambda > 0} \overline{\bigcup_{\lambda' > \lambda} \operatorname*{argmax}_{\beta > 0} R^{\gamma}_{-1/2}(\beta, \lambda')}$$

is infinite.

As mentioned earlier the case $\gamma = \sigma = \tau = 0$ was recently settled by Marshall and Steinerberger [19].

1.4. **Idea of proof.** The conjecture, as well as the proof of Theorems 1.3 and 1.4, is based on precise asymptotic expansions of $R^{\gamma}_{\sigma,\tau}(\beta,\lambda)$ as $\lambda \to \infty$. In [9, 10] the authors study the asymptotic behavior of $R_{-1/2}(1,\lambda/2) = \mathrm{Tr}((-\Delta+|x|^2)-\lambda)^{\gamma}_{-}$ in connection to Lieb–Thirring inequalities (see also [6, 15]). The calculations carried out there transfer without much change to what we study here, see Section 5.

Let $\zeta\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ denote the Hurwitz ζ-function. In the special case $\zeta(z,1)$ this is the Riemann ζ-function which we denote simply by $\zeta(z)$ [21, Chapter 25]. Let also $\{x\}$ denote the fractional part of $x\in\mathbb{R}$, i.e. $\{x\} = x - \lfloor x \rfloor$.

Theorem 1.6. *For any $\gamma>0$, $M\in\mathbb{N}$, $\delta>0$, $\beta\in\mathbb{R}_+$ and $\sigma,\tau>-1$, there are constants $\alpha_k=\alpha_k(\beta,\sigma,\tau,\gamma)$ such that*

$$R^{\gamma}_{\sigma,\tau}(\beta,\lambda) = \sum_{k=0}^{M+1} \alpha_k \lambda^{2-k+\gamma} + \mathrm{Osc}(\beta,\lambda) + o(\lambda^{-M+\gamma+\delta}),$$

as $\lambda\to\infty$. The coefficients α_k are continuous in β and $|\mathrm{Osc}(\beta,\lambda)|\le C_\beta(\lambda+1)$. Moreover, C_β and the implicit constant of the remainder term are uniformly bounded for β in compact subsets of $\mathbb{R}_+$.

Furthermore,

(*i*) *if $\beta=\frac{\mu}{\nu}\in\mathbb{Q}_+$, $\gcd(\mu,\nu)=1$, then, with $x=\sqrt{\mu\nu}\lambda-\mu\sigma-\nu\tau$,*

$$\begin{aligned}\mathrm{Osc}(\beta,\lambda) = {} & \frac{\zeta(-\gamma,\{x\})}{(\mu\nu)^{\frac{1+\gamma}{2}}}\lambda - \frac{\zeta(-1-\gamma,\{x\})}{(\mu\nu)^{1+\frac{\gamma}{2}}} - \frac{(1+2\sigma)\mu+(1+2\tau)\nu}{2(\mu\nu)^{1+\frac{\gamma}{2}}}\zeta(-\gamma,\{x\})\\ & + \frac{\nu^{\gamma/2}\Gamma(1+\gamma)}{\mu^{\gamma/2}(2\pi)^{1+\gamma}} \sum_{\substack{k\in\mathbb{N}\\ k/\nu\notin\mathbb{N}}} \frac{\sin(\pi k(2x-\mu)/\nu - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\frac{\mu}{\nu})}\\ & + \frac{\mu^{\gamma/2}\Gamma(1+\gamma)}{\nu^{\gamma/2}(2\pi)^{1+\gamma}} \sum_{\substack{k\in\mathbb{N}\\ k/\mu\notin\mathbb{N}}} \frac{\sin(\pi k(2x-\nu)/\mu - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\frac{\nu}{\mu})};\end{aligned}$$

(*ii*) *if $\beta\in\mathbb{R}_+\backslash\mathbb{Q}$, it holds that*

$$\begin{aligned}\mathrm{Osc}(\beta,\lambda) = {} & \frac{\beta^{-\gamma/2}\Gamma(1+\gamma)}{(2\pi)^{1+\gamma}} \sum_{k=1}^{\Lambda(\lambda)/\sqrt{\beta}} \frac{\sin(\pi k(2\lambda\sqrt{\beta}-(1+2\sigma)\beta-2\tau)-\frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\beta)}\\ & + \frac{\beta^{\gamma/2}\Gamma(1+\gamma)}{(2\pi)^{1+\gamma}} \sum_{k=1}^{\Lambda(\lambda)\sqrt{\beta}} \frac{\sin(\pi k(2\lambda/\sqrt{\beta}-2\sigma-(1+2\tau)/\beta)-\frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k/\beta)}\\ & + o(\lambda^{-M+\gamma+\delta}),\end{aligned}$$

where $\Lambda(\lambda) = O(\lambda^{\frac{M+2-\gamma}{\gamma}})$.

Remark 1.7. A couple of remarks are in order:

(1) If $\gamma\in\mathbb{N}$ then $\alpha_k=0$ for all $k>2+\gamma$.

(2) We emphasize that the amplitude of the oscillatory term $\operatorname{Osc}(\beta,\lambda)$ grows at most linearly in λ independently of the values of γ and β:
   - In the rational case (i) the only term of $\operatorname{Osc}(\beta,\lambda)$ that is not bounded is the first one,
$$\operatorname{Osc}\bigl(\tfrac{\mu}{\nu},\lambda\bigr)=\frac{\zeta(-\gamma,\{x\})}{(\mu\nu)^{\frac{1+\gamma}{2}}}\lambda+O(1),\quad \text{as } \lambda\to\infty.$$
   - In the irrational case (ii) we believe that $\operatorname{Osc}(\beta,\lambda)=o(\lambda)$. For $\gamma=0$ it follows that this is the case from the results in [19], but we are currently unable to prove this when $\gamma>0$. Whether or not this statement is true will be of little importance in what follows, but if one aims to prove (or disprove) Conjecture 1.5 it would most likely be necessary to understand $\operatorname{Osc}(\beta,\lambda)$ in greater detail.

For an explicit formula for the coefficients α_k see (29). For our purposes it will only be important that

$$\alpha_0=\frac{1}{(1+\gamma)(2+\gamma)},\qquad \alpha_1=-\frac{(1+2\sigma)\sqrt{\beta}+(1+2\tau)/\sqrt{\beta}}{2(1+\gamma)},$$
$$\alpha_2=\frac{(1+2\sigma)(1+2\tau)}{4}+\frac{(1+6\sigma(1+\sigma))\beta+(1+6\tau(1+\tau))/\beta}{12}.$$

The α_2 term will only be important in the case $\sigma=\tau=-1/2$, in which case $\alpha_1=0$ and $\alpha_2=-\frac{1+\beta^2}{24\beta}$.

Heuristically, Theorem 1.6 suggests that Theorems 1.3, 1.4 and Conjecture 1.5 should be true. Essentially, since the first order term is independent of β it is reasonable to conjecture that to asymptotically maximize $R^\gamma_{\sigma,\tau}$ one would want to choose β to maximize the next order term. The cases where we can prove that an asymptotic maximizer exists is precisely those where:

(i) the subleading polynomial term is asymptotically much larger than $\operatorname{Osc}(\beta,\lambda)$, and
(ii) the coefficient of this term is maximized at some $\beta\in\mathbb{R}_+$.

In the harmonic oscillator case, when $\alpha_1=0$, this means that the third term $\alpha_2\lambda^\gamma$ needs to be superlinear, and hence $\gamma>1$.

For the combinations of σ,τ and γ in Conjecture 1.5 the oscillatory parts of the expansion are of greater importance. It is suitable to consider the renormalized quantity

$$\frac{R^\gamma_{\sigma,\tau}(\beta,\lambda)-\alpha_0\lambda^{2+\gamma}}{\lambda}. \tag{12}$$

If σ,τ and γ are as in Conjecture 1.5 then in the limit $\lambda\to\infty$ (12) converges to a function which is periodic in λ and whose period and amplitude depend on β (for $\gamma>0$ this follows from Theorem 1.6 and for $\gamma=0$ from [19, Lemmas 4 and 5] by a change of variables). It is not unreasonable to believe that one can align these periods to construct a large set of limit points for $\beta^\gamma_{\sigma,\tau}(\lambda)$. In fact, this is the underlying idea in Marshall and Steinerberger's proof of the conjecture in the the case $\sigma=\tau=\gamma=0$ [19].

From Theorem 1.6 it is not difficult to conclude that any sequence of maximizers of $R^\gamma_{\sigma,\tau}$ must degenerate when $(\sigma,\tau)\in(-1,\infty)^2\setminus((-1/2,\infty)^2\cup\{(-1/2,-1/2)\})$. Indeed, for such shifts the second term of the asymptotic expansion is maximized when β tends either to 0 or ∞. Since the expansion is uniform on compact sets this implies that any maximizing sequence must eventually leave all compacts.

In the case of $H_{\sigma,\tau}$ similar reasoning can be used to conclude that any sequence of maximizers $\beta^H_{\sigma,\tau}$ must degenerate as $t\to 0^+$. Indeed, from Theorem 1.6 and (9) one finds that

$$H_{\sigma,\tau}(\beta,t)=\frac{1}{t^2}-\frac{(1+2\sigma)\sqrt{\beta}+(1+2\tau)/\sqrt{\beta}}{2t}+O(1),\quad \text{as } t\to 0^+,$$

where the remainder term is uniform for β on compact subsets of $\mathbb{R}_+$, which allows us to argue as above.

1.5. **Higher dimensions.** Using an idea of Laptev [14] and the bounds proved in Section 2 one can reduce the corresponding d-dimensional version of the problems considered here to lower dimensional ones. In [7] this strategy was applied to generalize the results of [2, 4, 5] to any dimension.

Providing asymptotic expansions similar to those in Theorem 1.6 in higher dimensions is possible using the techniques from [9, 10, 11], see also Section 5. Naturally the computations in general dimension are more difficult. However, for the cases where one would expect the existence of an asymptotic maximizer the formulas in Theorem 1.6 are more detailed than necessary. For the applications considered, it is sufficient to know the first and second non-vanishing polynomial term, and that the oscillatory part of the expansion is of lower order than the second polynomial term. In the d-dimensional case the oscillatory terms will generally be of magnitude $\sim\lambda^{d-1}$. Precise, and uniform, asymptotic expansions to sufficiently low order can be obtained following the argument in Section 5.3.

1.6. **Structure of the paper.** The remainder of the paper is structured as follows. In Section 2 we prove a number of bounds for $R^\gamma_{\sigma,\tau}$ which will enable us to exclude that there are sequences of maximizers which degenerate as $\lambda\to\infty$. In Section 3 we study the problem of maximizing $H_{\sigma,\tau}$ and prove Theorem 1.1. Section 4 is dedicated to the proofs of Theorems 1.3 and 1.4, which will rely on the bounds proved in Section 2 and Theorem 1.6. Finally in Section 5 we study the asymptotic behavior of $R^\gamma_{\sigma,\tau}(\beta,\lambda)$, as $\lambda\to\infty$, and prove Theorem 1.6.

2. Preliminaries

Before we continue we need to verify that we can actually talk about maximizers of $R^\gamma_{\sigma,\tau}(\,\cdot\,,\lambda)$ and $H_{\sigma,\tau}(\,\cdot\,,t)$. For $H_{\sigma,\tau}$ it is clear from (11) that the maximization problem is well posed, and hence we only need to prove that this is the case for $R^\gamma_{\sigma,\tau}$.

Lemma 2.1. *For each $\lambda\geq 0,\gamma>0$ and $\sigma,\tau>-1$ there exists a maximizing value $\beta^\gamma_{\sigma,\tau}(\lambda)$. If $\lambda\leq 2\sqrt{(1+\sigma)(1+\tau)}$ then $R^\gamma_{\sigma,\tau}(\beta,\lambda)=0$ for all $\beta>0$, and thus any β is a maximizer.*

If $\lambda > 2\sqrt{(1+\sigma)(1+\tau)}$ *then all maximizers satisfy*

$$\beta^\gamma_{\sigma,\tau}(\lambda) \in \Big(\frac{(1+\tau)^2}{\lambda^2}, \frac{\lambda^2}{(1+\sigma)^2}\Big).$$

Lemma 2.1 follows directly from [17, Lemma 9], but since our notation is different and the proof is simple we choose to include it.

Proof of Lemma 2.1. Note first that if we can prove the second part of the lemma, that there are no maximizers outside $\big(\frac{(1+\tau)^2}{\lambda^2}, \frac{\lambda^2}{(1+\sigma)^2}\big)$, then the existence of a maximizer is clear by the continuity of $R^\gamma_{\sigma,\tau}(\beta,\lambda)$ as a function of β.

That $R^\gamma_{\sigma,\tau}(\beta,\lambda) = 0$ for all β if $\lambda \le 2\sqrt{(1+\sigma)(1+\tau)}$ follows since the inequality

$$\lambda - (1+\sigma)\sqrt{\beta} - (1+\tau)/\sqrt{\beta} \le 0, \tag{13}$$

holds for all $\lambda \le 2\sqrt{(1+\sigma)(1+\tau)}$. Similarly, (13) holds if $\beta \le \frac{(1+\tau)^2}{\lambda^2}$ or $\beta \ge \frac{\lambda^2}{(1+\sigma)^2}$, and thus $R^\gamma_{\sigma,\tau}(\beta,\lambda) = 0$ for such β. However, if $\lambda > 2\sqrt{(1+\sigma)(1+\tau)}$ then $R^\gamma_{\sigma,\tau}(\beta^\gamma_{\sigma,\tau}(\lambda),\lambda) \ge R^\gamma_{\sigma,\tau}(\frac{1+\tau}{1+\sigma},\lambda) > 0$, which implies that $\beta \notin \big(\frac{(1+\tau)^2}{\lambda^2}, \frac{\lambda^2}{(1+\sigma)^2}\big)$ cannot be a maximizer. $\square$

To conclude that any sequence of maximizers of $R^\gamma_{\sigma,\tau}$, with $\lambda \to \infty$, remains in a compact subset of $\mathbb{R}_+$ we require better control than that provided by Lemma 2.1. When proving that this is in fact the case the following bounds will be useful:

Lemma 2.2. *We have that:*

(i) For $\sigma \ge -1/2$,

$$\sum_{k\ge1}(\lambda - (k+\sigma)\sqrt{\beta})_+ \le \frac{\lambda^2}{2\sqrt{\beta}}, \tag{14}$$

for all $\beta > 0$ *and* $\lambda \ge 0$.

(ii) For $\sigma > -1/2$ *there exist positive constants* c_1, c_2, b_0 *such that*

$$\sum_{k\ge1}(\lambda - (k+\sigma)\sqrt{\beta})_+ \le \frac{\lambda^2}{2\sqrt{\beta}} - c_1 b\lambda + c_2 b^2\sqrt{\beta}, \tag{15}$$

for all $\beta > 0, \lambda \ge 0$ *and* $b \in [0, b_0]$.

(iii) There exist positive constants c_1, c_2, b_0 *such that*

$$\sum_{k\ge1}(\lambda - (k-\tfrac12)\sqrt{\beta})^2_+ \le \frac{\lambda^3}{3\sqrt{\beta}} - c_1 b\sqrt{\beta}\lambda + c_2 b^{3/2}\beta, \tag{16}$$

for all $\beta > 0, \lambda \ge 0$ *and* $b \in [0, b_0]$.

Proof of Lemma 2.2. We begin by proving parts (i) and (ii) of the lemma. Clearly (ii) implies (i) when $\sigma > -1/2$. For $\sigma \geq -1/2$,

$$\begin{aligned}\sum_{k\geq 1}(\lambda-(k+\sigma)\sqrt{\beta})_+ &= \sum_{k=1}^{\lfloor \lambda/\sqrt{\beta}-\sigma\rfloor}(\lambda-(k+\sigma)\sqrt{\beta})\\ &= \frac{\lambda^2}{2\sqrt{\beta}}-\frac{1+2\sigma}{2}\lambda+\frac{r-r^2+\sigma+\sigma^2}{2}\sqrt{\beta},\end{aligned}\tag{17}$$

where $r=\big\{\frac{\lambda}{\sqrt{\beta}}-\sigma\big\}$. Maximizing the right-hand side of (17) with respect to $r\in[0,1)$ we find

$$\sum_{k\geq 1}(\lambda-(k+\sigma)\sqrt{\beta})_+ \leq \frac{\lambda^2}{2\sqrt{\beta}}-\frac{1+2\sigma}{2}\lambda+\frac{(1+2\sigma)^2}{8}\sqrt{\beta},\tag{18}$$

which implies (i) when $\sigma=-1/2$. Moreover, since the left-hand side of (18) is decreasing in σ we find (ii) with $c_1=1/2, c_2=1/8$ and $b_0=1+2\sigma$.

The proof of part (iii) is similar:

$$\begin{aligned}\sum_{k\geq 1}(\lambda-(k-\tfrac{1}{2})\sqrt{\beta})_+^2 &= \sum_{k=1}^{\lfloor \lambda/\sqrt{\beta}+1/2\rfloor}(\lambda-(k-\tfrac{1}{2})\sqrt{\beta})^2\\ &= \frac{\lambda^3}{3\sqrt{\beta}}-\frac{\sqrt{\beta}}{12}\lambda-\frac{r(1-r)(1-2r)}{6}\beta\\ &\leq \frac{\lambda^3}{3\sqrt{\beta}}-\frac{\sqrt{\beta}}{12}\lambda+\frac{\beta}{36\sqrt{3}},\end{aligned}$$

where we again maximized in $r=\big\{\frac{\lambda}{\sqrt{\beta}}+\frac{1}{2}\big\}$.

We aim for a bound on the form

$$\sum_{k\geq 1}(\lambda-(k-\tfrac{1}{2})\sqrt{\beta})_+^2 \leq \frac{\lambda^3}{3\sqrt{\beta}}-b\sqrt{\beta}\lambda+\frac{2}{3}b^{3/2}\beta.$$

The right-hand side is non-negative for $b,\beta>0$ and $\lambda\geq 0$, and hence the bound is trivially true when the left-hand side is zero, i.e. for $\lambda\leq\sqrt{\beta}/2$. It thus suffices to prove that

$$\frac{\lambda^3}{3\sqrt{\beta}}-\frac{\sqrt{\beta}}{12}\lambda+\frac{\beta}{36\sqrt{3}} \leq \frac{\lambda^3}{3\sqrt{\beta}}-b\sqrt{\beta}\lambda+\frac{2}{3}b^{3/2}\beta,$$

when b is small enough and $\lambda\geq\sqrt{\beta}/2$. The above inequality holds for all $\lambda\geq\sqrt{\beta}/2$ if and only if

$$b\leq 1/12 \quad\text{and}\quad -\frac{9}{2}+\sqrt{3}+54b-72b^{3/2}\leq 0,$$

which it is easy to check holds for all $b\in[0,1/12]$. This completes the proof of (iii) with $c_1=1, c_2=2/3$ and $b_0=1/12$, and hence the proof of Lemma 2.2. $\square$

Based on Lemma 2.2 we can adapt an idea from [14] (see also [7]) to reduce the proof of a good enough bound for the counting function to a bound for what is essentially a one-dimensional Riesz mean of order 1.

Lemma 2.3. *Fix $\sigma, \tau > -1/2$. There exist positive constants c_1, c_2, c_3, b_0 such that*

$$N_{\sigma,\tau}(\beta,\lambda) \le \frac{\lambda^2}{2} - c_1 b\frac{1+\beta}{\sqrt{\beta}}\lambda + c_2 b^2\frac{1+\beta^2}{\beta} + c_3(\lambda+1),$$

for all $\lambda \ge 0$, $\beta > 0$ and $b \in [0, b_0]$.

Remark 2.4. A similar bound appears in [17, Propositon 10]. However, for σ, τ small the linear term of that bound becomes positive. In what follows it will be essential for this term to be negative, which corresponds to the positivity of c_1 in Lemma 2.3.

Proof of Lemma 2.3. The bound is an easy consequence of Lemma 2.2. First observe that for all $\lambda \ge 0$, $\beta > 0$ and $\sigma' \in (-1/2, \min\{\sigma, \tau\}]$ we have

$$N_{\sigma,\tau}(\beta,\lambda) \le N_{\sigma'}(\beta,\lambda).$$

A straightforward estimate yields that

$$\begin{aligned} N_{\sigma'}(\beta,\lambda) &= \sum_{k\in\mathbb{N}^2} (\lambda - (k_1+\sigma')\sqrt{\beta} - (k_2+\sigma')/\sqrt{\beta})_+^0 \\ &= \sum_{k_1\ge 1} \lfloor (\lambda\sqrt{\beta} - (k_1+\sigma)\beta - \sigma')_+ \rfloor \\ &\le \sum_{k_1\ge 1} (\lambda\sqrt{\beta} - (k_1+\sigma')\beta + \sigma'_-)_+ \\ &= \sqrt{\beta}\sum_{k_1\ge 1} (\lambda + \sigma'_-/\sqrt{\beta} - (k_1+\sigma')\sqrt{\beta})_+. \end{aligned} \tag{19}$$

Applying (15) of Lemma 2.2 one obtains that

$$\begin{aligned} N_{\sigma'}(\beta,\lambda) &\le \frac{(\lambda+\sigma'_-/\sqrt{\beta})^2}{2} - c_1 b\sqrt{\beta}(\lambda + \sigma'_-/\sqrt{\beta}) + c_2 b^2\beta \\ &= \frac{\lambda^2}{2} - c_1 b\sqrt{\beta}\lambda + c_2 b^2\beta + \frac{\sigma'_-}{\sqrt{\beta}}\lambda + \frac{(\sigma'_-)^2}{2\beta} - c_1 b\sigma'_-. \end{aligned} \tag{20}$$

Arguing as above but switching the roles of k_1 and k_2 one correspondingly finds that

$$N_{\sigma'}(\beta,\lambda) \le \frac{\lambda^2}{2} - \frac{c_1 b}{\sqrt{\beta}}\lambda + \frac{c_2 b^2}{\beta} + \sigma'_-\sqrt{\beta}\lambda + \frac{(\sigma'_-)^2}{2}\beta - c_1 b\sigma'_-. \tag{21}$$

Together these two bounds imply that

$$N_{\sigma'}(\beta,\lambda) \le \frac{\lambda^2}{2} - \frac{c_1 b}{2}\frac{1+\beta}{\sqrt{\beta}}\lambda + c_2 b^2\frac{1+\beta^2}{\beta} + \sigma'_-\lambda + \frac{(\sigma'_-)^2}{2} - c_1 b\sigma'_-. \tag{22}$$

Indeed, for $\beta \ge 1$ the right-hand side of (22) is larger than that of (20), and for $\beta \le 1$ larger than that of (21). This completes the proof of the claimed bound with constants related to those in Lemma 2.2. $\square$

In the case of the harmonic oscillators we prove the following lemma, which will play the same role as Lemma 2.3 in what follows.

Lemma 2.5. *There exist positive constants c_1, c_2, b_0 such that*

$$R^1_{-1/2}(\beta,\lambda) \le \frac{\lambda^3}{6} - c_1 b\frac{1+\beta^2}{\beta}\lambda + c_2 b^{3/2}\frac{1+\beta^3}{\beta^{3/2}},$$

for all $\beta > 0, \lambda \ge 0$ and $b \in [0, b_0]$.

Proof of Lemma 2.5. Again the lemma is a simple consequence of Lemma 2.2. Applying first (14) and then (16) we find that

$$\begin{aligned} R^1_{-1/2}(\beta,\lambda) &= \sum_{k\in\mathbb{N}^2}(\lambda - (k_1-\tfrac12)\sqrt{\beta} - (k_2-\tfrac12)/\sqrt{\beta})_+ \\ &\le \frac{\sqrt{\beta}}{2}\sum_{k_1\ge1}(\lambda-(k_1-\tfrac12)\sqrt{\beta})_+^2 \\ &\le \frac{\lambda^3}{6} - \frac{c_1 b}{2}\beta\lambda + \frac{c_2 b^{3/2}}{2}\beta^{3/2}. \end{aligned}$$

Arguing identically but switching the roles of k_1 and k_2 we find that

$$R^1_{-1/2}(\beta,\lambda) \le \frac{\lambda^3}{6} - \frac{c_1 b}{2}\beta^{-1}\lambda + \frac{c_2 b^{3/2}}{2}\beta^{-3/2}.$$

Taking the average of the two bounds completes the proof of Lemma 2.5. $\square$

Combining Lemmas 2.3 and 2.5 with the Aizenman–Lieb Identity (7) one finds the following.

Corollary 2.6. *We have that:*

(i) For $\sigma, \tau > -1/2$ and $\gamma > 0$, there exist positive constants c_1, c_2, c_3, b_0 such that

$$R^\gamma_{\sigma,\tau}(\beta,\lambda) \le \frac{\lambda^{2+\gamma}}{(1+\gamma)(2+\gamma)} - c_1 b\frac{1+\beta}{\sqrt{\beta}}\lambda^{1+\gamma} + c_2 b^2\frac{1+\beta^2}{\beta}\lambda^\gamma + c_3(\lambda+1)\lambda^\gamma, \tag{23}$$

for all $\beta > 0, \lambda \ge 0$ and $b \in [0, b_0]$.

(ii) For $\gamma \ge 1$ there exist positive constants c_1, c_2, b_0 such that

$$R^\gamma_{-1/2}(\beta,\lambda) \le \frac{\lambda^{2+\gamma}}{(1+\gamma)(2+\gamma)} - c_1 b\frac{1+\beta^2}{\beta}\lambda^\gamma + c_2 b^{3/2}\frac{1+\beta^3}{\beta^{3/2}}\lambda^{\gamma-1}, \tag{24}$$

for all $\beta > 0, \lambda \ge 0$ and $b \in [0, b_0]$.

Remark 2.7. We note that the proofs above lift without much work to the corresponding d-dimensional problem. Using again the Aizenman–Lieb Identity (7) one finds a version of (14) for higher order Riesz means. When $\gamma \ge 1$ one can follow the lifting argument of [14] (used in a similar context in [7]): use the corresponding one-term bound to bound the first $d-1$ sums and then a bound similar to (15) to bound the final sum. For the case of the counting function one can mimic (19) reducing the problem to bound a Riesz mean of order one where the spectral parameter λ is slightly shifted. Bounding this Riesz mean can be carried out as described above.

3. Proof of Theorem 1.1

We now turn to the problem of maximizing $H_{\sigma,\tau}$. Due to the fact that we have a closed expression for $H_{\sigma,\tau}$ this is reduced to a maximization problem in one real variable. However solving this problem still turns out to be rather tedious.

Since

$$H_{\sigma,\tau}(\beta,t)=\frac{e^{-t(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)}$$

is non-negative, continuous in β and $\lim_{\beta\to 0}H_{\sigma,\tau}(\beta,t)=\lim_{\beta\to\infty}H_{\sigma,\tau}(\beta,t)=0$ for all $t>0$ and $\sigma,\tau>-1$, it follows that there is at least one maximizing β for each t.

Set $x=\sqrt{\beta}$ and note that

$$H_{\sigma,\tau}(x^2,t)=\frac{e^{-t((\sigma+1/2)x+(\tau+1/2)/x)}}{4t^2}\frac{tx/2}{\sinh(tx/2)}\frac{t/(2x)}{\sinh(t/(2x))}.$$

By the monotonicity of the logarithm we can equivalently consider maximizing $\log(H_{\sigma,\tau})$:

$$\begin{aligned}\log(H_{\sigma,\tau}(x^2,t))&=-t((\sigma+1/2)x+(\tau+1/2)/x)\\&\quad-\log\Big(\frac{\sinh(tx/2)}{tx/2}\Big)-\log\Big(\frac{\sinh(t/(2x))}{t/(2x)}\Big)-\log(4t^2).\end{aligned}$$

By recalling that $\log(\sinh(x)/x)$ is increasing and strictly convex on $\mathbb{R}_+$ it follows that

$$\frac{\partial^2}{\partial x^2}\log(H_{\sigma,\tau}(x^2,t))<-t\frac{\tau+1/2}{2x^3}.$$

Hence, if $\tau\geq -1/2$ the function $\log(H_{\sigma,\tau}(x^2,t))$ is concave in x. Since $\log(H_{\sigma,\tau}(x^2,t))$ also tends to $-\infty$ when $x\to 0$ or ∞ it has a unique maximum. Since $H_{\sigma,\tau}(x^2,t)=H_{\tau,\sigma}(1/x^2,t)$ we can conclude that the same is true if instead $\sigma\geq -1/2$. Moreover, when $\sigma=\tau\geq -1/2$ the symmetry $H_\sigma(x^2,t)=H_\sigma(1/x^2,t)$ implies that $x=1$ must be the unique maximizer.

As the function $x\mapsto\log(H_{\sigma,\tau}(x^2,t))$ is smooth any maximizing $x^*(t)$ must satisfy

$$\frac{\partial}{\partial x}\log(H_{\sigma,\tau}(x^2,t))=-\frac{t}{2x^2}\Big[\Big(1+2\sigma+\coth\Big(\frac{tx}{2}\Big)\Big)x^2-\Big(1+2\tau+\coth\Big(\frac{t}{2x}\Big)\Big)\Big]=0. \tag{25}$$

When $t\to 0^+$ it is easy to see that this equation has a solution which converges to $\sqrt{\frac{1+2\tau}{1+2\sigma}}$. Similarly when $t\to\infty$ we see that there is a solution converging to $\sqrt{\frac{1+\tau}{1+\sigma}}$. When $\max\{\sigma,\tau\}>-1/2$ this concludes the proof of the theorem since we know that the solution is unique.

When σ and τ are both less than $-1/2$ maximizers are no longer necessarily unique when t is small. However, when $t\to\infty$ any sequence of maximizers converges. If there is some solution $x^*(t)$ of (25) which remains in a compact subset of $\mathbb{R}_+$ as $t \to \infty$, we must have that

$$\lim_{t\to\infty} x^*(t) = \sqrt{\frac{1+\tau}{1+\sigma}},$$

since otherwise the expression in the brackets is bounded away from zero when t is large enough.

What remains is to conclude that there can be no maximizers which degenerate, thus implying that the asymptotically stable stationary point is indeed an asymptotic maximizer. Since $H_{\sigma,\tau}(x^2, t) = H_{\tau,\sigma}(1/x^2, t)$ any sequence of maximizers tending to infinity as $t \to \infty$ implies the existence of a sequence of maximizers tending to zero for the problem where σ and τ have been interchanged. Therefore it is sufficient to show that we cannot have maximizers degenerating to zero.

Assume for contradiction that we have a sequence of maximizers $x^* = x^*(t)$ such that $\lim_{t\to\infty} x^* = 0$. Since the factor in front of the parenthesis is non-zero, (25) implies that

$$\lim_{t\to\infty} \coth\Big(\frac{tx^*(t)}{2}\Big)x^*(t)^2 = 2 + 2\tau.$$

But this is a contradiction since

$$0 \le \coth\Big(\frac{tx^*(t)}{2}\Big)x^*(t)^2 \le \Big(1 + \frac{2}{tx^*(t)}\Big)x^*(t)^2 \to 0 \quad \text{as } t \to \infty,$$

which completes the proof of Theorem 1.1.

4. Proof of Theorems 1.3 and 1.4

We now turn our attention to the main results of the paper, namely Theorems 1.3 and 1.4. As the proofs of the two theorems are essentially identical we will write out only the former in detail. The main idea is to combine the bounds in Corollary 2.6 with Theorem 1.6 following the strategy of [2], with some modifications resembling those in [7].

Fix $\sigma, \tau > -1/2$ and $\gamma > 0$. For notational convenience we will write $R(\beta, \lambda) = R_{\sigma,\tau}(\beta, \lambda)$, $\beta = \beta^\gamma_{\sigma,\tau}(\lambda)$ and $\beta^* = \frac{1+2\tau}{1+2\sigma}$ throughout the proof.

By the maximality of β and (23) of Corollary 2.6 we have that

$$R(\beta^*, \lambda) \le R(\beta, \lambda) \le \frac{\lambda^{2+\gamma}}{(1+\gamma)(2+\gamma)} - c_1 b\frac{1+\beta}{\sqrt{\beta}}\lambda^{1+\gamma} + c_2 b^2\frac{1+\beta^2}{\beta}\lambda^\gamma + c_3(\lambda+1)\lambda^\gamma.$$

Using the asymptotic expansion of the left-hand side given by Theorem 1.6, rearranging and using that $\frac{1+\beta^2}{1+\beta} \le 1 + \beta$ we find

$$c_1 b\frac{1+\beta}{\sqrt{\beta}}\Big(1 - b\frac{c_2(1+\beta)}{c_1\sqrt{\beta}\lambda}\Big) \le C + O(\lambda^{-\min\{1,\gamma\}}), \tag{26}$$

as $\lambda \to \infty$.

From Lemma 2.1 we know that $\frac{1+\beta}{\sqrt{\beta}\lambda} \leq \frac{1}{1+\tau} + \frac{1}{1+\sigma}$, and hence we can choose b small enough so that the left-hand side of (26) is positive. Therefore we conclude that

$$\limsup_{\lambda\to\infty} \frac{1+\beta}{\sqrt{\beta}} \leq C,$$

and hence $\beta = \beta^\gamma_{\sigma,\tau}(\lambda)$ remains uniformly bounded away from zero and infinity.

As we now know that all maximizers are contained in a compact subset of $\mathbb{R}_+$ we can use Theorem 1.6 to expand both sides of the inequality $R(\beta^*, \lambda) \leq R(\beta, \lambda)$ with remainder terms independent of β. After rearranging this yields that

$$(1+2\sigma)\sqrt{\beta} + (1+2\tau)/\sqrt{\beta} \leq (1+2\sigma)\sqrt{\beta^*} + (1+2\tau)/\sqrt{\beta^*} + O(\lambda^{-\min\{\gamma,1\}}). \tag{27}$$

Since β^* is the unique minimizer of the function $x \mapsto (1+2\sigma)\sqrt{x} + (1+2\tau)/\sqrt{x}$ and the remainder term is independent of β, (27) implies that

$$\beta = \beta^* + o(1) \quad \text{as } \lambda \to \infty,$$

which concludes the proof of Theorem 1.3.

The proof of Theorem 1.4 is almost identical with the only change being the application of (24) instead of (23) in the first part of the proof.

5. Proof of Theorem 1.6

What remains is to prove Theorem 1.6. The calculations follow those of Helffer and Sjöstrand in [11] for the isotropic harmonic oscillator $\beta = 1$ and $\sigma = \tau = -1/2$ (see also [9, 10]). The key idea is to use the Laplace transform to rewrite $R^\gamma_{\sigma,\tau}$ as an integral which opens up for use of the residue theorem. For any $c > 0$,

$$\begin{aligned}
R^\gamma_{\sigma,\tau}(\beta,\lambda) &= \sum_{k\in\mathbb{N}^2} (\lambda - (k_1+\sigma)\sqrt{\beta} - (k_2+\tau)/\sqrt{\beta})^\gamma_+ \\
&= \sum_{k\in\mathbb{N}^2} \frac{\Gamma(1+\gamma)}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{t(\lambda-(k_1+\sigma)\sqrt{\beta}-(k_2+\tau)/\sqrt{\beta})} t^{-1-\gamma}\,dt \\
&= \frac{\Gamma(1+\gamma)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{t(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)} t^{-1-\gamma}\,dt.
\end{aligned}$$

The integrand in the last expression is a meromorphic function of t outside of $(-\infty, 0]$, with poles at $t = 2\pi i k\sqrt{\beta}$ and $t = 2\pi i l/\sqrt{\beta}$, for $k, l \in \mathbb{Z}\setminus\{0\}$. If β is irrational all of these poles are simple. If $\beta \in \mathbb{Q}$ say $\beta = \frac{\mu}{\nu}$, with $\gcd(\mu,\nu) = 1$, then there are degree-two poles whenever k, l are related by $\mu k = \nu l$. The remaining poles remain simple. That is, degree-two poles at $t = 2\pi i\sqrt{\mu\nu}\, m$ for $m \in \mathbb{Z}\setminus\{0\}$, and simple poles at $t = 2\pi i\sqrt{\frac{\mu}{\nu}}k_1$ and $t = 2\pi i\sqrt{\frac{\nu}{\mu}}k_2$ for $k_1, k_2 \in \mathbb{Z}\setminus\{0\}$ such that $\beta k_1 = \frac{\mu k_1}{\nu} \notin \mathbb{Z}$ and $\frac{k_2}{\beta} = \frac{\nu k_2}{\mu} \notin \mathbb{Z}$.

Letting $f(t) = \frac{e^{t(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)}t^{-1-\gamma}$ and formally using the residue theorem, one would obtain that

$$R^\gamma_{\sigma,\tau}(\beta,\lambda) = \Gamma(1+\gamma)\sum_{t\in\mathcal{P}(f)}\mathrm{Res}(f,t) + \frac{\Gamma(1+\gamma)}{2\pi i}\int_{\Gamma_1} f(t)\,dt, \tag{28}$$

where $\mathcal{P}(f)$ denotes the poles of f and Γ_1 is a contour oriented counter-clockwise which encircles the negative real axis but none of the poles of f. However, to make this rigorous we need that the sum of residues is absolutely convergent. We shall prove that this is the case when $\beta\in\mathbb{Q}_+$ but possibly not when $\beta\notin\mathbb{Q}_+$.

It is no big surprise that the contributions to the asymptotic expansion coming from the residues is the most complicated part to analyse. It is this part which accounts for the oscillatory terms in the expansion and the number theoretic dependence on β. In contrast the integral over the contour Γ_1 has an asymptotic expansion in λ to arbitrary order as λ tends to infinity.

The proof will be split into two parts, first treating $\beta\in\mathbb{Q}_+$ and then $\beta\notin\mathbb{Q}_+$. Much of the work done in the first case will turn out to be useful also in the second.

5.1. **Rational** β**.** In this case it turns out that the use of the residue theorem above is justified. This will be verified once we prove that the sum of residues is absolutely convergent. However, we begin by studying the non-oscillatory part of the expansion, that is, the contribution from the contour integral in (28).

Non-oscillatory part. Let $\varepsilon\in(0,\min\{\sqrt{\beta},1/\sqrt{\beta}\}]$, and let $\Gamma_1=\Gamma_-\cup\Gamma_0\cup\Gamma_+$ with

$$\Gamma_\pm = (-\infty\pm i0, -\varepsilon\pm i0],$$
$$\Gamma_0 = \varepsilon e^{i\theta},\quad \theta\in(-\pi,\pi).$$

For $\lambda>\sigma\sqrt{\beta}+\tau/\sqrt{\beta}$ and any $\varepsilon\in(0,1)$, we see that

$$\Big|\int_{\Gamma_\pm} f(t)\,dt\Big| \le \frac{e^{\varepsilon(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})}}{\gamma(e^{-\sqrt{\beta}}-1)(e^{-1/\sqrt{\beta}}-1)}e^{-\varepsilon\lambda}\varepsilon^{-2-\gamma}.$$

Returning to the integral over Γ_0,

$$\int_{\Gamma_0} f(t)\,dt = \int_{\Gamma_0}\frac{e^{t\lambda}}{t^{3+\gamma}}\frac{t^2e^{-t(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)}\,dt.$$

For small enough ε and any $M\in\mathbb{N}$ we have a uniform expansion

$$\frac{t^2e^{-t(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})}}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)} = \sum_{k=0}^{M-1}a_k(\beta,\sigma,\tau)t^k + O(t^M),$$

where the implicit constant is uniform for β in compact subsets of $\mathbb{R}_+$. The $a_k(\beta,\sigma,\tau)$ are explicitly given by

$$a_k(\beta,\sigma,\tau) = \sum_{l=0}^{k}\frac{(-1)^l}{l!}(\sigma\sqrt{\beta}+\tau/\sqrt{\beta})^l b_{k-l}(\beta),$$

where the $b_k(\beta)$ are the coefficients in the expansion

$$\frac{t^2}{(e^{t\sqrt{\beta}}-1)(e^{t/\sqrt{\beta}}-1)} = \sum_{k=0}^{M-1} b_k(\beta)t^k + O(t^M).$$

The first few coefficients are given by

$$b_0(\beta) = 1, \qquad b_1(\beta) = -\frac{1+\beta}{2\sqrt{\beta}}, \qquad b_2(\beta) = \frac{1+3\beta+\beta^2}{12\beta},$$
$$b_3(\beta) = -\frac{1+\beta}{24\sqrt{\beta}}, \qquad b_4(\beta) = -\frac{1-5\beta^2+\beta^4}{720\beta^2}, \qquad b_5(\beta) = \frac{1+\beta^3}{1440\beta^{3/2}}.$$

Thus we find that

$$\begin{aligned}\int_{\Gamma_0} f(t)\,dt &= \sum_{k=0}^{M-1} a_k(\beta,\sigma,\tau)\int_{\Gamma_0} e^{t\lambda}t^{k-3-\gamma}\,dt + e^{\varepsilon\lambda}O(\varepsilon^{M-2-\gamma}) \\ &= \sum_{k=0}^{M-1} a_k(\beta,\sigma,\tau)\int_{\Gamma_1} e^{\lambda t}t^{k-3-\gamma}\,dt + e^{\varepsilon\lambda}O(\varepsilon^{M-2-\gamma}) + e^{-\varepsilon\lambda}O(\varepsilon^{-2-\gamma}),\end{aligned}$$

where we used that

$$\int_{\varepsilon}^{\infty} e^{-\lambda t}t^{k-3-\gamma}\,dt \le \sup_{t\ge\varepsilon}(e^{-\lambda t}t^k)\int_{\varepsilon}^{\infty} t^{-3-\gamma}\,dt = \frac{e^{-\varepsilon\lambda}\varepsilon^{k-2-\gamma}}{2+\gamma},$$

provided $\varepsilon\lambda \ge k$.

Recall Hankel's integral representation for the reciprocal Γ function [21, eq. 5.9.2]:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int e^t t^{-z}\,dt,$$

where the integral is over a contour which encircles the origin in the positively oriented direction, beginning and returning to $-\infty$ while respecting the branch cut along the negative real axis. By a change of variables we find that $\int_{\Gamma_1} e^{\lambda t}t^{k-3-\gamma}\,dt = \frac{2\pi i\lambda^{2+\gamma-k}}{\Gamma(3+\gamma-k)}$.

Therefore we conclude that

$$\begin{aligned}\frac{\Gamma(1+\gamma)}{2\pi i}\int_{\Gamma_1} f(t)\,dt = \sum_{k=0}^{M-1} & a_k(\beta,\sigma,\tau)\frac{\Gamma(1+\gamma)}{\Gamma(3+\gamma-k)}\lambda^{2-k+\gamma} \\ &+ e^{-\varepsilon\lambda}O(\varepsilon^{-2-\gamma}) + e^{\varepsilon\lambda}O(\varepsilon^{M-2-\gamma}).\end{aligned}$$

Choose $\varepsilon = \varepsilon(\lambda)$ to solve $e^{-\lambda\varepsilon} = \varepsilon^{M/2}$. For large enough λ this choice satisfies the requirements above and the error terms become

$$O(\varepsilon(\lambda)^{M/2-2-\gamma}) = o(\lambda^{-M/2+2+\gamma+\delta}), \quad \forall\delta > 0,$$

since $\varepsilon(\lambda) = O(\log(\lambda)/\lambda) = o(\lambda^{-1+\delta})$ for any $\delta > 0$.

Moving unnecessary parts into the error term, we have for any $M' \in \mathbb{N}$ and $\delta > 0$ that

$$\frac{\Gamma(1+\gamma)}{2\pi i}\int_{\Gamma_1} f(t)\,dt = \sum_{k=0}^{M'+1} \alpha_k \lambda^{2-k+\gamma} + o(\lambda^{-M'+\gamma+\delta}),$$

where

$$\alpha_k(\beta,\sigma,\tau,\gamma) = a_k(\beta,\sigma,\tau)\frac{\Gamma(1+\gamma)}{\Gamma(3+\gamma-k)}. \tag{29}$$

Oscillatory part. We now turn our attention to the sum of residues

$$\Gamma(1+\gamma)\sum_{t\in\mathcal{P}(f)} \operatorname{Res}(f,t).$$

Simple poles. If $t = 2\pi i k\sqrt{\beta}$, with $k \in \mathbb{Z}\setminus\{0\}$ such that $\beta k \notin \mathbb{Z}$, then it is straightforward to calculate the residue of f at t, yielding:

$$\operatorname{Res}(f, 2\pi i k\sqrt{\beta}) = \beta^{-\gamma/2}\frac{e^{2\pi i k(\lambda\sqrt{\beta}-\sigma\beta-\tau)}}{(2\pi i k)^{1+\gamma}(e^{2\pi i k\beta}-1)}.$$

If instead $t = 2\pi i k/\sqrt{\beta}$, with $k \in \mathbb{Z}\setminus\{0\}$ such that $k/\beta \notin \mathbb{Z}$, then an almost identical calculation leads to:

$$\operatorname{Res}(f, 2\pi i k/\sqrt{\beta}) = \beta^{\gamma/2}\frac{e^{2\pi i k(\lambda/\sqrt{\beta}-\sigma-\tau/\beta)}}{(2\pi i k)^{1+\gamma}(e^{2\pi i k/\beta}-1)}.$$

Let $x_1 = \lambda\sqrt{\beta} - \sigma\beta - \tau$ and $x_2 = \lambda/\sqrt{\beta} - \sigma - \tau/\beta$. Combining the contributions from k and $-k$ one obtains that

$$\begin{aligned}
\sum_{t\in\mathcal{P}_1}\operatorname{Res}(f,t) &= \frac{\beta^{-\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ \beta k\notin\mathbb{N}}}\frac{1}{k^{1+\gamma}}\left(\frac{e^{2\pi i k x_1}}{e^{i\pi(1+\gamma)/2}(e^{2\pi i k\beta}-1)} + \frac{e^{-2\pi i k x_1}}{e^{-i\pi(1+\gamma)/2}(e^{-2\pi i k\beta}-1)}\right)\\
&\quad + \frac{\beta^{\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ k/\beta\notin\mathbb{N}}}\frac{1}{k^{1+\gamma}}\left(\frac{e^{2\pi i k x_2}}{e^{i\pi(1+\gamma)/2}(e^{2\pi i k/\beta}-1)} + \frac{e^{-2\pi i k x_2}}{e^{-i\pi(1+\gamma)/2}(e^{-2\pi i k/\beta}-1)}\right)\\
&= \frac{\beta^{-\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ \beta k\notin\mathbb{N}}}\frac{1}{k^{1+\gamma}}\left(\frac{\sin(\pi k(2x_1-\beta)-\frac{\pi}{2}(1+\gamma))}{\sin(\pi k\beta)}\right)\\
&\quad + \frac{\beta^{\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ k/\beta\notin\mathbb{N}}}\frac{1}{k^{1+\gamma}}\left(\frac{\sin(\pi k(2x_2-1/\beta)-\frac{\pi}{2}(1+\gamma))}{\sin(\pi k/\beta)}\right),
\end{aligned}$$

where $\mathcal{P}_1$ denotes the simple poles of f.

Let $\beta = \frac{\mu}{\nu}$, with $\gcd(\mu,\nu) = 1$, we shall show that the first of the above series is absolutely convergent, the second can be treated identically. Since $\gcd(\mu,\nu) = 1$ we have that $\frac{\mu}{\nu}k \notin \mathbb{N}$

if and only if $\frac{k}{\nu} \notin \mathbb{N}$. We find that

$$\begin{aligned}\frac{\nu^{\gamma/2}}{\mu^{\gamma/2}} \sum_{\substack{k\in\mathbb{N}\\ k/\nu\notin\mathbb{N}}} \Big|\frac{\sin(\pi k(2x_1-\mu/\nu)-\frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\mu/\nu)}\Big| &\le \frac{\nu^{\gamma/2}}{\mu^{\gamma/2}} \sum_{\substack{k\in\mathbb{N}\\ k/\nu\notin\mathbb{N}}} \frac{1}{k^{1+\gamma}}\frac{1}{|\sin(\pi k\mu/\nu)|}\\
&= \frac{\nu^{\gamma/2}}{\mu^{\gamma/2}} \sum_{l=1}^{\nu-1}\sum_{j=0}^{\infty} \frac{1}{(j\nu+l)^{1+\gamma}}\frac{1}{|\sin(\pi l\mu/\nu)|}\\
&\le \frac{\nu^{\gamma/2}}{\mu^{\gamma/2}} \sum_{l=1}^{\nu-1} \frac{1}{|\sin(\pi l\mu/\nu)|}\left(\frac{1}{l^{1+\gamma}}+\sum_{j=1}^{\infty}\frac{1}{(j\nu)^{1+\gamma}}\right)\\
&= \frac{\nu^{\gamma/2}}{\mu^{\gamma/2}} \sum_{l=1}^{\nu-1} \frac{1}{|\sin(\pi l\mu/\nu)|}\left(\frac{1}{l^{1+\gamma}}+\frac{\zeta(1+\gamma)}{\nu^{1+\gamma}}\right),\end{aligned}$$

implying that the series is absolutely convergent.

Degree-two poles. When $\beta = \frac{\mu}{\nu}$ then f has poles of degree two at $t = 2\pi i\sqrt{\mu\nu}\,k$, for $k \in \mathbb{Z}\setminus\{0\}$. The residues at these poles can be calculated:

$$\operatorname{Res}(f, 2\pi i\sqrt{\mu\nu}\,k) = \frac{e^{2i\pi k(\lambda\sqrt{\mu\nu}-\mu\sigma-\nu\tau)}(\gamma-2\pi ik(\lambda\sqrt{\mu\nu}-\mu(\sigma+\frac12)-\nu(\tau+\frac12))+1)}{(2\pi)^{\gamma+2}(ik)^{\gamma}k^2(\mu\nu)^{1+\gamma/2}}.$$

It is clear that the sum of these residues is absolutely convergent, which validates our use of the residue theorem in (28) in the case of rational β.

Letting $x_3 = \{\sqrt{\mu\nu}\,\lambda - \mu\sigma - \nu\tau\}$ we find that

$$\begin{aligned}\sum_{t\in\mathcal{P}_2}\operatorname{Res}(f,t) &= -\frac{(1+\gamma)}{(\mu\nu)^{1+\gamma/2}}\left[e^{-\frac{i\pi}{2}(2+\gamma)}\sum_{k=1}^{\infty}\frac{e^{2\pi ikx_3}}{(2\pi k)^{2+\gamma}} + e^{\frac{i\pi}{2}(2+\gamma)}\sum_{k=1}^{\infty}\frac{e^{-2\pi ikx_3}}{(2\pi k)^{2+\gamma}}\right]\\
&\quad+\frac{\lambda\sqrt{\mu\nu}-\mu\big(\sigma+\frac12\big)-\nu\big(\tau+\frac12\big)}{(\mu\nu)^{1+\gamma/2}}\left[e^{-\frac{i\pi}{2}(1+\gamma)}\sum_{k=1}^{\infty}\frac{e^{2\pi ikx_3}}{(2\pi k)^{1+\gamma}} + e^{\frac{i\pi}{2}(1+\gamma)}\sum_{k=1}^{\infty}\frac{e^{-2\pi ikx_3}}{(2\pi k)^{1+\gamma}}\right]\\
&= \frac{\lambda\zeta(-\gamma,x_3)-\zeta(-1-\gamma,x_3)/\sqrt{\mu\nu}-\Big(\big(\sigma+\frac12\big)\sqrt{\frac{\mu}{\nu}}+\big(\tau+\frac12\big)/\sqrt{\frac{\mu}{\nu}}\Big)\zeta(-\gamma,x_3)}{(\mu\nu)^{(1+\gamma)/2}\Gamma(1+\gamma)},\end{aligned}$$

where we made use of [21, eq. 25.12.13], and $\mathcal{P}_2$ denotes the set of degree-two poles of f.

5.2. **Irrational** β**.** For $\beta\in\mathbb{R}_+\backslash\mathbb{Q}$ the calculation leading to the precise asymptotic expansion of $R^{\gamma}_{\sigma,\tau}$ is slightly more complicated. The complication stems from the fact that we do not know if the sum of residues in (28) is absolutely convergent. Hence we cannot justify our use of the residue theorem as above. However, by choosing a λ-dependent contour where we only use the residue theorem for bounded contours one can obtain the desired result.

Fix $\lambda > \sigma\sqrt{\beta} + \tau/\sqrt{\beta}$. From the residue theorem we find that for $c > 0$ and $\Lambda > 1$ to be chosen later

$$\begin{aligned}
R^{\gamma}_{\sigma,\tau}(\beta,\lambda) &= \frac{\Gamma(1+\gamma)}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(t)\,dt \\
&= \frac{\Gamma(1+\gamma)}{2\pi i}\left(\int_{\Gamma_0} f(t)\,dt + \int_{\Gamma^{\pm}_{\Lambda,\infty}} f(t)\,dt + \int_{\Gamma^{\pm}_{\varepsilon,\Lambda}} f(t)\,dt + \int_{\Gamma^{\pm}_{\varepsilon,c}} f(t)\,dt\right) \\
&\quad + \Gamma(1+\gamma)\sum_{\substack{t\in\mathcal{P}(f)\\ |\Im(t)|\in(0,\Lambda)}} \operatorname{Res}(f,t),
\end{aligned}$$

where Γ_0, ε are as before and

$$\begin{aligned}
\Gamma^{\pm}_{\Lambda,\infty} &= (c \pm i\Lambda, c \pm i\infty), \\
\Gamma^{\pm}_{\varepsilon,\Lambda} &= (-\varepsilon \pm i0, -\varepsilon \pm i\Lambda), \\
\Gamma^{\pm}_{\varepsilon,c} &= (-\varepsilon \pm i\Lambda, c \pm i\Lambda).
\end{aligned}$$

The integral over Γ_0 can be computed precisely as in the case of rational β:

$$\frac{\Gamma(1+\gamma)}{2\pi i}\int_{\Gamma_0} f(t)\,dt = \sum_{k=0}^{M+1}\alpha_k\lambda^{2-k+\gamma} + o(\lambda^{-M+\gamma+\delta}),$$

for any $M \in \mathbb{N}$, $\delta > 0$.

There are now only simple poles, the residues at which can be calculated as before:

$$\begin{aligned}
\sum_{\substack{t\in\mathcal{P}(f)\\ |\Im(t)|\in(0,\Lambda)}} \operatorname{Res}(f,t) &= \frac{\beta^{-\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ 2\pi k\sqrt{\beta}<\Lambda}} \frac{\sin(\pi k(2\lambda\sqrt{\beta} - (1+2\sigma)\beta - 2\tau) - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k\beta)} \\
&\quad + \frac{\beta^{\gamma/2}}{(2\pi)^{1+\gamma}}\sum_{\substack{k\in\mathbb{N}\\ 2\pi k/\sqrt{\beta}<\Lambda}} \frac{\sin(\pi k(2\lambda/\sqrt{\beta} - 2\sigma - (1+2\tau)/\beta) - \frac{\pi}{2}(1+\gamma))}{k^{1+\gamma}\sin(\pi k/\beta)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\int_{\Gamma^{\pm}_{\Lambda,\infty}} f(t)\,dt\right| &\le \frac{e^{c(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{(e^{c\sqrt{\beta}}-1)(e^{c/\sqrt{\beta}}-1)}\int_{c\pm i\Lambda}^{c\pm i\infty} |t|^{-1-\gamma}\,dt \\
&\le \frac{e^{c(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{c^2}\int_{\Lambda}^{\infty} t^{-1-\gamma}\,dt \\
&= \frac{e^{c(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{\gamma c^2}\Lambda^{-\gamma},
\end{aligned}$$

since $e^x - 1 \geq x$, for $x \geq 0$. Furthermore,

$$
\begin{aligned}
\left|\int_{\Gamma^{\pm}_{\varepsilon,\Lambda}} f(t)\,dt\right| &\leq \frac{e^{-\varepsilon(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{(1-e^{-\varepsilon\sqrt{\beta}})(1-e^{-\varepsilon/\sqrt{\beta}})}\int_0^{\Lambda}(t^2+\varepsilon^2)^{-(1+\gamma)/2}\,dt \\
&\leq \frac{4e^{-\varepsilon(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{\varepsilon^2}\varepsilon^{-\gamma}\int_0^{\frac{\Lambda}{\sqrt{\Lambda^2+\varepsilon^2}}}(1-z^2)^{-1+\gamma/2}\,dz \\
&\leq \frac{2\sqrt{\pi}\,\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{1+\gamma}{2})}e^{-\varepsilon(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}\varepsilon^{-2-\gamma} \\
&= o(\lambda^{-M+\gamma+\delta}),
\end{aligned}
$$

where we used that $1-e^{-x} \geq x/2$, for $x \geq 0$, and the change of variables $t = \frac{\varepsilon z}{\sqrt{1-z^2}}$.

Finally, for the two last segments of the contour we firstly have that by changing Λ by something smaller than $2\pi\min\{\sqrt{\beta}, 1/\sqrt{\beta}\}$ we can choose Λ so that $\operatorname{dist}(i\Lambda, \mathcal{P}(f)) \geq \frac{\pi}{2}\min\{\sqrt{\beta}, 1/\sqrt{\beta}\}$, that is $\operatorname{dist}(\Lambda, 2\pi\sqrt{\beta}\mathbb{Z}\cup 2\pi/\sqrt{\beta}\mathbb{Z}) \geq \frac{\pi}{2}\min\{\sqrt{\beta}, 1/\sqrt{\beta}\}$. Hence

$$
\begin{aligned}
&\operatorname{dist}(\Lambda\sqrt{\beta}, 2\pi\mathbb{Z}) \geq \operatorname{dist}(\Lambda\sqrt{\beta}, 2\pi\beta\mathbb{Z}\cup 2\pi\mathbb{Z}) \geq \tfrac{\pi}{2}\sqrt{\beta}\min\{\sqrt{\beta}, \tfrac{1}{\sqrt{\beta}}\} = \tfrac{\pi}{2}\min\{\beta, 1\}, \\
&\operatorname{dist}(\tfrac{\Lambda}{\sqrt{\beta}}, 2\pi\mathbb{Z}) \geq \operatorname{dist}(\tfrac{\Lambda}{\sqrt{\beta}}, 2\pi\mathbb{Z}\cup \tfrac{2\pi}{\beta}\mathbb{Z}) \geq \tfrac{\pi}{2\sqrt{\beta}}\min\{\sqrt{\beta}, \tfrac{1}{\sqrt{\beta}}\} = \tfrac{\pi}{2}\min\{1, \tfrac{1}{\beta}\}.
\end{aligned}
$$

For $\Re(z) \geq -\log(2)$,

$$
|e^z - 1|^2 = e^{2\Re(z)} - 2e^{\Re(z)}\cos(\Im(z)) + 1 \geq 1 - \cos(\Im(z)) \geq \frac{2}{\pi^2}\operatorname{dist}(\Im(z), 2\pi\mathbb{Z})^2.
$$

Here the first inequality follows from that $g(x,y) = e^{2x} - (2e^x - 1)\cos(y)$ is non-negative when $x \geq -\log(2)$. Indeed, if $\cos(y) \leq 0$ this is clearly the case, and if $\cos(y) \geq 0$ this can be seen by writing g as $(e^x - \cos(y))^2 + (1-\cos(y))\cos(y)$.

For $t \in \Gamma^{\pm}_{\varepsilon,c}$, we thus have that $|(1-e^{t\sqrt{\beta}})(1-e^{t/\sqrt{\beta}})| \geq \frac{1}{2}\min\{\beta, 1\}\min\{1, 1/\beta\} = \frac{1}{2}\min\{\beta, 1/\beta\}$. Therefore

$$
\begin{aligned}
\left|\int_{\Gamma^{\pm}_{\varepsilon,c}} f(t)\,dt\right| &\leq \frac{2e^{c(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{\min\{\beta, 1/\beta\}}\int_{-\varepsilon\pm i\Lambda}^{c\pm i\Lambda}|t|^{-1-\gamma}\,dt \\
&\leq \frac{2e^{c(\lambda-\sigma\sqrt{\beta}-\tau/\sqrt{\beta})}}{\min\{\beta, 1/\beta\}}\Lambda^{-1-\gamma}(c+\varepsilon).
\end{aligned}
$$

What remains is to choose Λ, c appropriately. If $c = O(\lambda^{-\alpha})$ and $\Lambda = O(\lambda^{\beta})$, for some $\alpha \geq 1$ and $\beta > 0$, then the errors are of orders

$$
(\lambda^{-\alpha} + \log(\lambda)\lambda^{-1})\lambda^{-\beta(1+\gamma)} \sim \log(\lambda)\lambda^{-1-\beta(1+\gamma)}, \qquad \lambda^{2\alpha-\beta\gamma}, \qquad \lambda^{-M+\gamma+\delta}.
$$

The errors contributing are thus only the last two. Hence larger α only makes things worse so we choose $\alpha = 1$, and β so that $2 - \beta\gamma = -M + \gamma$, that is $\beta = \frac{M+2-\gamma}{\gamma}$. This choice yields the desired expansion with the claimed remainder term $o(\lambda^{-M+\gamma+\delta})$, for any $\delta > 0$.

5.3. **Bounding** $\operatorname{Osc}(\beta,\lambda)$**.** The only remaining part to complete the proof of Theorem 1.6 is to prove that the sum of oscillatory terms is $O(\lambda)$ uniformly for β in compact subsets of $\mathbb{R}_+$. To this end we make use of the following one-dimensional asymptotic expansion:

Lemma 5.1 ([11, Lemma 2.1])**.** *For $\gamma > 0$ we have an expansion*

$$\sum_{k=1}^{\infty}(\lambda-k)_+^\gamma = \sum_{k=0}^{\lceil\gamma\rceil}\rho_k(\gamma)\lambda^{1+\gamma-k}+O(1),$$

as $\lambda\to\infty$.

Using Lemma 5.1 we find that

$$\begin{aligned}
R^\gamma_{\sigma,\tau}(\beta,\lambda) &= \sum_{k\in\mathbb{N}^2}(\lambda-(k_1+\sigma)\sqrt{\beta}-(k_2+\tau)/\sqrt{\beta})_+^\gamma\\
&= \frac{1}{\beta^{\gamma/2}}\sum_{k_1=1}^{\lfloor\lambda/\sqrt{\beta}-\tau/\beta-\sigma\rfloor}\Big(\sum_{n=0}^{\lceil1+\gamma\rceil}\rho_n(\gamma)(\sqrt{\beta}\lambda-\tau-(k_1+\sigma)\beta)^{1+\gamma-n}+O(1)\Big)\\
&= \sum_{n=0}^{\lceil1+\gamma\rceil}\Big(\beta^{1-n+\gamma/2}\rho_n(\gamma)\sum_{k_1\geq1}(\lambda/\sqrt{\beta}-\tau/\beta-\sigma-k_1)_+^{1+\gamma-n}\Big)+O(\lambda)\\
&= \sum_{\substack{n,m\geq0\\ m+n<2+\gamma}}\beta^{(n+m)/2}\rho_n(\gamma)\rho_m(1+\gamma-n)\lambda^{2+\gamma-n-m}\Big(1-\frac{\sigma\beta+\tau}{\lambda\sqrt{\beta}}\Big)_+^{2+\gamma-n-m}\\
&\quad+O(\lambda),
\end{aligned}$$

where the error is uniform for β on compact subsets of $\mathbb{R}_+$. By expanding the $(1-c/\lambda)^\eta$ terms in the sum up to $O(\lambda^{-1-\gamma+n+m})$ we obtain an asymptotic expansion of $R^\gamma_{\sigma,\tau}$ up to $O(\lambda)$. Comparing this to the precise asymptotics we obtained above leads us to conclude that the $\operatorname{Osc}(\beta,\lambda)=O(\lambda)$ locally uniformly in β.

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References

1. M. Aizenman and E. H. Lieb, *On semiclassical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. A **66** (1978), no. 6, 427–429.
2. P. R. S. Antunes and P. Freitas, *Optimal spectral rectangles and lattice ellipses*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **469** (2013), no. 2150.
3. S. Ariturk and R. S. Laugesen, *Optimal stretching for lattice points under convex curves*, Portugaliae Mathematica **74** (2017), no. 2, 91–114.

4. M. van den Berg, D. Bucur, and K. Gittins, *Maximising Neumann eigenvalues on rectangles*, Bull. Lond. Math. Soc. **48** (2016), no. 5, 877–894.
5. M. van den Berg and K. Gittins, *Minimising Dirichlet eigenvalues on cuboids of unit measure*, Mathematika **63** (2017), 468–482.
6. R. de la Bretèche, *Preuve de la conjecture de Lieb–Thirring dans le cas des potentiels quadratiques strictement convexes*, Ann. Inst. H. Poincaré Phys. Théor. **70** (1999), no. 4, 369–380.
7. K. Gittins and S. Larson, *Asymptotic behaviour of cuboids optimising Laplacian eigenvalues*, Integral equations and operator theory **89** (2017), no. 5, 607–629.
8. J. Guo and W. Wang, *Lattice points in stretched model domains of finite type in* $\mathbb{R}^d$, J. Number Theory **191** (2017), 273–288.
9. B. Helffer and D. Robert, *Riesz means of bound states and semiclassical limit connected with a Lieb-Thirring's conjecture*, Asymptotic Anal. **3** (1990), no. 2, 91–103.
10. B. Helffer and D. Robert, *Riesz means of bounded states and semi-classical limit connected with a Lieb-Thirring conjecture. II*, Ann. Inst. H. Poincaré Phys. Théor. **53** (1990), no. 2, 139–147.
11. B. Helffer and J. Sjöstrand, *On diamagnetism and de Haas-van Alphen effect*, Ann. Inst. H. Poincaré Phys. Théor. **52** (1990), no. 4, 303–375.
12. A. Henrot (ed.), *Shape optimization and spectral theory*, de Gruyter Open, Warsaw, 2017.
13. A. Ivić, E. Krätzel, M. Kühleitner, and W. G. Nowak, *Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic*, Elementare und analytische Zahlentheorie, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, vol. 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006, pp. 89–128.
14. A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151** (1997), no. 2, 531–545.
15. A. Laptev, *On the Lieb–Thirring conjecture for a class of potentials*, The Maz′ya anniversary collection, Vol. 2 (Rostock, 1998), Oper. Theory Adv. Appl., vol. 110, Birkhäuser, Basel, 1999, pp. 227–234.
16. R. S. Laugesen and S. Liu, *Optimal stretching for lattice points and eigenvalues*, Arkiv för matematik **56** (2018), 111–145.
17. R. S. Laugesen and S. Liu, *Shifted lattices and asymptotically optimal ellipses*, The Journal of Analysis **26** (2018), 71–102.
18. N. F. Marshall, *Stretching convex domains to capture many lattice points*, Int. Math. Res. Not. IMRN, published online (2018).
19. N. F. Marshall and S. Steinerberger, *Triangles capturing many lattice points*, Mathematika **64** (2018), 551–582.
20. W. G. Nowak, *Integer points in large bodies*, Topics in mathematical analysis and applications, Springer Optim. Appl., vol. 94, Springer, Cham, 2014, pp. 583–599.
21. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders (eds.), *NIST Digital Library of Mathematical Functions*, Release 1.0.19.

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Paper G

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Exclusion Bounds for Extended Anyons

Simon Larson & Douglas Lundholm

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Abstract

We introduce a rigorous approach to the many-body spectral theory of extended anyons, that is quantum particles confined to two dimensions that interact via attached magnetic fluxes of finite extent. Our main results are many-body magnetic Hardy inequalities and local exclusion principles for these particles, leading to estimates for the ground-state energy of the anyon gas over the full range of the parameters. This brings out further non-trivial aspects in the dependence on the anyonic statistics parameter, and also gives improvements in the ideal (non-extended) case.

Contents

1. Introduction

In many-body quantum mechanics, the notion of particle indistinguishability and statistics plays a fundamental role. Namely, particles of the same kind are typically logically identical and fall into two classes: bosons or fermions, giving rise to such diverse phenomena as Bose–Einstein condensation and coherent propagation of light in the former case, and the Fermi sea with its implications for conduction bands, atomic structure, etc., in the latter. However, while these are the only two options for fundamental particles that propagate in three-dimensional space, for quantum systems confined to lower dimensions there is a possibility for effective particles (quasiparticles) escaping the usual boson/fermion dichotomy. We shall here consider the two-dimensional case where the quantum state of a system of N particles at positions $\mathbf{x}_j \in \mathbb{R}^2$ may be described by a square-integrable, normalized, complex wave function $\Psi : \mathbb{R}^{2N} \to \mathbb{C}$, where $|\Psi(\mathrm{x})|^2$ is interpreted as the probability density of finding the particles at positions $\mathrm{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$.[1] If the particles are indistinguishable the density needs to be symmetric under permutations of the particle labels:

$$|\Psi(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots, \mathbf{x}_N)|^2 = |\Psi(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_j, \dots, \mathbf{x}_N)|^2, \quad j \neq k. \tag{1.1}$$

However, the exact phase of Ψ is not an observable quantity and therefore (1.1) leaves room for an exchange phase:

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_k, \dots, \mathbf{x}_N) = e^{i\alpha\pi}\Psi(\mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_j, \dots, \mathbf{x}_N), \quad j \neq k, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ ($2\mathbb{Z}$-periodic) is called the statistics parameter. If $\alpha = 0$ the particles are called bosons (symmetric Ψ), and if $\alpha = 1$ they are fermions (antisymmetric Ψ). Because of the antisymmetry, fermions obey Pauli's exclusion principle [61] leading to Fermi–Dirac statistics, while bosons do not, leading to Bose–Einstein statistics. These are indeed the familiar possibilities found in introductory quantum mechanics textbooks, however, upon investigating the argument more carefully one realizes that one needs to be more precise with what is meant with the exchange $j \leftrightarrow k$ in (1.1)–(1.2). Namely, the exchange should in fact be viewed as a continuous loop in the manifold of positions x of N identical particles, and then topology plays a crucial role. Thus we *define* (1.2) to mean a continuous simple exchange of a single pair of particles (in two dimensions counterclockwise and with no other particles enclosed; furthermore the exchange phase can be shown to be independent of which pair of particles is considered). In three dimensions and higher, the direction of the exchange does not matter and a double exchange is topologically the same as no exchange; therefore the group of continuous exchanges reduces to the group of permutations and one ends up with the usual bosons or fermions. In two spatial dimensions, on the other hand, the exchange group is the braid group and it then turns out that *any* phase $e^{i\alpha\pi} \in \mathrm{U}(1)$ in (1.2) is allowed [23,31,76,77,79] (see

[1] Here we restrict to the simplest case of $\mathbb{C}$-valued wave functions corresponding to *abelian* anyons, while $\mathbb{C}^n$-valued, possibly *non-abelian*, anyons are also possible [20,58].

also [65, p. 386]). The corresponding particles are therefore called anyons [77]. We refer to [20,28,33,57,58,60,68,78] for extensive reviews on this topic.

The relative change of phase of the wave function Ψ with respect to changes of the coordinates may be geometrically understood as due to the curvature of a corresponding complex line bundle of which Ψ is a section. This is naturally described by a magnetic field, and in the case of anyons one may indeed model the above statistics phase as induced by a magnetic field of Aharonov–Bohm type. Namely, one could start with $\Psi \in L^2_{\mathrm{sym}}((\mathbb{R}^2)^N)$ (or $\Psi \in L^2_{\mathrm{asym}}((\mathbb{R}^2)^N)$) being bosonic (fermionic) and then attach magnetic fluxes to the particles so that their winding around each other gives rise to the correct phase (1.2). This is commonly called the magnetic gauge picture for anyons, and it is actually in this form that they may most realistically arise in a real physical system. The most promising such realization is in the context of the fractional quantum Hall effect (FQHE) [21,22,27,30,69], a strongly correlated planar electron (or bosonic atom [5,10,55,62,73]) system in a strong transverse external magnetic field, where particles have the freedom to bind magnetic flux and thereby become anyons [1,30,48]. However, in this scenario the flux typically has some extent determined by the experimental conditions, and one therefore talks about *extended* anyons [9,47,52,71] as opposed to the purely theoretical (but conceptually attractive) ideal anyons which are purely pointlike.[2] Denoting the size of the flux, say its radius if disk-shaped, by $R \geq 0$ we can thus talk about R-extended anyons, and one may also introduce a dimensionless parameter $\bar{\gamma} := R\bar{\varrho}^{1/2}$ to describe the state of the system, where $\bar{\varrho}$ denotes the average density of the particles. The parameter $\bar{\gamma}$ is the ratio of the magnetic dimension to the average interparticle distance and has therefore been called the magnetic filling ratio in [71,72].

Our interest in this paper is to study a free gas of such extended anyons, that is ignoring any additional interactions as a simplifying first step, and focusing on the most basic aspect: its ground-state energy. We consider this in the thermodynamic limit (cf. [6,36]), that is the limit as both the number of particles N and the volume (area) of the system V tends to infinity while keeping the density $\bar{\varrho} = N/V$ fixed. In the ideal non-interacting case, the quantum gas consisting of a large number of bosons or fermions in a large volume at fixed density has been completely understood since the early days of quantum mechanics and is nowadays often given as a textbook exercise, as it only amounts to adding up eigenvalues of a one-body operator. However, the purely anyonic case $\alpha \in (0, 1)$ still remains an unsolved problem after almost four decades, owing to the fact that the statistical many-body interaction cannot be completely removed in favor of a one-body description as for bosons and fermions. The simplest case of two anyons can be solved exactly [2,31,77], that of three and four anyons has been studied numerically [56,66,67], and beyond that various approximative descriptions have been proposed [7,8,17,25,64,71,72,74,75,78]. One of these is called average-field theory (cf. mean-field theory [47,78]) whereby the magnetic flux of the anyons is seen as sufficiently

[2] By ideal in this context we mean that the only interaction is statistical and independent of any energy, momentum or length scale (cf. [28, p. 146]).

spread out (in other words $\bar{\gamma}$ should be sufficiently large) so that the particles are effectively moving in a (locally) uniform magnetic field, say $B(\mathbf{x}) \sim 2\pi\alpha\varrho(\mathbf{x})$ where $2\pi\alpha$ is the flux of each anyon and $\varrho(\mathbf{x})$ the local density, and therefore have a definite magnetic ground-state energy given by that of the lowest Landau level, hence proportional to $|B| \sim 2\pi|\alpha|\varrho$. In other words the energy per particle in this approximation is given by

$$2\pi|\alpha|\bar{\varrho} \tag{1.3}$$

in the case of the homogeneous gas. Another approximation has been to assume that the gas is so dilute that only two-particle interactions are relevant [2,54].

Except for a small number of results concerning the mathematical formulation of the many-anyon problem [3,12,13,42], there has not been much progress on the rigorous mathematical side until recently. In [49] the case of ideal anyons was considered using a local approach involving a relative magnetic Hardy inequality and a local exclusion principle, leading to a first set of non-trivial rigorous bounds for the ground-state energy of the ideal anyon gas. These bounds, which will be outlined below, have an interesting non-trivial dependence on the statistics parameter α in that they depend, in the many-body limit, solely on the quantity

$$\alpha_* := \inf_{p,q\in\mathbb{Z}} \left|(2p+1)\alpha - 2q\right|, \tag{1.4}$$

which is zero unless α is an odd-numerator fraction $\alpha = \mu/\nu \in \mathbb{Q}$ (reduced, with $\nu \geq 1$) and in which case $\alpha_* = 1/\nu$. In [51] a fundamental question concerning operator domains for ideal anyons was settled and applications of the local energy bounds to interacting systems were considered. Also, the validity of an average-field approximation for the case of almost-bosonic ($\alpha \to 0$) R-extended anyons was proved in [47] (see also [11]).

Here we shall consider the homogeneous R-extended anyon gas in the thermodynamic limit and build on the local approach of [49] to prove a lower bound for the ground-state energy per particle with statistics parameter $\alpha \in \mathbb{R}\setminus\{0\}$ and magnetic filling ratio $\bar{\gamma} = R\bar{\varrho}^{1/2} \geq 0$ of the form

$$Ce(\alpha, \bar{\gamma})\bar{\varrho},$$

where $C > 0$ is a universal constant and (see Fig. 1 below for intermediate values)

$$e(\alpha, \gamma) \sim \begin{cases} \frac{2\pi}{|\ln \gamma|} + \pi(j'_{\alpha_*})^2 \geq 2\pi\alpha_*, & \gamma \to 0, \\ 2\pi|\alpha|, & \gamma \gtrsim 1. \end{cases}$$

Here j'_ν denotes the first positive zero of the derivative of the Bessel function J_ν (and $j'_0 := 0$). This bound effectively interpolates between a dilute regime involving (1.4) and a high-density regime with a dependence on α matching that of average-field theory (1.3). Also in the case of even-numerator α, where $\alpha_* = 0$, the bound is strictly positive but vanishes in the dilute limit in a way similar to that of a dilute Bose gas in two dimensions [41,63]. This may however not be so surprising in the case that $\alpha \in 2\mathbb{Z}$ (composite bosons; cf. [27]), considering the periodicity in the statistics parameter for ideal anyons.

1.1. The Extended Anyons Model

In order to state our results precisely we need to introduce some notation that will be used throughout the paper.

We take as our concrete model for R-extended anyons a set of N identical bosons, to each of which has been attached a magnetic field in the shape of a disk with radius R and total flux $2\pi\alpha$, and which is felt by all the other particles (cf. [9,47,48,52,71]). Such flux centered at the origin can be given explicitly by the magnetic vector potential $\alpha\mathbf{A}_0$ with

$$\mathbf{A}_0(\mathbf{x}) := \frac{(\mathbf{x}-\cdot)^\perp}{|\mathbf{x}-\cdot|^2} * \frac{\mathbb{1}_{B_R(0)}}{\pi R^2} = \frac{\mathbf{x}^\perp}{|\mathbf{x}|_R^2}, \qquad \operatorname{curl} \mathbf{A}_0(\mathbf{x}) = 2\pi \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}).$$

Here $(x, y)^\perp := (-y, x)$, that is a $\pi/2$ counterclockwise rotation, $B_R(\mathbf{x})$ denotes the open ball/disk of radius R centered at $\mathbf{x} \in \mathbb{R}^2$, and

$$|\mathbf{x}|_R := \max\{|\mathbf{x}|, R\},$$

which can be interpreted as a regularized distance. Starting from a conventional magnetic Hamiltonian formulation, the (non-relativistic) free kinetic energy operator is then

$$\hat{T}_\alpha := \sum_{j=1}^N D_j^2, \tag{1.5}$$

where we have normalized physical units so that $\hbar^2/(2m) = 1$ and the magnetically coupled momentum operator for each particle j is given by

$$D_j := -i\nabla_{\mathbf{x}_j} + \alpha\mathbf{A}_j(\mathbf{x}_j),$$

where

$$\mathbf{A}_j(\mathbf{x}) := \frac{(\mathbf{x}-\cdot)^\perp}{|\mathbf{x}-\cdot|^2} * \sum_{k\neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} = \sum_{k\neq j} \frac{(\mathbf{x}-\mathbf{x}_k)^\perp}{|\mathbf{x}-\mathbf{x}_k|_R^2},$$

corresponding to the total magnetic field felt by the particle $\mathbf{x}_j$

$$\operatorname{curl} \alpha\mathbf{A}_j = 2\pi\alpha \sum_{k\neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2} \xrightarrow{R\to 0} 2\pi\alpha \sum_{k\neq j} \delta_{\mathbf{x}_k}. \tag{1.6}$$

We note that this form for the magnetic interaction is not only convenient but also realistic from the perspective of the FQHE [48]. Also note that we allow for any $\alpha \in \mathbb{R}$ here.

The operator (1.5) acts on the bosonic Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ as an unbounded operator. Let us denote by $\mathscr{D}^N_{\alpha,R}$ the natural (minimal as well as maximal [51, Theorem 5]) domain of the magnetic gradient

$$\begin{aligned} D\colon L^2_{\mathrm{sym}}(\mathbb{R}^{2N};\mathbb{C}) &\to L^2(\mathbb{R}^{2N};\mathbb{C}^N) \\ \Psi &\mapsto D\Psi = (-i\nabla + \alpha\mathbf{A})\Psi = \big((-i\nabla_j + \alpha\mathbf{A}_j)\Psi\big)_{j=1}^N, \end{aligned}$$

then this is also the natural form domain of (1.5), and $\hat{T}_\alpha := D^*D$. In the case $R > 0$ (as well as for $\alpha = 0$) we have $\mathscr{D}^N_{\alpha,R} = H^1_{\mathrm{sym}}(\mathbb{R}^{2N})$, since $\mathbf{A}$ is then a bounded perturbation of $-i\nabla$. On the other hand, if $R = 0$ then $\mathbf{A}$ is singular and these spaces are typically different (see [51, Section 2.2]). For $R = 0$ and $\alpha \in 2\mathbb{Z}$ (respectively $\alpha \in 2\mathbb{Z}+1$), however, $\mathscr{D}^N_{\alpha,0}$ is gauge-equivalent to $\mathscr{D}^N_{0,0} = H^1_{\mathrm{sym}}(\mathbb{R}^{2N})$ (respectively $\mathscr{D}^N_{1,0} = U^{-1}H^1_{\mathrm{asym}}(\mathbb{R}^{2N})$):

$$D_{(\alpha+2n)} = U^{-2n}D_{(\alpha)}U^{2n}, \qquad \mathscr{D}^N_{\alpha+2n,0} = U^{-2n}\mathscr{D}^N_{\alpha,0}, \qquad n \in \mathbb{Z}, \tag{1.7}$$

where U is the isometry (singular gauge transformation)

$$U\colon L^2_{\mathrm{sym/asym}} \to L^2_{\mathrm{asym/sym}}, \qquad (U\Psi)(\mathrm{x}) := \prod_{1\le j<k\le N} \frac{z_j - z_k}{|z_j - z_k|}\Psi(\mathrm{x}),$$

with z_j the complex coordinate representatives of $\mathbf{x}_j$ given by identifying $\mathbb{R}^2$ with $\mathbb{C}$. In other words, for ideal anyons the spectrum of the operator $\hat{T}_\alpha$ is 2-periodic in α, however we will find that this is not the case for extended anyons.

We define the one-body density associated with any normalized state $\Psi \in L^2(\mathbb{R}^{2N})$ by

$$\varrho_\Psi(\mathbf{x}) := \sum_{j=1}^N \int_{\mathbb{R}^{2(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{k\neq j} d\mathbf{x}_k,$$

with $\int_\Omega \varrho_\Psi$ the expected number of particles to be found on $\Omega \subseteq \mathbb{R}^2$, while $\bar{\varrho} := N/|Q_0|$ denotes the average density if confined to a domain (typically a square) $Q_0 \subseteq \mathbb{R}^2$, that is for states Ψ with $\operatorname{supp}\Psi \subseteq Q_0^N$. Furthermore, with

$$\mathbb{\Delta} := \left\{\mathrm{x} \in (\mathbb{R}^2)^N : \exists\, j \neq k \text{ s.t. } \mathbf{x}_j = \mathbf{x}_k\right\},$$

the fat diagonal of the configuration space $(\mathbb{R}^2)^N$, we note that we may use the density of $C^\infty_c(\mathbb{R}^{2N}\setminus\mathbb{\Delta}) \cap L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ in the domain $\mathscr{D}^N_{\alpha,R}$ (again, see [51, Theorem 5]).

1.2. Main Bounds

We are now ready to state our main results for R-extended anyons. For the reader's convenience we outline and compare to the previously studied ideal case, which is also improved in several aspects in this work.

Our study of the homogeneous anyon gas relies on two key insights which were brought together in [49] for ideal anyons. On the one hand, we follow an idea originally used by Dyson and Lenard in their proof of the stability of matter for fermionic Coulomb systems [16] (see also [15,32]). They realized that the Pauli exclusion principle is strong enough (for many purposes, including the stability of matter) acting only between pairs or small numbers of particles. It is in fact sufficient that the local kinetic energy is strictly positive for two particles and that it grows at least linearly with the number of particles, in contrast to the true ground-state energy for fermions which grows with N according to the Weyl asymptotics for the sum of Laplacian eigenvalues, that is as $N^{1+2/d}$ in dimension d. We refer to such a bound as a local exclusion principle, and the method has recently been generalized to interacting bosonic gases with the Pauli principle replaced by repulsive interactions [45,46,49–51], and to point-interacting fermionic gases [19]. Essentially the idea is based on splitting the full domain to which the gas is confined into subdomains whose size is chosen so that the expected number of particles in each domain is not too large or, for that matter, too small. By estimating the local contribution to the energy from each subdomain one can obtain bounds for the total energy of the gas which are of the correct order.

The second key idea that we will use is based on the observation that a pair of fermions, due to their relative antisymmetry, experience an effective repulsion. This may be concretized in the following many-particle Hardy inequality for fermions [24, Theorem 2.8]:

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j \Psi|^2 \, dx \geq \frac{d^2}{N} \sum_{1 \leq j < k \leq N} \int_{\mathbb{R}^{dN}} \frac{|\Psi(x)|^2}{|\mathbf{x}_j - \mathbf{x}_k|^2} \, dx, \tag{1.8}$$

valid for any N-body state $\Psi \in H^1_{\text{asym}}(\mathbb{R}^{dN})$ in any dimension $d \geq 1$. Antisymmetry is in fact crucial here, as the inequality is not valid for bosons (the corresponding optimal Hardy constant vanishes in two dimensions). A local version of (1.8), given below, was obtained in [49] for ideal anyons, that is with $d = 2$ and with the right-hand side remaining linear in N, thus providing a local exclusion principle for anyons. It was shown that this inequality may be combined with the Dyson–Lenard approach to yield global bounds for the energy of the gas depending on the statistics parameter.

We start with an observation which is only helpful in the sufficiently extended case. Namely, for ideal anyons the singular magnetic potential $\mathbf{A}$ effectively excludes the diagonals $\triangle$ from the configuration space, much like a strong repulsive point interaction. For R-extended anyons we have instead the following effective repulsive short-range interaction of soft-disk type:

Lemma 1.1. (Short-range magnetic interaction) *For any $\alpha \in \mathbb{R}$, $R > 0$, $N \geq 1$, and $\Psi \in \mathscr{D}^N_{\alpha,R} = H^1_{\text{sym}}(\mathbb{R}^{2N})$ we have that*

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 \, dx \geq 2\pi|\alpha| \sum_{j \neq k} \int_{\mathbb{R}^{2N}} \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k) \, |\Psi|^2 \, dx. \tag{1.9}$$

Note that this repulsion is not at all as powerful as (1.8) upon taking the limit $R \to 0$ (or equivalently $\bar{\varrho} \to 0$), because functions in $H^1(\mathbb{R}^{2N})$ may be approximated by smooth functions supported away from diagonals as $R \to 0$ [51, Lemma 3], such that the right-hand side of (1.9) vanishes identically. However the inequality will be useful in the case that $R \sim \bar{\varrho}^{-1/2}$, that is $\bar{\gamma} \sim 1$.

Now, defining (denoted $C_{\alpha,N}$ in [49])

$$\alpha_N := \min_{p \in \{0,1,\ldots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|, \qquad \alpha_* := \inf_{N \geq 2} \alpha_N = \lim_{N \to \infty} \alpha_N, \tag{1.10}$$

we may state the following local many-particle magnetic Hardy inequality for ideal anyons which was given in [49, Theorem 4]:

Theorem 1.2. *Let $\alpha \in \mathbb{R}$, $R = 0$, $N \geq 1$ and $\Omega \subseteq \mathbb{R}^2$ be open and convex. Then, for any $\Psi \in \mathscr{D}^N_{\alpha,0}$,*

$$\sum_{j=1}^{N} \int_{\Omega^N} |D_j \Psi|^2 \, \mathrm{dx} \;\geq\; \frac{\alpha_N^2}{N} \sum_{j<k} \int_{\Omega^N} \frac{|\Psi|^2}{r_{jk}^2} \mathbb{1}_{\Omega \circ \Omega}(\mathbf{x}_j, \mathbf{x}_k) \, \mathrm{dx},$$

with the reduced support $\mathbb{1}_{\Omega\circ\Omega}(\mathbf{x}_j, \mathbf{x}_k) := \mathbb{1}_{B_{\delta(\mathbf{X}_{jk})}(0)}(\mathbf{r}_{jk})$, and

$$\mathbf{r}_{jk} := (\mathbf{x}_j - \mathbf{x}_k)/2, \quad \mathbf{X}_{jk} := (\mathbf{x}_j + \mathbf{x}_k)/2, \quad r_{jk} := |\mathbf{r}_{jk}|, \quad \delta(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \partial\Omega)$$

pairwise coordinates and distances.

For fermions, with $\alpha = 1$ and $\alpha_N = \alpha_* = 1$, considered on the full two-dimensional plane $\Omega = \mathbb{R}^2$, this is exactly the inequality (1.8). For anyons the dependence on the statistics parameter α comes in via the expressions (1.10) as will be explained below.

Our first main result is the following improvement and extension of Theorem 1.2 to R-extended anyons, thereby providing us with a concrete (and indeed useful) measure of the long-range effect of the statistical magnetic interaction:

Theorem 1.3. (Long-range magnetic interaction) *Let $\alpha \in \mathbb{R}$, $R \geq 0$, $N \geq n \geq 1$ and $\Omega \subseteq \mathbb{R}^2$ be open and convex. Then, for any $\Psi \in \mathscr{D}^N_{\alpha,R}$ and $\kappa \in [0, 1)$,*

$$\begin{aligned}
&\sum_{j=1}^{n} \int_{\Omega^n} |D_j \Psi|^2 \, \mathrm{dx} \\
&\quad \geq \frac{1}{n} \int_{\Omega^n} \left| \sum_{j=1}^{n} D_j \Psi \right|^2 \mathrm{dx} \\
&\qquad + \frac{1}{n} \sum_{j<k} \int_{\Omega^n} \left((1-\kappa) \left| \partial_{r_{jk}} |\Psi| \right|^2 + c(\kappa)^2 \frac{\alpha_N^2}{r_{jk}^2} \mathbb{1}_A(\mathbf{x}_j, \mathbf{x}_k) \, |\Psi|^2 \right) \mathrm{dx} \\
&\quad \geq 4\pi(1-\kappa) \frac{1}{n} \sum_{j<k} \int_{\Omega^n} g\left(\frac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \frac{3R/\delta(\mathbf{X}_{jk})}{1 - 3R/\delta(\mathbf{X}_{jk})} \right)^2 \frac{\mathbb{1}_A(\mathbf{x}_j, \mathbf{x}_k)}{4\pi \delta(\mathbf{X}_{jk})^2} |\Psi|^2 \, \mathrm{dx},
\end{aligned}$$

where D_j may depend on the positions of all N particles $\mathrm{x} \in \mathbb{R}^{2N}$, the support

$$\mathbb{1}_A(\mathbf{x}_j, \mathbf{x}_k) := \mathbb{1}_{B_{\delta(\mathbf{X}_{jk})-3R}(0) \setminus B_{3R}(0)}(\mathbf{r}_{jk})$$

describes a maximal annulus contained in Ω (with some R-dependent margins) in terms of the relative coordinate, and $g(\nu, \gamma)$ for $\nu \in \mathbb{R}_+$ and $0 \leq \gamma < 1$ is the square root of the smallest positive solution λ associated with the Bessel equation $-u'' - u'/r + \nu^2 u/r^2 = \lambda u$ on the interval $[\gamma, 1]$ with Neumann boundary conditions, while $g(\nu, \gamma) := \nu$ for $\gamma \geq 1$.

In the ideal case $R = 0$ the inequality is valid with $c(\kappa) \equiv 1$ (hence take $\kappa = 0$), while for any $R \geq 0$ it holds at least for $c(\kappa) = 4.7 \cdot 10^{-4}\kappa/(1 + 2\kappa)$.

Moreover, the function g has the following properties:

$$\nu \leq g(\nu, \gamma) \leq j'_\nu, \qquad g(\nu, \gamma) \sim \begin{cases} j'_\nu \geq \sqrt{2\nu}, & \gamma \to 0, \\ \nu, & \gamma \to 1, \end{cases}$$

where j'_ν denotes the first positive zero of the derivative of the Bessel function J_ν (and $j'_0 := 0$).

Theorem 1.3 will be applied to study the energy of the homogeneous anyon gas according to the local strategy outlined above. In such a setting Ω is typically not the domain to which our gas is confined, but rather a subdomain thereof, and n is the number of particles present in Ω while N is the total number of particles in the gas. This more complicated division of particles is needed in the statement of the theorem because the magnetic derivatives depend on *all* particles, not just those in Ω, which is even more relevant in the extended case.

We note that the above inequality may in some sense be viewed as a refinement (with respect to the angular dependence in pairwise relative coordinates) of the usual (pointwise) diamagnetic inequality:

Lemma 1.4. (Diamagnetic inequality) *For any $\alpha \in \mathbb{R}$, $R \geq 0$, $N \geq 1$ and $\Psi \in \mathscr{D}^N_{\alpha,R}$ we have that*

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 \,\mathrm{d}x \;\geq\; \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} \big|\nabla_j |\Psi|\big|^2 \,\mathrm{d}x.$$

For $R > 0$, the vector potential satisfies $\mathbf{A} \in L^\infty(\mathbb{R}^{2N}) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^{2N})$ and hence it is covered by standard theorems; see for example [34, Theorem 7.21]. For $R = 0$ it is not, but the above diamagnetic inequality still holds in this case, as was proved in [51, Lemma 4] (and actually our understanding of the form domain $\mathscr{D}^N_{\alpha,0}$ alluded to above depends on this general formulation of the inequality).

Note that $|\Psi| \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$. Therefore the diamagnetic inequality of Lemma 1.4 says that the kinetic energy for anyons is always higher than that for bosons, while the short-range inequality of Lemma 1.1 tells us that the anyons also feel an effective repulsion proportional to $|\alpha|$ whenever they overlap. Taking a combination of these two bounds would then correspond to a two-dimensional soft-disk repulsive Bose gas, whose energy in the dilute limit tends to zero logarithmically with the density (here the magnetic filling ratio $\bar{\gamma} := R\bar{\varrho}^{1/2} \to 0$) [41]. On the other hand, Theorem 1.3 provides a local bound for the energy in the form of a long-range inverse-square repulsion similar to (1.8), and whose strength depends on the fractionality of α via $\alpha_N \to \alpha_*$. While this 'statistical repulsion' does not change the above repulsive picture much in the regime of high densities ($\bar{\gamma} \gtrsim 1$) where the anyons already feel each other's magnetic fields by (partially) overlapping, it makes a significant difference in the dilute limit, actually resulting in a uniform bound for the energy from below in terms of $(j'_{\alpha_*})^2 \geq 2\alpha_*$.

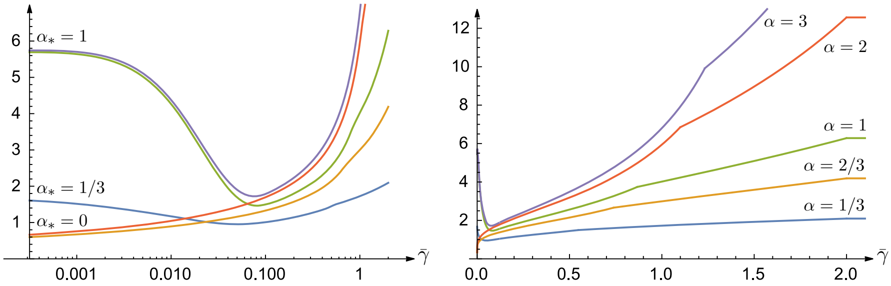


Fig. 1. The universal lower bound for $e(\alpha, \bar{\gamma})$ plotted as a function of $\bar{\gamma}$ for some fixed values of α, in the hypothetical case $C = 1$, $c = 1/\sqrt{3}$ for illustrative purposes. The figure to the right shows the general behavior over the full range of $\bar{\gamma}$, while that on the left shows the behavior in the dilute limit plotted in logarithmic scale where the long-range dependence on α_* becomes relevant

As discussed in [49], and further in [43], the reason for the dependence on $\alpha_N \geq \alpha_*$ and not directly α in the bounds of Theorems 1.2 and 1.3 is the local gauge invariance of the pairwise relative magnetic potential. Namely, in an exchange of a pair of particles additional flux may also be enclosed. Apart from the flux corresponding to the simple exchange (1.2), enclosing p other particles in such an exchange loop contributes an additional $2p$ multiples of the exchange flux, yielding the factor $2p + 1$ in (1.10). At the same time, any even multiple of a unit flux may be compensated for (gauged away) by an opposite and equally large orbital angular momentum of that same particle pair, thus explaining the subtraction of an arbitrary even integer $2q$ in (1.10). However, for odd-numerator rational α there can never be a complete cancellation of this type, and therefore there is always some long-range pair repulsion, $\alpha_* > 0$ [49, Proposition 5].

All these effects are summarized in the following theorem concerning the R-extended anyon gas, which is our second main result (see Fig. 1 for an illustration):

Theorem 1.5. (Universal bounds for the homogeneous anyon gas) *Let $e(\alpha, \bar{\gamma})$, where $\bar{\gamma} = R\bar{\varrho}^{1/2}$, denote the ground-state energy per particle and unit density of the extended anyon gas in the thermodynamic limit at fixed $\alpha \in \mathbb{R}$, $R \geq 0$ and density $\bar{\varrho} > 0$ where Dirichlet boundary conditions have been imposed, that is*

$$e(\alpha, \bar{\gamma}) := \liminf_{\substack{N, |Q_0| \to \infty \\ N/|Q_0| = \bar{\varrho}}} \left(\frac{1}{\bar{\varrho} N} \inf_{\substack{\Psi \in \mathscr{D}^N_{\alpha,R} \cap C_c^\infty(Q_0^N) \\ \|\Psi\|_2 = 1}} \langle \Psi, \hat{T}_\alpha \Psi \rangle \right).$$

Then

$$e(\alpha, \bar{\gamma}) \geq C \Bigg(2\pi \frac{|\alpha| \min\{2(1 - \bar{\gamma}^2/4)^{-1}, K_\alpha\}}{K_\alpha + 2|\alpha| \ln(2/\bar{\gamma})} \mathbb{1}_{\bar{\gamma} < 2} + 2\pi |\alpha| \mathbb{1}_{\bar{\gamma} \geq 2}$$
$$+ \pi g(c\alpha_*, 12\bar{\gamma}/\sqrt{2})^2 (1 - 12\bar{\gamma}/\sqrt{2})_+^3 \Bigg),$$

for some universal constants $C, c > 0$, $K_\alpha \geq 2$ (is defined in Lemma 5.1), and g as in Theorem 1.3. Furthermore, for any $\alpha \in \mathbb{R}$ we have for the ideal anyon gas that

$$e(\alpha, 0) \geq \frac{1}{2} 2\pi\alpha_* \big(1 - O\big(\alpha_*^{1/3}\big)\big). \tag{1.11}$$

As mentioned, our approach to obtain the above theorem is to first formulate the effects of the short- and long-range interactions in the form of local exclusion principles, an approach that goes back to Dyson and Lenard's original proof of the stability of matter for fermionic Coulomb systems [16]. This method was further developed in [45,46,49,51], not only to treat homogeneous gases but also to prove Lieb–Thirring inequalities (that is uniform kinetic energy bounds in accord with the Thomas–Fermi approximation for the inhomogeneous Fermi gas; cf. [35,37,38]) with the usual Pauli exclusion principle for fermions replaced by more general repulsive interactions for bosons. The reason for the factor $1/2$ in (1.11) compared to the expected value $2\pi\alpha_*$ (at least if comparing to the Fermi gas at $\alpha = \alpha_* = 1$ and assuming a linear interpolation to small α such that $\alpha = \alpha_*$) is that the long-range exclusion principle, which is applied locally on boxes of a tunable size, only increases linearly with the number of particles and is strongest on a scale where about two particles fit in each box. We provide further bounds for $e(\alpha, \bar{\gamma})$ in various parameter regimes in Theorem 6.1.

It should be remarked that our local exclusion principles also can be used to prove Lieb–Thirring inequalities. We postpone the extended case to future work but note that the ideal case is directly improved by the present results, namely replacing [49, Lemma 8] with the local exclusion principle of Lemma 5.3 below yields the following bounds for ideal anyons, where the constant $(j'_{\alpha_N})^2 \geq 2\alpha_N \geq \alpha_N^2$ improves the one in [49, Theorems 1 and 11]:

Theorem 1.6. (Lieb–Thirring inequality for ideal anyons) *With $\alpha \in \mathbb{R}$, $R = 0$, $N \geq 1$ and $\Psi \in \mathscr{D}^N_{\alpha,0}$ we have that*

$$\left\langle \Psi, \hat{T}_\alpha \Psi \right\rangle \geq C \left(j'_{\alpha_N}\right)^2 \int_{\mathbb{R}^2} \varrho_\Psi(\mathbf{x})^2 \, d\mathbf{x},$$

and if $V\colon \mathbb{R}^2 \to \mathbb{R}$ is an external one-body potential, acting by $\hat{V}(\mathrm{x}) := \sum_{j=1}^N V(\mathbf{x}_j)$, then

$$\left\langle \Psi, \left(\hat{T}_\alpha + \hat{V}\right) \Psi \right\rangle \geq -C' \left(j'_{\alpha_N}\right)^{-2} \int_{\mathbb{R}^2} V_-(\mathbf{x})^2 \, d\mathbf{x},$$

for some positive universal constants C and $C' = (4C)^{-1}$, and $V_\pm := \max\{\pm V, 0\}$.

The question concerning optimality of the above bounds with respect to their dependence on α in the dilute limit is a very difficult one, and will be discussed elsewhere [43]. However, we would like to point out that it was suggested in [50] (see also [44]) that a class of FQHE-inspired trial states with a clustering behavior could minimize the energy for certain fractions, and here we find additional support for this claim; cf. Fig. 3 below. Furthermore, there was in [50], then based on the weaker bounds of [49], a slight inconsistency in the behavior with respect to odd-numerator α which is remedied by the improved bounds presented here.

The structure of the paper is as follows. We lay the foundations in Sections 2 and 3 by proving the short-range bound of Lemma 1.1, and the basis for the long-range bound in the form of a relative magnetic Hardy inequality with symmetry. Then the main body of the paper, Section 4, is concerned with the application of this Hardy inequality to prove the long-range bound of Theorem 1.3. This turns out to become surprisingly challenging in the extended case due to the oscillatory nature of an effective potential, and in fact takes up the largest part of the proofs section. In Section 5 the long- and short-range bounds are applied to prove local exclusion principles for anyons, and finally in Section 6 we discuss the homogeneous anyon gas in the thermodynamic limit.

2. Short-Range Interaction

The short-range interaction given by Lemma 1.1 comes as a simple consequence of the well-known magnetic inequality (see for example [18, Lemma 1.4.1] or [4, p. 171])

$$\int_\Omega |(\nabla + i\mathbf{A})u|^2 \geq \pm \int_\Omega \operatorname{curl} \mathbf{A}\, |u|^2, \qquad u \in H_0^1(\Omega), \quad \Omega \subseteq \mathbb{R}^2. \tag{2.1}$$

This inequality also follows directly from integrating the straightforward identity

$$|(\nabla + i\mathbf{A})u|^2 = |((\partial_1 + iA_1) \pm i(\partial_2 + iA_2))u|^2 \pm \operatorname{curl} \mathbf{J}[u] \pm \mathbf{A} \cdot \nabla^\perp |u|^2,$$

with $\mathbf{J}[u] := \frac{i}{2}(u\nabla\bar{u} - \bar{u}\nabla u)$.

Proof of Lemma 1.1. Splitting the coordinates according to $\mathrm{x} = (\mathbf{x}_j; \mathrm{x}')$ for each particle j, we write for the left-hand side of (1.9)

$$\begin{aligned}
&\sum_{j=1}^N \int_{\mathbb{R}^{2(N-1)}} \int_{\mathbb{R}^2} |\left(\nabla_j + i\alpha\mathbf{A}_j\left(\mathbf{x}_j\right)\right)\Psi\left(\mathbf{x}_j; \mathrm{x}'\right)|^2 \,d\mathbf{x}_j d\mathrm{x}' \\
&\qquad \geq \sum_{j=1}^N \int_{\mathbb{R}^{2(N-1)}} \int_{\mathbb{R}^2} 2\pi|\alpha| \sum_{k\neq j} \frac{\mathbb{1}_{B_R(\mathbf{x}_k)}}{\pi R^2}\left(\mathbf{x}_j\right) |\Psi\left(\mathbf{x}_j; \mathrm{x}'\right)|^2 \,d\mathbf{x}_j d\mathrm{x}',
\end{aligned}$$

where we used the expression (1.6) for $\operatorname{curl} \alpha\mathbf{A}_j(\mathbf{x}_j)$ in (2.1). We have thus obtained the right-hand side of (1.9). □

We note that the Dirichlet boundary conditions on Ψ and u are in fact necessary here since the bound (2.1) is otherwise invalid, as can be seen by taking $\mathbf{A} = \beta\mathbf{A}_0$, $\beta \to 0$, and the trial state $u = 1$. Similarly, had we considered the inequality (1.9) locally on a small enough domain (compared to R) we would have found a contradiction as $\alpha \to 0$, unless Dirichlet boundary conditions are enforced.

3. Relative Magnetic Inequality

For the long-range statistical interaction between anyons we take the same starting point as in [49], namely, the core observation is the validity of a relative magnetic Hardy inequality which respects the symmetry of the anyon problem. Non-symmetric versions of this inequality were introduced and studied in [29] (one-particle version) and in [24, Theorem 2.7] (many-particle version); see also [53], [4, Chapter 5.5] and references therein. However, as was pointed out in [49], symmetry is crucial in order to obtain non-trivial bounds in the many-particle limit. We formulate the following version of the inequality quite generally.

Initially, consider a magnetic field $b\colon B_R(0) \to \mathbb{R}$ defined on a disk of radius $R > 0$, and assumed to be determined by a suitable continuous vector potential $\mathbf{a}\colon B_R(0) \to \mathbb{R}^2$ as $b = \operatorname{curl} \mathbf{a}$. Then the normalized flux inside a smaller disk of radius $r \in [0, R)$ is given by

$$\hat{\Phi}(r) := \frac{1}{2\pi} \int_{B_r(0)} b = \frac{1}{2\pi} \int_{\partial B_r(0)} \mathbf{a} \cdot d\mathbf{r}'. \tag{3.1}$$

Note that if we were only given $\mathbf{a}\colon \Omega \to \mathbb{R}^2$ on some annulus $\Omega = B_R(0)\setminus \bar{B}_{R'}(0)$, with $0 < R' < R$, that is if we only knew b on Ω (so that only the right-hand side of (3.1) makes sense for $r \in (R', R)$), then b can nevertheless be extended (non-uniquely) to the full interior $B_{R'}(0)$, for example by taking

$$b|_{B_{R'}(0)} = \frac{2\pi\hat{\Phi}(R')}{\pi(R')^2} \qquad \text{or} \qquad b|_{B_{R'}(0)} = 2\pi\hat{\Phi}(R')\delta_0,$$

with $\Phi(R')$ here defined in terms of $\mathbf{a}$ as in (3.1) (note that we are not considering extending $\mathbf{a}$). Then both expressions for $\hat{\Phi}(r)$ in (3.1) are well defined and agree for all $r \in (R', R)$. We also note that if the magnetic field is antipodal-symmetric on Ω, that is $b(-\mathbf{r}) = b(\mathbf{r})$ for all $\mathbf{r} \in \Omega$, then the corresponding potential must (if gauge-normalized correctly) be antipodal-*anti*symmetric, $\mathbf{a}(-\mathbf{r}) = -\mathbf{a}(\mathbf{r})$, $\mathbf{r} \in \Omega$, and vice versa.

Lemma 3.1. (Magnetic Hardy inequality with symmetry) *Let $\Omega = B_{R_2}(0)\setminus \bar{B}_{R_1}(0)$, with $R_2 > R_1 \geq 0$, be an annular domain in $\mathbb{R}^2$. Let $\mathbf{a}\colon \Omega \to \mathbb{R}^2$ be a continuous vector potential corresponding to a magnetic field b, $b|_\Omega = \operatorname{curl} \mathbf{a}$, that is defined on the entire disk $B_{R_2}(0)$ such that the normalized flux $\hat{\Phi}(r)$ given by (3.1) is finite for all $r \in (R_1, R_2)$. Furthermore, assume that $\mathbf{a}$ is antipodal-**antisymmetric** resp. b is antipodal-**symmetric** on Ω, that is $\mathbf{a}(-\mathbf{r}) = -\mathbf{a}(\mathbf{r})$ resp. $b(-\mathbf{r}) = b(\mathbf{r})$ for $\mathbf{r} \in \Omega$.*

*Then, for any antipodal-**symmetric** $u \in C^\infty(\Omega)$, that is with $u(-\mathbf{r}) = u(\mathbf{r})$ for all $\mathbf{r} \in \Omega$,*

$$\int_\Omega |(-i\nabla + \mathbf{a})u|^2 \, d\mathbf{r} \;\geq\; \int_\Omega \left(\big|\partial_r |u|\big|^2 + \inf_{k\in\mathbb{Z}} \big|\hat{\Phi}(r) - 2k\big|^2 \frac{|u|^2}{r^2} \right) d\mathbf{r}.$$

*Alternatively, if instead u is antipodal-**antisymmetric**, $u(-\mathbf{r}) = -u(\mathbf{r})$ for all $\mathbf{r} \in \Omega$, then*

$$\int_\Omega |(-i\nabla + \mathbf{a})u|^2 \,\mathrm{d}\mathbf{r} \geq \int_\Omega \left(\big|\partial_r |u|\big|^2 + \inf_{k\in\mathbb{Z}} \big|\hat{\Phi}(r) - (2k+1)\big|^2 \frac{|u|^2}{r^2} \right) \mathrm{d}\mathbf{r}.$$

Proof. We apply the techniques from [29], with symmetry taken into account as in [49, Lemma 2]. We start by letting $h[\mathbf{a}]$ denote the magnetic quadratic form on Ω,

$$\begin{aligned} h[\mathbf{a}](u) &:= \int_\Omega |(-i\nabla + \mathbf{a})u|^2 \,\mathrm{d}\mathbf{r} \\ &= \int_{R_1}^{R_2} \int_0^{2\pi} \big(|(-i\partial_r + a_r)u|^2 + r^{-2} |(-i\partial_\varphi + r a_\varphi)u|^2 \big) r \,\mathrm{d}\varphi \mathrm{d}r, \end{aligned}$$

where $a_r := r^{-1}\mathbf{r} \cdot \mathbf{a}$ and $a_\varphi := r^{-1}\mathbf{r}^\perp \cdot \mathbf{a}$. For the first term above we use the diamagnetic inequality $|(\partial_r + i a_r)u| \geq \big|\partial_r |u|\big|$, while for the second we can for each $r \in (R_1, R_2)$ explicitly diagonalize the self-adjoint operator $K_\varphi(r) := -i\partial_\varphi + r a_\varphi(r,\varphi)$ acting on $L^2(\mathbb{S}^1)$. The corresponding eigenvalues and normalized eigenfunctions of this operator are given by:

$$\begin{aligned} \lambda_k(r) &= -k + (2\pi)^{-1} r \int_0^{2\pi} a_\varphi(r,\varphi)\,\mathrm{d}\varphi = -k + \hat{\Phi}(r), \\ \psi_k(r,\varphi) &= (2\pi)^{-1/2} e^{i\left(\varphi\lambda_k(r) - r\int_0^\varphi a_\varphi(r,\eta)\,\mathrm{d}\eta\right)}, \end{aligned}$$

for $k \in \mathbb{Z}$. Because of the antipodal-antisymmetry of $\mathbf{a}$, implying antipodal-symmetry of a_φ, that is $a_\varphi(r,\varphi) = a_\varphi(r, \varphi+\pi)$, we have that

$$\psi_k(r, \varphi + \pi) = (-1)^k \psi_k(r,\varphi).$$

Therefore, only the even/odd terms will contribute upon expanding $u \in L^2_{\mathrm{sym/asym}}(\Omega)$ as

$$u(r,\varphi) = \sum_{k\in\mathbb{Z}} u_k(r)\psi_k(r,\varphi) = \sum_{k\in\mathbb{Z}_{e/o}} u_k(r)\psi_k(r,\varphi),$$

with $\mathbb{Z}_e := 2\mathbb{Z}$ and $\mathbb{Z}_o := 2\mathbb{Z}+1$.

By the above remarks and Parseval's identity we find that

$$\begin{aligned} h[\mathbf{a}](u) &= \int_{R_1}^{R_2} \int_0^{2\pi} |(\partial_r + i a_r)u|^2 \, r \,\mathrm{d}\varphi \mathrm{d}r + \int_{R_1}^{R_2} \sum_{k\in\mathbb{Z}_{e/o}} |\lambda_k(r)|^2 |u_k(r)|^2 \, r^{-1} \,\mathrm{d}r \\ &\geq \int_{R_1}^{R_2} \int_0^{2\pi} \big|\partial_r |u|\big|^2 \, r \,\mathrm{d}\varphi \,\mathrm{d}r + \int_{R_1}^{R_2} \inf_{k\in\mathbb{Z}_{e/o}} |\lambda_k(r)|^2 \sum_{k\in\mathbb{Z}_{e/o}} |u_k(r)|^2 \, r^{-1} \,\mathrm{d}r \\ &= \int_{R_1}^{R_2} \int_0^{2\pi} \big(\big|\partial_r |u|\big|^2 + r^{-2} \inf_{k\in\mathbb{Z}_{e/o}} |\lambda_k(r)|^2 |u|^2 \big) r \,\mathrm{d}\varphi \,\mathrm{d}r. \end{aligned}$$

Thus the estimate we are left with is

$$h[\mathbf{a}](u) \geq \int_{R_1}^{R_2} \int_0^{2\pi} \left(\left| \partial_r |u| \right|^2 + r^{-2} \inf_{k \in \mathbb{Z}_{e/o}} \left| \hat{\Phi}(r) - k \right|^2 |u|^2 \right) r \, \mathrm{d}\varphi \, \mathrm{d}r,$$

which proves the lemma. □

The above lemma not only extends the inequality of [49, Lemma 2] to more general (extended) magnetic fields, but also improves it by keeping the radial derivative. This turns out to be crucial in order to obtain an improved dependence on α in the dilute limit. We note that in [24] the radial derivatives were effectively discarded in two dimensions.

4. Analysis of the Long-Range Interaction

We set out to prove Theorem 1.3 and first note that by the remarks in Section 1.1 we may assume without loss of generality that $\Psi \in C_c^\infty(\mathbb{R}^{2N} \setminus \triangle)$. Proceeding as was done in [49] for the non-extended case $R = 0$, we start from the expression for the kinetic energy on a domain $\Omega \subseteq \mathbb{R}^2$,

$$\int_{\Omega^n} \sum_{j=1}^n |D_j \Psi|^2 \, \mathrm{dx}, \qquad \text{where} \qquad D_j = -i \nabla_{\mathbf{x}_j} + \alpha \sum_{\substack{k=1 \\ k \neq j}}^N \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2},$$

and we are considering the first n particles $\mathbf{x}_{j=1,\ldots,n} \in \Omega$ while the remaining $N-n$ ones may reside anywhere in $\mathbb{R}^2$. Using that, for any $\mathrm{z} = (\mathbf{z}_j)_j \in \mathbb{C}^n$,

$$\sum_{j=1}^n |\mathbf{z}_j|^2 = \frac{1}{n} \sum_{1 \leq j < k \leq n} |\mathbf{z}_j - \mathbf{z}_k|^2 + \frac{1}{n} \left| \sum_{j=1}^n \mathbf{z}_j \right|^2,$$

we have that

$$\begin{aligned} \int_{\Omega^n} \sum_{j=1}^n |D_j \Psi|^2 \, \mathrm{dx} &= \frac{1}{n} \sum_{1 \leq j < k \leq n} \int_{\Omega^{n-2}} \int_{\Omega^2} |(D_j - D_k)\Psi|^2 \, \mathrm{d}\mathbf{x}_j \mathrm{d}\mathbf{x}_k \prod_{l \neq j,k} \mathrm{d}\mathbf{x}_l \\ &\quad + \frac{1}{n} \int_{\Omega^n} \left| \sum_{j=1}^n D_j \Psi \right|^2 \mathrm{dx}, \end{aligned} \tag{4.1}$$

where we also note that the magnetic field present in the last (total momentum) term simplifies to

$$\sum_{j=1}^n D_j = -i \sum_{j=1}^n \nabla_{\mathbf{x}_j} + \alpha \sum_{j=1}^n \sum_{k=n+1}^N \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|_R^2},$$

by the antisymmetry of the vector potential, and thus vanishes if $n = N$.

We now study the inner integral in (4.1) for the $j < k$ particle pair, and introduce relative coordinates:

$$\mathbf{r}_{jk} := (\mathbf{x}_j - \mathbf{x}_k)/2, \qquad \mathbf{X}_{jk} := (\mathbf{x}_j + \mathbf{x}_k)/2, \qquad r_{jk} := |\mathbf{r}_{jk}|,$$

giving

$$\begin{aligned}
&\int_{\Omega^2} |(D_j - D_k)\Psi|^2 \, d\mathbf{x}_j d\mathbf{x}_k \\
&= \int_{\Omega^2} \Big| \Big(-i(\nabla_{\mathbf{x}_j} - \nabla_{\mathbf{x}_k}) + \alpha \sum_{l \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_l)^\perp}{|\mathbf{x}_j - \mathbf{x}_l|_R^2} - \alpha \sum_{l \neq k} \frac{(\mathbf{x}_k - \mathbf{x}_l)^\perp}{|\mathbf{x}_k - \mathbf{x}_l|_R^2} \Big) \Psi \Big|^2 d\mathbf{x}_j d\mathbf{x}_k \\
&= \int_{\Omega^2} \Big| \Big(-i\nabla_{\mathbf{r}_{jk}} + \alpha \mathbf{a}_0(\mathbf{r}_{jk}) + \alpha \sum_{l \neq j,k} (\mathbf{a}_l(\mathbf{X}_{jk}, \mathbf{r}_{jk}) - \mathbf{a}_l(\mathbf{X}_{jk}, -\mathbf{r}_{jk})) \Big) \Psi \Big|^2 d\mathbf{x}_j d\mathbf{x}_k,
\end{aligned} \tag{4.2}$$

where the relative vector potentials are given by

$$\mathbf{a}_0(\mathbf{r}) := \frac{4\mathbf{r}^\perp}{|2\mathbf{r}|_R^2} = \frac{\mathbf{r}^\perp}{|\mathbf{r}|_{R/2}^2} \quad \text{and} \quad \mathbf{a}_l(\mathbf{X}, \mathbf{r}) := \frac{(\mathbf{X} + \mathbf{r} - \mathbf{x}_l)^\perp}{|\mathbf{X} + \mathbf{r} - \mathbf{x}_l|_R^2}.$$

Hence, for any positions $\mathrm{x}' = (\mathbf{x}_1, \ldots, \cancel{\mathbf{x}_j}, \ldots, \cancel{\mathbf{x}_k}, \ldots, \mathbf{x}_N) \in \mathbb{R}^{2(N-2)}$ of the other particles and for each center-of-mass coordinate $\mathbf{X} = \mathbf{X}_{jk} \in \Omega$ of the particle pair, we observe that the resulting magnetic vector potential

$$\mathbf{a}(\mathbf{r}) := \alpha \mathbf{a}_0(\mathbf{r}) + \alpha \sum_{l \neq j,k} (\mathbf{a}_l(\mathbf{X}, \mathbf{r}) - \mathbf{a}_l(\mathbf{X}, -\mathbf{r}))$$

is antipodal-antisymmetric on the relative disk

$$\Omega_{\mathbf{X}} := B_{\delta(\mathbf{X})}(0), \qquad \delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial\Omega),$$

with a corresponding antipodal-symmetric magnetic field

$$b := \operatorname{curl} \mathbf{a} = 2\pi\alpha \left(\frac{\mathbb{1}_{B_{R/2}(0)}}{\pi(R/2)^2} + \sum_{l \neq j,k} \left(\frac{\mathbb{1}_{B_R(\mathbf{x}_l - \mathbf{X})}}{\pi R^2} + \frac{\mathbb{1}_{B_R(-(\mathbf{x}_l - \mathbf{X}))}}{\pi R^2} \right) \right) \tag{4.3}$$

(given here for $R > 0$). Also, the smooth function defined relative to $\mathbf{X}$ and x' by

$$u(\mathbf{r}) := \Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j = \mathbf{X} + \mathbf{r}, \ldots, \mathbf{x}_k = \mathbf{X} - \mathbf{r}, \ldots, \mathbf{x}_n, \ldots, \mathbf{x}_N)$$

is antipodal-symmetric on $\Omega_{\mathbf{X}}$. Hence, we may apply the relative magnetic Hardy inequality of Lemma 3.1 (for $R = 0$ we split into concentric annuli avoiding the $\mathbf{x}_l$ as in [49, Theorem 4]) to obtain that

$$\begin{aligned}
\int_{\Omega^2} |(D_j - D_k)\Psi|^2 \, d\mathbf{x}_j d\mathbf{x}_k &\geq \int_\Omega \int_{\Omega_{\mathbf{X}}} |(-i\nabla + \mathbf{a})u|^2 \, 4 \, d\mathbf{r} d\mathbf{X} \\
&\geq \int_\Omega \int_{\Omega_{\mathbf{X}}} \left(\big|\partial_r |u|\big|^2 + \frac{\rho(r)}{r^2} |u|^2 \right) 4 \, d\mathbf{r} d\mathbf{X},
\end{aligned} \tag{4.4}$$

where

$$\rho(r) := \inf_{q \in \mathbb{Z}} \left| \hat{\Phi}(r) - 2q \right|^2, \tag{4.5}$$

and $\hat{\Phi}(r)$ here, and in what follows, denotes the flux through the disk $B_r(\mathbf{X})$ of the magnetic field (4.3):

$$\hat{\Phi}(r) = \frac{1}{2\pi} \int_{\partial B_r(0)} \mathbf{a} \cdot \mathrm{d}\mathbf{r}' = \frac{1}{2\pi} \int_{B_r(0)} b.$$

Note that the magnetic field is induced by the particle configuration $(\mathrm{x}'; \mathbf{x}_j, \mathbf{x}_k)$, and the only dependence which remains after fixing x' in (4.1) and $\mathbf{X} = \mathbf{X}_{jk}$ in (4.4) is that of the relative coordinate $\mathbf{r} = \mathbf{r}_{jk}$. With the remaining particle positions expressed relative to the coordinate $\mathbf{X}$, $\mathbf{y}_l := \mathbf{x}_l - \mathbf{X}$, we can write the normalized flux $\hat{\Phi}(r)$ as:

$$\hat{\Phi}(r) = \alpha \left(\int_{B_r(0)} \frac{\mathbb{1}_{B_{R/2}(0)}}{\pi (R/2)^2} + 2 \sum_{l \neq j,k} \int_{B_r(0)} \frac{\mathbb{1}_{B_R(\mathbf{y}_l)}}{\pi R^2} \right). \tag{4.6}$$

Hence $\rho(r)$ depends only on the arbitrary but fixed configuration $(\mathbf{y}_l)_l \in \mathbb{R}^{2(N-2)}$.

By the above discussion, the problem of bounding the kinetic energy (4.1) has been reduced to studying the radial Schrödinger operator in (4.4) with explicit *scalar* interaction potential $\rho(r)/r^2$. This potential is essentially an inverse-square repulsion, modulated with a coupling strength $\rho(r)$ which measures how well the normalized flux $\hat{\Phi}(r)$ stabilizes away from the even integers. In the dilute situation the flux and hence also ρ would for the most part be constant, however we could have significant oscillations of $\rho(r)$ between one and zero whenever many particles are enclosed over short differences in the radial variable r (see Fig. 2). Controlling these oscillations turns out to be a significant challenge, and the entire remainder of this section shall be concerned with proving the following theorem, from which Theorem 1.3 follows.

Theorem 4.1. *For any* $0 \leq R \leq L/6$, $\kappa \in [0, 1]$, $u \in W^{1,2}([R, L], r\mathrm{d}r)$, *and* ρ *defined in* (4.5)–(4.6) *with* $(\mathbf{y}_l)_l \in \mathbb{R}^{2(N-2)}$ *arbitrary, we have that*

$$\int_R^L \left(|u'|^2 + \frac{\rho(r)}{r^2} |u|^2 \right) r \, \mathrm{d}r \geq \int_R^L \left((1-\kappa)|u'|^2 + c(\kappa)^2 \frac{\alpha_N^2}{r^2} \mathbb{1}_{[3R, L-3R]} |u|^2 \right) r \, \mathrm{d}r,$$

with $c(\kappa) = 4.7 \cdot 10^{-4} \kappa/(1 + 2\kappa)$. *In the case* $R = 0$ *we may take* $c(\kappa) \equiv 1$.

Remark 4.2. The margins which appear here as a cut-off for the potential are not optimal and could be improved with more work, to the cost of an even weaker constant. The main reason for the weakness of the constant $c(\kappa)$ is the fact that we have chosen to control the above form by means of filling the gaps around the zeros of the potential by smearing it over longer (but not too long) intervals, and that in the worst possible situation there are very large regions of intense oscillation and many such zeros.

By considering the special case $\alpha = \alpha_N = 1$ and densely packed, overlapping particles (that is when $\bar{\gamma}$ is large) distributed so that the effective magnetic field

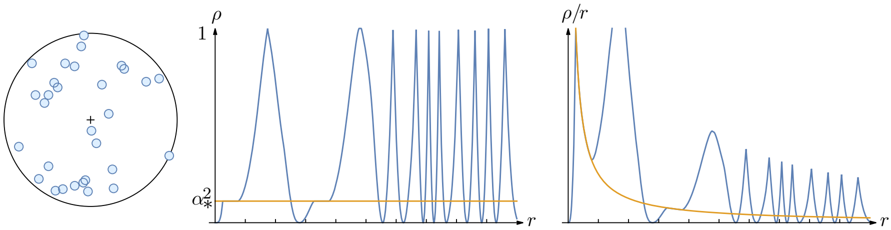


Fig. 2. The function $\rho(r)$ and the effective potential $\rho(r)/r$ for a random (uniformly distributed) configuration of 30 particles in a disk of radius $L = 20R$ with $\alpha = \alpha_* = 1/3$ plotted from $r = 0$ to $r = L$, where α_*^2 resp. α_*^2/r are shown for comparison. As one can see, the effective potential is generally quite a lot larger than α_*^2/r

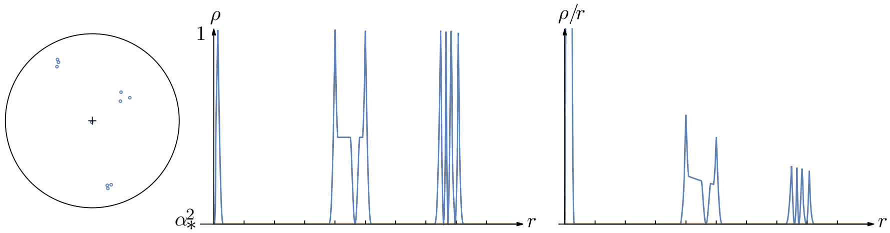


Fig. 3. The same as in Fig. 2, now for 10 particles in a disk of radius $L = 60R$ with $\alpha = 2/3$ ($\alpha_* = 0$), and with a single particle close to our center of mass and the remaining nine in clusters of three. Note that in this case the effective potential can become identically zero on long intervals

is approximately constant, we find that $c(\kappa)$ cannot be greater than $1/\sqrt{3}$, which is what the corresponding constant would be if one applied the same argument to the case of a homogeneous magnetic field (see below). However, for $\alpha_* \leq 1/2$ (or small enough so that ρ is larger than α_*^2 for a sufficiently large set of radii), we expect that the ground-state energy of the left-hand side (though difficult to compute in general) should in almost all situations be bounded by that with $\rho(r)$ replaced by α_N^2 (compare Fig. 2). We discuss further possible improvements to the constant $c(\kappa)$ at the end of Section 4.5.

Proof of Theorem 1.3. Inserting the bound of Theorem 4.1 with $L = \delta(\mathbf{X})$ into the expressions (4.4), (4.2), and (4.1), we obtain the first bound of the theorem. Furthermore, by rescaling $v(r) := u((L - 3R)r)$ and considering the minimizer v which is the solution of the Bessel equation

$$-v''(r) - v'(r)/r + \nu^2 v(r)/r^2 = \lambda v(r), \qquad v'(\gamma) = 0,\ v'(1) = 0,$$

with the minimal eigenvalue $\lambda = g\Big(\nu = \frac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \gamma = \frac{3R/L}{1-3R/L}\Big)^2 \geq 0$, one obtains

$$\int_{3R}^{L-3R} \left((1-\kappa)|u'|^2 + c(\kappa)^2 \frac{\alpha_N^2}{r^2} |u|^2 \right) r \, \mathrm{d}r = (1-\kappa) \int_\gamma^1 \left(|v'|^2 + \frac{c(\kappa)^2}{1-\kappa} \frac{\alpha_N^2}{r^2} |v|^2 \right) r \, \mathrm{d}r$$

$$\geq (1-\kappa) g\left(\frac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \gamma \right)^2 \int_\gamma^1 |v|^2 r \, \mathrm{d}r = (1-\kappa) \frac{g\left(\frac{c(\kappa)\alpha_N}{\sqrt{1-\kappa}}, \gamma \right)^2}{L^2(1-3R/L)^2} \int_{3R}^{L-3R} |u|^2 r \, \mathrm{d}r,$$

and therefore, after the simplifying estimate $(1-3R/L)^{-2} \geq 1$, the second bound of Theorem 1.3. The properties of g described in the theorem are direct consequences of Proposition A.1 and A.2. $\square$

Before continuing with the proof of Theorem 4.1 we note that, although this method involving the magnetic Hardy inequality turns out to be sufficient and indeed well-suited for our purposes, it does not deal well with strong magnetic fields (hence also the presence of a large external field), as the following example shows. The strong magnetic fields arising from a large overlap between the particles will instead be handled by the short-range part of the interaction, Lemma 1.1.

Proposition 4.3. (Constant magnetic field on a disk) *The ground-state energy $\lambda_1(\beta)$ for the Neumann form (with no symmetry imposed) with a constant magnetic field $b(\mathbf{r}) = \beta \geq 0$ on the unit disk,*

$$\lambda_1(\beta) := \inf_{\|u\|_2=1} \int_{B_1(0)} \left| (-i\nabla + \beta \mathbf{r}^\perp/2) u \right|^2 \mathrm{d}\mathbf{r},$$

satisfies

$$\lambda_1(\beta) \sim \Theta_0 \beta \quad as \quad \beta \to \infty, \quad where \quad \Theta_0 \approx 0.59.$$

However, the ground-state energy for the corresponding lower bound obtained from the Hardy inequality,

$$\mu_1(\beta) := \inf_{\|u\|_2=1} \int_{B_1(0)} \left(\left| \partial_r |u| \right|^2 + \inf_{k\in\mathbb{Z}} |k - \beta r^2/2|^2 \frac{1}{r^2} |u|^2 \right) \mathrm{d}\mathbf{r},$$

is bounded from above by $g(1/2, 0)^2 = (j'_{1/2})^2$ independent of β.

Proof. The first estimate follows for example from [18, Theorem 5.3.1], while the second from bounding the infimum by $1/4$ and taking as a trial state the Bessel function $u(\mathbf{r}) = J_{1/2}(j'_{1/2} r)$. $\square$

4.1. A One-Dimensional Projection Bound

Our strategy in order to find a uniform bound for the scalar interaction of Theorem 4.1 will be to borrow a bit of the radial kinetic energy to smear ρ over intervals whenever it has critical oscillations. As a preliminary to the proceeding analysis we therefore study the localized effective quadratic form

$$h_{I,\rho}(u) := \int_I \left(\kappa |u'|^2 + \frac{\rho}{r^2} |u|^2 \right) r \, \mathrm{d}r, \quad \kappa \in [0, 1],$$

on an interval $I = (r_1, r_2) \subseteq \mathbb{R}_+$, and our goal is to find a bound of the form

$$h_{I,\rho}(u) \gtrsim \int_I \frac{|u|^2}{r}\,\mathrm{d}r,$$

that is corresponding to ρ being constant.

Lemma 4.4. *Let I be an interval (r_1, r_2), such that $r_1 \geq R$ and $|r_2 - r_1| \leq 2R$, and let $\rho \in L^\infty(I)$ be non-negative with $\|\rho\|_\infty \leq 1$. Then for any $\kappa \in [0, 1]$ we have that*

$$\int_I \Big(\kappa|u'|^2 + \frac{\rho}{r^2}|u|^2\Big) r\,\mathrm{d}r \geq \frac{\kappa\bar{\rho}}{\beta(\kappa)} \int_I \frac{|u|^2}{r}\,\mathrm{d}r,$$

where $\bar{\rho}$ denotes the weighted mean on I,

$$\bar{\rho} := \int_I \frac{\rho}{r}\,\mathrm{d}r \Big/ \int_I \frac{\mathrm{d}r}{r},$$

and $\beta(\kappa)$ is an explicit function satisfying $\kappa < \beta(\kappa) < \kappa + 1/4$.

Remark 4.5. This lemma can be proven under more general conditions; the only condition on I needed for our proof is that r_2/r_1 is sufficiently small. The current setting is simply what we require later.

Proof of Lemma 4.4. By the change of variables $r = e^t$ and with $\tilde{u}(t) = u(e^t)$ we find that

$$h_{I,\rho}(u) = \tilde{h}(\tilde{u}) := \int_{\ln(I)} \big(\kappa|\tilde{u}'|^2 + \tilde{\rho}|\tilde{u}|^2\big)\,\mathrm{d}t.$$

For this quadratic form we can perform a projection-type argument to bound the first eigenvalue of the associated operator $\widetilde{H} := -\kappa\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \tilde{\rho}$ (with Neumann boundary conditions), which in turn will imply a bound of the desired form.

Let P denote the orthogonal projection onto the ground state $\psi_0 \equiv 1/\sqrt{|\ln(I)|}$ of $-\mathrm{d}^2/\mathrm{d}t^2$, where $|\ln(I)| = \ln(r_2/r_1)$, and let $P^\perp = \mathbb{1} - P$. Then $\big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big)P = 0$ and $\big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big)P^\perp \geq \pi^2/|\ln(I)|^2 P^\perp$ (the first non-zero Neumann eigenvalue of $-\mathrm{d}^2/\mathrm{d}t^2$).

Since $\tilde{\rho} \geq 0$, an application of Cauchy–Schwarz' and Young's inequalities yields, for any $u \in L^2(\ln(I))$ and $\mu > 0$, that

$$\begin{aligned}\big|\langle u, (P\tilde{\rho}P^\perp + P^\perp\tilde{\rho}P)u\rangle\big| &= \big|\langle\tilde{\rho}^{1/2}Pu, \tilde{\rho}^{1/2}P^\perp u\rangle + \langle\tilde{\rho}^{1/2}P^\perp u, \tilde{\rho}^{1/2}Pu\rangle\big| \\ &\leq \mu\|\tilde{\rho}^{1/2}Pu\|_2^2 + \mu^{-1}\|\tilde{\rho}^{1/2}P^\perp u\|_2^2 \\ &= \langle u, (\mu P\tilde{\rho}P + \mu^{-1}P^\perp\tilde{\rho}P^\perp)u\rangle.\end{aligned}$$

Hence we see that

$$\tilde{\rho} = (P + P^\perp)\tilde{\rho}(P + P^\perp) \geq (1-\mu)P\tilde{\rho}P + (1-\mu^{-1})P^\perp\tilde{\rho}P^\perp.$$

The operator $P\tilde{\rho}P$ is equal to $\|\tilde{\rho}\|_1/|\ln(I)|P$, where

$$\|\tilde{\rho}\|_1 = \int_{\ln(I)} \tilde{\rho}\,\mathrm{d}t = \int_I \rho(r)r^{-1}\,\mathrm{d}r,$$

and $P^{\perp}\tilde{\rho}P^{\perp}$ we bound from above by $\|\tilde{\rho}\|_{\infty}P^{\perp}$.

We find that for any $\mu \in (0, 1)$ the operator $\widetilde{H}$ satisfies

$$\begin{aligned}\widetilde{H} &\geq \kappa\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)P + \kappa\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big)P^{\perp} + (1-\mu)P\tilde{\rho}P + (1-\mu^{-1})P^{\perp}\tilde{\rho}P^{\perp} \\ &\geq \frac{(1-\mu)}{|\ln(I)|}\|\tilde{\rho}\|_1 P + \left(\frac{\kappa\pi^2}{|\ln(I)|^2} + (1-\mu^{-1})\|\tilde{\rho}\|_{\infty}\right)P^{\perp} \\ &\geq \min\left\{\frac{(1-\mu)\|\tilde{\rho}\|_1}{|\ln(I)|}, \frac{\kappa\pi^2}{|\ln(I)|^2} + (1-\mu^{-1})\|\tilde{\rho}\|_{\infty}\right\}.\end{aligned}$$

With $|r_2 - r_1| = |I| \leq 2R$ and $r_1 \geq R$ we find that

$$|\ln(I)| = \ln\Big(\frac{r_2}{r_1}\Big) = \ln\Big(1 + \frac{|I|}{r_1}\Big) \leq \ln\Big(1 + \frac{2R}{R}\Big) = \ln(3).$$

Hence, writing $\mu = 1 - \kappa/\beta$, $\beta > \kappa$, and using that $\|\tilde{\rho}\|_{\infty} = \|\rho\|_{\infty} \leq 1$, $\|\tilde{\rho}\|_1/|\ln(I)| \leq 1$ we have that

$$\begin{aligned}&\min\left\{\frac{(1-\mu)\|\tilde{\rho}\|_1}{|\ln(I)|}, \frac{\kappa\pi^2}{|\ln(I)|^2} + (1-\mu^{-1})\|\tilde{\rho}\|_{\infty}\right\} \\ &\quad\geq \kappa\frac{\|\tilde{\rho}\|_1}{|\ln(I)|}\min\left\{\frac{1}{\beta}, \frac{\pi^2}{\ln(3)^2} - \frac{1}{\beta-\kappa}\right\},\end{aligned}$$

where we assumed the positivity of the second argument (this will be clear by the choice of β below). Note that the first argument of the minimum is decreasing in $\beta > \kappa$ while the second one is increasing. Thus to find the maximizing β we only need to solve the equation $1/\beta = \pi^2/\ln(3)^2 - 1/(\beta - \kappa)$. Plugging the solution, given by

$$\beta(\kappa) = \frac{\pi^2\kappa + \sqrt{\pi^4\kappa^2 + 4\ln(3)^4} + 2\ln(3)^2}{2\pi^2} > \kappa,$$

into the above yields

$$\tilde{H} \geq \frac{\kappa}{\beta(\kappa)}\frac{\|\tilde{\rho}\|_1}{|\ln(I)|} = \frac{\kappa\bar{\rho}}{\beta(\kappa)}.$$

Finally, since $\beta(\kappa)$ is a convex function for $\kappa \in [0, 1]$ we can simplify this expression using

$$\beta(\kappa) \leq \beta(0) + (\beta(1) - \beta(0))\kappa =: L_{\beta}(\kappa),$$

and by simple numerical estimates one finds that $L_{\beta}(\kappa) < \kappa + 1/4$. $\square$

4.2. Number-Theoretic Structure of the Effective Scalar Potential

To proceed with the analysis we will need a more precise understanding of how ρ depends on the positions of the other particles. Note first that we may assume that $\alpha > 0$ using the reflection-conjugation symmetry. We then begin by writing for the normalized flux

$$\hat{\Phi}(r) = \alpha(1 + 2\mathcal{N}(r)), \quad r \geq R/2,$$

where we introduce the *particle counting function*

$$\mathcal{N}(r) := \sum_{l=1}^{N-2} \int_{B_r(0)} \frac{\mathbb{1}_{B_R(\mathbf{y}_l)}}{\pi R^2}. \tag{4.7}$$

Recall that in the expression (4.6) for the flux $\hat{\Phi}$, all particles are treated relative to the fixed center of mass $\mathbf{X}$ of the considered particle pair, and have also been renumbered for convenience: $\mathbf{y}_l := \mathbf{x}_l - \mathbf{X} \in \mathbb{R}^2$, with $l \in \{1, \ldots, N-2\}$.

In terms of the function $\mathcal{N}$ we have that

$$\rho(r) = \min_{q \in \mathbb{Z}} \big(\alpha(1 + 2\mathcal{N}(r)) - 2q\big)^2, \qquad \mathcal{N}(r) = \frac{1}{2\alpha}\hat{\Phi}(r) - \frac{1}{2}, \tag{4.8}$$

and we may cover the interval $[R/2, L]$ by smaller intervals J_q labeled by the minimizer $q \in \mathbb{N}$ (note the monotonicity of the function $\mathcal{N}(r)$, and that we might already have $q \gg 1$ on the first such interval at $r = R/2$). Each J_q contains, except possibly for the first and last such interval, exactly one zero of ρ which we denote by r_q:

$$\rho(r_q) = (\alpha(1 + 2\mathcal{N}(r_q)) - 2q)^2 = 0 \quad \Leftrightarrow \quad \mathcal{N}(r_q) = \frac{q}{\alpha} - \frac{1}{2},$$

so that

$$|\mathcal{N}(r_q) - p| = \frac{1}{2\alpha}|(2p+1)\alpha - 2q| \geq \frac{\alpha_N}{2\alpha} \quad \forall p \in \{0, 1, \ldots, N-2\}. \tag{4.9}$$

We then also have the very useful identity

$$\begin{aligned} \rho(r) &= |\alpha(1 + 2\mathcal{N}(r)) - 2q|^2 = |\alpha(1 + 2\mathcal{N}(r)) - \alpha(1 + 2\mathcal{N}(r_q))|^2 \\ &= 4\alpha^2 |\mathcal{N}(r) - \mathcal{N}(r_q)|^2, \end{aligned} \tag{4.10}$$

whenever $r \in J_q$. Let us denote by e_q^- and e_q^+ the nearest points to the left resp. right of r_q where $\rho(r) = 1$, then

$$\rho(e_q^\pm) = 1, \quad \text{and} \quad \rho(r) = 4\alpha^2(\mathcal{N}(r) - \mathcal{N}(r_q))^2 < 1 \quad \forall r \in (e_q^-, e_q^+) \subseteq J_q.^{3}$$

[3] Typically we have that $e_q^+ = e_{q+1}^-$ and $J_q = [e_q^-, e_q^+]$ unless ρ stabilizes at 1 on some interval between r_q and r_{q+1}, in which case $e_q^+ < e_{q+1}^-$ and the intervals J_q and J_{q+1} overlap.

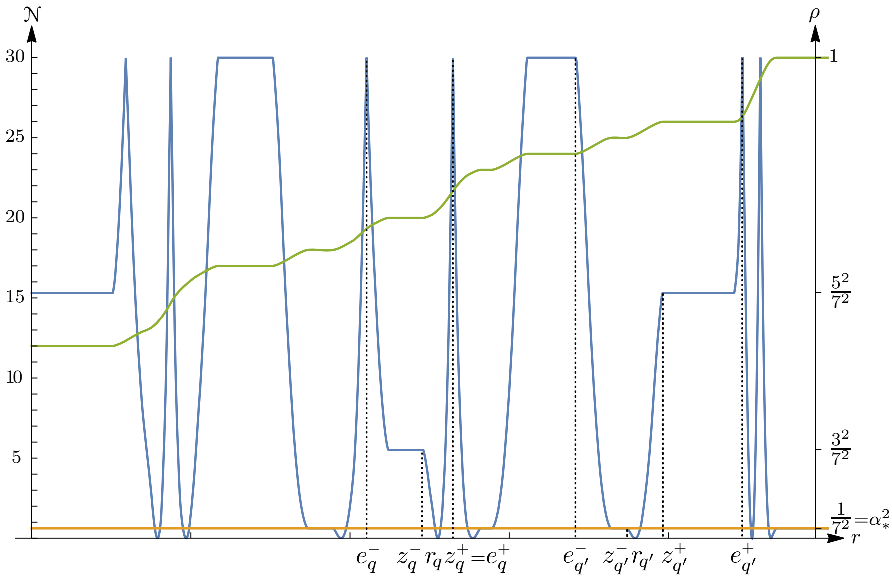


Fig. 4. The function $\mathcal{N}(r)$ (*green*) together with $\rho(r)$ (*blue*) and α_*^2 (*yellow*), for $\alpha = 3/7$, over an interval where the enclosed number of particles increases from 12 to 30. Two separate zeros r_q and $r_{q'}$ of ρ, with $q' = q + 2$, are indicated together with the corresponding points $z_q^\pm, e_q^\pm$ and $z_{q'}^\pm, e_{q'}^\pm$

Finally, we also denote by z_q^- and z_q^+ the nearest points to the left resp. right of r_q where $\mathcal{N}(z_q^-), \mathcal{N}(z_q^+) \in \mathbb{Z}$, and hence $\mathcal{N}(z_q^+) - \mathcal{N}(z_q^-) = 1$, and we observe due to (4.9), (4.10) and monotonicity that

$$\rho(r) \geq \alpha_N^2 \qquad \forall r \in J_q \backslash \big(z_q^-, z_q^+\big). \tag{4.11}$$

Recall that this constant depends in a non-trivial way on number-theoretic aspects of the parameter α, and that it remains bounded away from zero for all N if and only if α is an odd-numerator rational number (see [49, Proposition 5]). To clarify the above definitions, two sets of points r_q, $e_q^\pm$, $z_q^\pm$ are illustrated in Fig. 4 for a particular particle configuration.

Hence, we can reduce our problem to studying precisely those smaller intervals around each zero of ρ not covered by (4.11). To this end we let I_q denote the interval (z_q^-, z_q^+) around the zero $r_q \in [R/2, L]$. When considering a fixed I_q we may for notational simplicity drop the subscripts q when referring to its endpoints. Observe by the size of each particle that $|I_q| \leq 2R$, and furthermore that there is always at least one particle covering the entire interval:

Lemma 4.6. *If $r_q \geq R/2$ is a zero of ρ then with I_q constructed as above there exists a particle centered at $\mathbf{y}_l$, at a distance $d = |\mathbf{y}_l| = |\mathbf{x}_l - \mathbf{X}|$, such that $I_q \subseteq [d - R, d + R]$. In other words, the angular projection of some particle completely covers I_q.*

Proof. Let $I_q = (z^-, z^+)$ and let $\widetilde{\mathcal{N}}(r)$ be the particle counting function corresponding to our particle configuration but where we remove all particles (seen as closed disks $\bar{B}_R(\mathbf{y}_l)$) that have empty intersection with the closed disk $\bar{B}_{z^-}(0)$, that is we remove all particles that are centered at a distance strictly larger than $z^- + R$ from the origin. By the construction of I_q, there is at least one particle that has non-empty intersection with $\partial B_{z^-}(0)$ (not counting any fully enclosed ones), since otherwise $\mathcal{N}(r)$ would be constant here which contradicts the choice of z^-. Let now r' be the radius such that all the particles that intersected $\partial B_{z^-}(0)$ are completely contained in the closed disk $\bar{B}_{r'}(0)$. By the construction of $\widetilde{\mathcal{N}}(r)$, its value at r' is an integer which, since there were particles intersecting $\partial B_{z^-}(0)$, is strictly larger than $\widetilde{\mathcal{N}}(z^-) = \mathcal{N}(z^-)$, but then, since $\widetilde{\mathcal{N}}(r) \le \mathcal{N}(r)$, the function $\mathcal{N}(r)$ must take at least one integer value on $(z^-, r']$. Thus, by the definition of z^+ we conclude that $z^+ \le r'$, which completes the proof. □

4.3. Geometric Structure of the Particle Counting Function

To proceed we will need more information on the local behavior of the particle counting function $\mathcal{N}(r)$. We note that

$$\mathcal{N}(r) = \sum_{l=1}^{N-2} \frac{|B_r(0) \cap B_R(\mathbf{y}_l)|}{\pi R^2},$$

where $\mathbf{y}_l$ are the centers (in relative coordinates) of the $N-2$ particles not in our presently studied pair.

To analyze $\mathcal{N}(r)$ we thus need to work with the area of the intersection of pairs of disks. An elementary, although slightly tedious, calculation yields the following expression.

Proposition 4.7. *Let $B_1 = B_{r_1}(\mathbf{x}_1)$ and $B_2 = B_{r_2}(\mathbf{x}_2)$ be disks of radii r_1, r_2, with $r_1 \le r_2$, centered at the points $\mathbf{x}_1$ and $\mathbf{x}_2$. Then with $d = |\mathbf{x}_1 - \mathbf{x}_2|$ we have for the area of intersection, in the non-trivial regime $d \le r_1 + r_2$ and $d + r_1 \ge r_2$, that*

$$\begin{aligned} |B_1 \cap B_2| = r_1^2 \arccos\Big(\frac{d^2 + r_1^2 - r_2^2}{2dr_1}\Big) + r_2^2 \arccos\Big(\frac{d^2 + r_2^2 - r_1^2}{2dr_2}\Big) \\ - \frac{1}{2}\sqrt{(-d + r_1 + r_2)(d + r_1 - r_2)(d - r_1 + r_2)(d + r_1 + r_2)}. \end{aligned}$$

If $d > r_1 + r_2$ the area is zero and if $d + r_1 < r_2$ the area is πr_1^2.

Differentiating the flux contribution from a single particle located at $\mathbf{y}_l \in \mathbb{R}^2$, given by

$$F(|\mathbf{y}_l|, r) := |B_r(0) \cap B_R(\mathbf{y}_l)|/(\pi R^2),$$

we find for arbitrary $d, r \ge 0$ that

$$f(d, r) := \frac{\partial}{\partial r} F(d, r) = \begin{cases} 2r/R^2, & \text{if } r \le R - d, \\ 0, & \text{if } r > R + d \text{ or } r < d - R, \\ \frac{2r}{\pi R^2} \arccos\Big(\frac{d^2+r^2-R^2}{2dr}\Big), & \text{otherwise.} \end{cases} \tag{4.12}$$

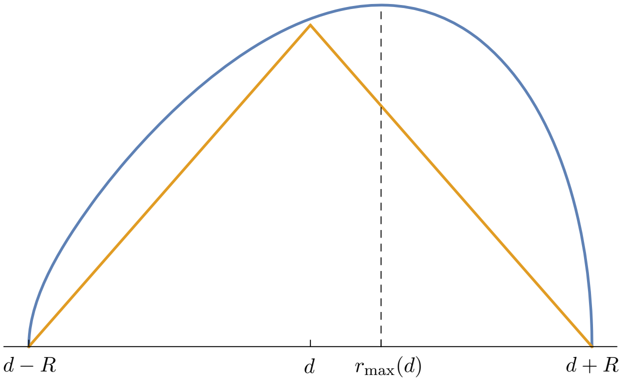


Fig. 5. The one-particle profile $f(d,\cdot)$ and its lower bound $f_\wedge(d,\cdot)$ plotted over the support of f. The profile depicted is for $d = 3R/2$, while as d increases this profile more and more resembles the upper half of a disk

In what follows we will frequently use that $f(d,\cdot)$ is essentially concave on its support (compare Fig. 5); the precise statement and its proof is found in Appendix B. Furthermore, it satisfies some simple bounds:

Lemma 4.8. *With $f(d,\cdot)$ denoting the one-particle profile (4.12) we have for $d \geq 0$ and $r \geq R$ the following bounds:*

$$f(d,r) \leq f_\sqcap(d,r) := \frac{2}{R}\mathbb{1}_{(d-R,d+R)}(r),$$
$$f(d,r) \geq f_\wedge(d,r) := \frac{2(R-d+r)}{\pi R^2}\mathbb{1}_{(d-R,d)}(r) + \frac{2(d+R-r)}{\pi R^2}\mathbb{1}_{[d,d+R)}(r).$$

Proof of Lemma 4.8. The upper bound for f given by the lemma is clear from the geometric construction of f and F. The value of f is equal to the length of the circle segment $\partial B_r(0) \cap B_R(\mathbf{x})$ where $|\mathbf{x}| = d$, divided by πR^2, and clearly this cannot exceed $2/R$. For the lower bound we use concavity.

For $d \geq R$ the function $f(d,\cdot)$ is concave on its support $[d-R, d+R]$ (see Appendix B). Moreover, $f_\wedge(d,\cdot)$ is continuous, piecewise linear and has the same support as $f(d,\cdot)$. By the construction of $f_\wedge$ and the concavity of $f(d,\cdot)$ it suffices to prove that the inequality holds at the maximum of $f_\wedge(d,\cdot)$, that is that $f(d,d) \geq f_\wedge(d,d)$, which is clear: for $d \geq R$ we have that $f(d,d)$ is a decreasing function and that $\lim_{d\to\infty} f(d,d) = \frac{2}{\pi R} = f_\wedge(d,d)$.

For $d < R$ we have that $f(d,\cdot)$ and $f_\wedge(d,\cdot)$ are concave on $[R, d+R]$ and zero otherwise (see Appendix B). By the linearity of $f_\wedge(d,\cdot)$ on this interval it is sufficient to prove that $f(d,R) \geq f_\wedge(d,R)$, which follows since $f(d,R) = \frac{2}{\pi R}\arccos\left(\frac{d}{2R}\right) \geq \frac{2d}{\pi R^2} = f_\wedge(d,R)$. $\square$

The following lemma captures in a convenient form essential aspects of the shape of the particle profile, and will play an important role in the analysis on intervals of oscillation below:

Lemma 4.9. (Shape lemma) *If $r \in [r_1, r_2]$ with $r_1 \geq R$ and $r_2 - r_1 \leq R/2$, we have that*

$$\mathcal{N}'(r_1) + \mathcal{N}'(r_2) \geq \mathcal{N}'(r).$$

Remark 4.10. The assumption $r_2 - r_1 \leq R/2$ can be relaxed slightly by instead requiring that r_1 is sufficiently large. In particular, in the limit $r_1 \to \infty$ the one-particle profile approaches a half disk and it is then geometrically clear that the statement holds whenever $r_2 - r_1 \leq R$.

Proof of Lemma 4.9. By linearity it is sufficient to prove that the inequality holds with $\mathcal{N}'$ replaced by the one-particle profile $f(d, \cdot)$ for any $d \geq 0$.

The proof utilizes that the profile $f(d, \cdot)$ is concave on its support intersected with $[R, \infty)$, which is shown in Appendix B. If in addition $d \geq R$ the profile is concave on its full support $(d - R, d + R)$, also shown in Appendix B. Thus, whenever (r_1, r_2) does not contain the maximum of $f(d, \cdot)$ the statement is clear, since if this is the case $f(d, \cdot)$ is monotone here and thus has its maximum value in either r_1 or r_2.

Thus we may assume that the unique maximum of $f(d, \cdot)$ is attained at a point $r_{\max}(d)$ in (r_1, r_2). Moreover, by the concavity of $f(d, \cdot)$ it suffices to consider the case when $|r_2 - r_1| = R/2$. The inequality we wish to prove can now be written as

$$f(d, r_{\max}(d)) \leq f(d, r_1) + f(d, r_1 + R/2), \tag{4.13}$$

which should hold for all $r_1 \geq R$ such that $r_{\max}(d) \in (r_1, r_1 + R/2)$.

Case 1: $d \geq R$**.** In this case it holds that $(r_1, r_1 + R/2) \subseteq (d - R, d + R)$, since $r_{\max}(d) \in (d - R/2, d + R/2)$. This can be verified by considering $\frac{\partial}{\partial r} f(d, r)|_{r=d+R/2}$, which can be shown by straightforward computation to be decreasing in d and moreover it is negative at $d = R$. Similarly, $\frac{\partial}{\partial r} f(d, r)|_{r=d-R/2}$ can be verified to be positive, and hence $d - R/2 < r_{\max}(d) < d + R/2$.

This implies that the right-hand side of (4.13) is a concave function of r_1, and hence its minimum value is attained at one of the extremal points of the allowed intervals. But this is precisely when either r_1 or r_2 is equal to $r_{\max}(d)$, in which case the statement is trivial by the non-negativity of f.

Case 2: $d \leq 2R/3$**.** By similar calculations as in Case 1, we have that $\frac{\partial}{\partial r} f(d, r)|_{r=R} < 0$ for $d \leq 2R/3$. Then by concavity $f(d, \cdot)$ is a monotonically decreasing function on $[R, d + R]$. Thus $f(d, r_1) \geq f(d, r)$ and the statement follows.

Case 3: $2R/3 < d < R$**.** Again the function $f(d, \cdot)$ is concave on $(R, d + R)$. Thus we again only need to consider the extremal cases of the intervals (r_1, r_2) containing the maximum of $f(d, \cdot)$ on this interval. This reduces to three different options. Either $r_1 = R$, or $r_2 = d + R$, or one of the endpoints of the interval is located at the maximum. In the last case the statement is trivially true.

If we were in the second option then $(r_1, r_2) = (d + R/2, d + R)$. Through a similar computation as above one checks that on this interval $f(d, \cdot)$ is monotone, and hence the statement follows.

If however $(r_1, r_2) = (R, 3R/2)$ the inequality is reduced to

$$f(d, r) \leq f(d, R) + f(d, 3R/2).$$

By scaling we may without loss of generality assume that $R = 1$. Using the explicit expression of f we need to show that

$$\frac{2r}{\pi}\arccos\left(\frac{d^2+r^2-1}{2dr}\right) \leq \frac{2}{\pi}\arccos\left(\frac{d}{2}\right) + \frac{3}{\pi}\arccos\left(\frac{d^2+5/4}{3d}\right).$$

Since for $d \leq R = 1$ we have that $r_{\max}(d) \leq 3/2$, it follows that

$$f(d, r_{\max}(d)) \leq \frac{3}{\pi}\arccos\left(\frac{d^2+r_{\max}(d)^2-1}{2dr_{\max}(d)}\right),$$

but the function $\frac{3}{\pi}\arccos\left(\frac{d^2+r^2-1}{2dr}\right)$ is decreasing in r, for $1 \leq r \leq d+1$, and thus we only need to verify the inequality

$$\frac{3}{\pi}\arccos\left(\frac{d}{2}\right) \leq \frac{2}{\pi}\arccos\left(\frac{d}{2}\right) + \frac{3}{\pi}\arccos\left(\frac{d^2+5/4}{3d}\right).$$

This is equivalent to $\arccos\left(\frac{d}{2}\right) \leq 3\arccos\left(\frac{d^2+5/4}{3d}\right)$. We observe that the left-hand side of this inequality is decreasing whilst the right is increasing. Thus it suffices to check the validity at $d = 2/3$, which is a simple numerical evaluation. □

4.4. Local Bounds for the Mean Potential

In this subsection we use the explicit form of $\mathcal{N}(r)$ uncovered above for $r \in (R, L)$ and the projection argument of Lemma 4.4 to locally replace the effective one-dimensional potential $\rho(r)/r$ with some constant times α_N^2/r. By Lemma 4.4 it suffices to prove that given an interval $I \subseteq (R, L)$ of small enough measure we have a suitable bound for the weighted mean $\bar{\rho}$ on I. On intervals (4.11) where ρ is already larger than α_N^2 we need not perform any detailed analysis. Thus the only intervals that remain are those of the form $I_q = (z_q^-, z_q^+) \subseteq J_q$ close to the zeros of ρ. The analysis is split into several parts depending on the behavior of ρ near a specific zero. Our first bound provides a general estimate for $\bar{\rho}$ on any subinterval of the J_q constructed above (Section 4.2) which contains the unique zero of ρ on this interval.

Lemma 4.11. *Let (r_1, r_2), with $r_1 \geq R/2$, be such that on this interval $\rho(r) = |\hat{\Phi}(r) - 2q|^2$ for some fixed $q \in \mathbb{Z}$ and such that there exists some $r_0 \in (r_1, r_2)$ with $\rho(r_0) = 0$. Then, with $\delta(r) := \min\{r - r_1, r_2 - r\}$, we have that*

$$\int_{r_1}^{r_2} \frac{\rho(r)}{r}\,\mathrm{d}r \geq \frac{2\alpha^2}{r_2(r_2 - r_1)}\left(\int_{r_1}^{r_2} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r\right)^2,$$

where as before $\mathcal{N}(r)$ denotes the particle counting function (4.7).

Proof of Lemma 4.11. On such an interval (r_1, r_2) we can, according to (4.10), express ρ in terms of $\mathcal{N}$ as

$$\rho(r) = |\alpha(1 + 2\mathcal{N}(r)) - \alpha(1 + 2\mathcal{N}(r_0))|^2 = 4\alpha^2|\mathcal{N}(r) - \mathcal{N}(r_0)|^2.$$

Inserting this into the integral we wish to bound and using the trivial estimate $1/r \geq 1/r_2$, we have that

$$\int_{r_1}^{r_2} \frac{\rho(r)}{r}\,\mathrm{d}r \geq \frac{4\alpha^2}{r_2}\int_{r_1}^{r_2} |\mathcal{N}(r) - \mathcal{N}(r_0)|^2\,\mathrm{d}r.$$

We split the above integral into two parts,

$$\begin{aligned}\int_{r_1}^{r_2} |\mathcal{N}(r) - \mathcal{N}(r_0)|^2\,\mathrm{d}r &= \int_{r_1}^{r_0} (\mathcal{N}(r) - \mathcal{N}(r_0))^2\,\mathrm{d}r + \int_{r_0}^{r_2} (\mathcal{N}(r_0) - \mathcal{N}(r))^2\,\mathrm{d}r \\ &= \int_{r_1}^{r_0}\left(\int_{r}^{r_0} \mathcal{N}'(t)\,\mathrm{d}t\right)^2\mathrm{d}r + \int_{r_0}^{r_2}\left(\int_{r_0}^{r} \mathcal{N}'(t)\,\mathrm{d}t\right)^2\mathrm{d}r.\end{aligned}$$

Using the Cauchy–Schwarz inequality and changing the order of integration one finds that

$$\begin{aligned}&\int_{r_1}^{r_2} |\mathcal{N}(r) - \mathcal{N}(r_0)|^2\,\mathrm{d}r \\ &\geq \frac{1}{r_2-r_1}\left(\left(\int_{r_1}^{r_0}\int_{r}^{r_0} \mathcal{N}'(t)\,\mathrm{d}t\mathrm{d}r\right)^2 + \left(\int_{r_0}^{r_2}\int_{r_0}^{r} \mathcal{N}'(t)\,\mathrm{d}t\mathrm{d}r\right)^2\right) \\ &= \frac{1}{r_2-r_1}\left(\left(\int_{r_1}^{r_0}\int_{r_1}^{t} \mathcal{N}'(t)\,\mathrm{d}r\mathrm{d}t\right)^2 + \left(\int_{r_0}^{r_2}\int_{t}^{r_2} \mathcal{N}'(t)\,\mathrm{d}r\mathrm{d}t\right)^2\right) \\ &= \frac{1}{r_2-r_1}\left(\left(\int_{r_1}^{r_0} \mathcal{N}'(t)(t-r_1)\,\mathrm{d}t\right)^2 + \left(\int_{r_0}^{r_2} \mathcal{N}'(t)(r_2-t)\,\mathrm{d}t\right)^2\right).\end{aligned}$$

To obtain the desired estimate we combine the above with the observation that both $t - r_1$ and $r_2 - t$ are larger than $\delta(t)$, and the elementary inequality $2(a^2+b^2) \geq (a+b)^2$,

$$\begin{aligned}\int_{r_1}^{r_2} \frac{\rho(r)}{r}\,\mathrm{d}r &\geq \frac{4\alpha^2}{r_2(r_2-r_1)}\left(\left(\int_{r_1}^{r_0} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r\right)^2 + \left(\int_{r_0}^{r_2} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r\right)^2\right) \\ &\geq \frac{2\alpha^2}{r_2(r_2-r_1)}\left(\int_{r_1}^{r_2} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r\right)^2.\end{aligned}$$

□

We now study $\bar{\rho}$ on the intervals $I_q = (z_q^-, z_q^+)$ constructed earlier around zeros of ρ, with $\mathcal{N}(z_q^\pm) \in \mathbb{Z}$. We begin with a lemma providing a bound for the local weighted mean on a certain subclass of these intervals where the potential is in some sense well behaved.

Lemma 4.12. (Good intervals) *Let $I_q = (z^-, z^+)$ be one of the intervals constructed above which satisfies $z^- \geq R$. Then if either*

$$|I_q| \geq CR \quad \text{or} \quad \frac{\inf_{I_q} \mathcal{N}'}{\sup_{I_q} \mathcal{N}'} \geq \frac{C^2}{\pi}$$

for some $0 < C \leq 1$, *we have that*

$$\bar{\rho}_{I_q} := \int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \bigg/ \int_{I_q} \frac{\mathrm{d}r}{r} \geq \frac{\alpha^2 C^4}{24\pi^2}.$$

Remark 4.13. We will later see that for our treatment of intervals I_q that are not covered by this lemma we will need to choose C rather small, approximately $C \approx 1/10$.

Proof of Lemma 4.12. By Lemma 4.11 we may estimate the integral of the potential by

$$\int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \geq \frac{2\alpha^2}{z^+(z^+ - z^-)} \left(\int_{I_q} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r \right)^2.$$

By Lemma 4.6 the interval I_q is covered by at least one particle. Thus for $r \in I_q$ we can bound $\mathcal{N}'(r)$ from below by using our lower bound for the one-particle profile $f(d,r)$ and minimizing over particle positions d such that $I_q \subseteq (d - R, d + R)$. Let as before $f_\wedge(d,r)$ denote the lower bound for f given by Lemma 4.8. We conclude that

$$\int_{I_q} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r \geq \inf_{d \in (z^- - 2R, z^+ + 2R)} \int_{I_q} f_\wedge(d,r)\delta(r)\,\mathrm{d}r.$$

As this integrand is piecewise linear in d we must have that the integral is minimized in one of the extremal points: a particle starting at z^-, a particle ending at z^+ or a particle centered at $(z^+ - z^-)/2$. By symmetry the last alternative maximizes the integral and thus we can discard this option. Moreover, the same symmetry implies that the first two alternatives are equal. Through a straightforward calculation we find that

$$\int_{I_q} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r \geq \frac{1}{4\pi} \begin{cases} |I_q|^3/R^2, & \text{if } |I_q| \leq R, \\ |I_q|, & \text{if } |I_q| > R. \end{cases}$$

Thus if $|I_q| \geq CR$, $0 < C \leq 1$, the above yields

$$\int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \geq \frac{\alpha^2 C^4}{8\pi^2 z^+} |I_q|.$$

If instead of $|I_q| \geq CR$ we have that

$$\frac{\inf_{I_q} \mathcal{N}'}{\sup_{I_q} \mathcal{N}'} \geq \frac{C^2}{\pi}$$

we can obtain the same bound. Namely, if we again consider the bound given by Lemma 4.11,

$$\int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \geq \frac{2\alpha^2}{z^+(z^+ - z^-)} \left(\int_{I_q} \mathcal{N}'(r)\delta(r)\,\mathrm{d}r \right)^2,$$

we find, using $\int_{I_q} \delta(r)\,\mathrm{d}r = |I_q|^2/4$, that

$$\int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \ge \frac{\alpha^2}{8z^+}(\inf_{I_q} \mathcal{N}')^2|I_q|^3 \ge \frac{\alpha^2(\inf_{I_q} \mathcal{N}')^2}{8z^+(\sup_{I_q} \mathcal{N}')^2}|I_q| \ge \frac{\alpha^2 C^4}{8\pi^2 z^+}|I_q|,$$

where we also used that $(\sup_{I_q} \mathcal{N}')|I_q| \ge \int_{I_q} \mathcal{N}' = 1$ for each q.

For the weighted mean we now find that

$$\bar{\rho}_{I_q} = \frac{\int_{I_q} \rho(r)/r\,\mathrm{d}r}{\int_{I_q} 1/r\,\mathrm{d}r} \ge \frac{z^-}{|I_q|}\int_{I_q} \frac{\rho(r)}{r}\,\mathrm{d}r \ge \frac{z^-\alpha^2 C^4}{z^+ 8\pi^2} \ge \frac{\alpha^2 C^4}{24\pi^2},$$

where we used that $|I_q| \le 2R$ and $z^- \ge R$ implies that $z^-/z^+ \ge 1/3$. □

The previous lemma does not cover the scenario where $\mathcal{N}(r)$ increases rapidly, resulting in rapid oscillations on many short intervals I_q. In the next lemma we consider the remaining intervals I_q and use our geometric knowledge of $\mathcal{N}(r)$ to show that these intervals cannot cover too much of our large-scale interval $[R, L]$. To achieve this we first cover the remaining collection of intervals I_q with a collection of intervals J_l such that $|J_l| = R/2$ for all l.

Lemma 4.14. (Bad intervals) *Let $J \subseteq (R, L]$ be an interval of length $R/2$. Then the fraction of J covered by intervals I_q satisfying both*

$$|I_q| < CR \quad \text{and} \quad \frac{\inf_{I_q} \mathcal{N}'}{\sup_{I_q} \mathcal{N}'} < \frac{C^2}{\pi}, \tag{4.14}$$

with $C < \sqrt{\pi/2}$, is less than

$$\frac{8C(\pi - C^2)}{\pi - 2C^2}.$$

Proof. Let $\{I_k\}_{k=1}^m$ denote the subset of the intervals I_q for which (4.14) is satisfied and $J \cap I_k \ne \emptyset$ for each $k = 1, \ldots, m$, and ordered from left to right (note in particular that throughout this proof the labeling of the intervals differs from that described below (4.8)). For further notational convenience we will let $\inf_k$ and $\sup_k$ denote $\inf_{I_k} \mathcal{N}'$ and $\sup_{I_k} \mathcal{N}'$, respectively. We will also denote by i_k and s_k a (fixed) choice of points in each I_k such that $\mathcal{N}'(i_k) = \inf_k$ and $\mathcal{N}'(s_k) = \sup_k$.

We begin by showing that we may assume that the distance between any two points in two consecutive intervals is less than $R/2$, allowing us to apply Lemma 4.9. If, for some $k \in \{1, \ldots, m\}$, $I_k = (z_k^+, z_k^-)$ and $I_{k+1} = (z_{k+1}^+, z_{k+1}^-)$ are such that $z_{k+1}^+ - z_k^- > R/2$, then since both intervals have non-empty intersection with J we must have that $m = 2$. But this implies that $|J \cap (\cup_{k=1}^m I_k)| \le 2CR$ and the statement follows. Similarly the statement is true if $m = 1$.

Suppose that there exists a j such that $i_j < s_j < s_{j+1} < i_{j+1}$. Then, since by the above we may assume that $i_{j+1} - i_j < R/2$, Lemma 4.9 implies that

$$\max\{\sup_j, \sup_{j+1}\} \le \inf_j + \inf_{j+1},$$

but combined with (4.14) this leads to a contradiction:

$$\max\{\sup_j, \sup_{j+1}\} \le \inf_j + \inf_{j+1} \\ \le 2\max\{\inf_j, \inf_{j+1}\} < \frac{2C^2}{\pi}\max\{\sup_j, \sup_{j+1}\},$$

which is impossible since $\frac{2C^2}{\pi} < 1$.

Let us say that an interval I_k where $s_k < i_k$ is of type A, and one where instead $i_k < s_k$ is of type B (note that $i_k \ne s_k$ by the assumption on I_k). We let $\mathcal{A}$ and $\mathcal{B}$ denote the collections of intervals of type A and type B respectively.

The above contradiction argument yields that an interval of type A cannot follow one of type B, that is if we for some j have that $I_j \in \mathcal{A}$ then $I_k \in \mathcal{A}$ for all $k < j$, and similarly, if $I_j \in \mathcal{B}$ then $I_k \in \mathcal{B}$ for all $k > j$. We conclude that there is at most one k such that I_k and I_{k+1} are of different type, and I_k must then be of type A.

As we will now show, it turns out that the sequence of lengths $|I_k|$ of consecutive intervals starting at any interval of type A and going to the left, resp. type B and going to the right, is monotonically decreasing and bounded from above by a geometric sequence. By assumption (4.14), all $|I_k| < CR$, and in particular this holds for the first interval in any such sequence. Using these observations we will be able to bound the total measure of $\cup_k I_k$.

We begin by studying a sequence starting at an interval of type A and going to the left (note that such a sequence may not exist if all $I_k \in \mathcal{B}$). We wish to prove that $|I_k|$ decreases along this sequence.

Let j be such that $I_j \in \mathcal{A}$. Then $i_{j-1} < s_j < i_j$, and by Lemma 4.9 we have that $\sup_j \le \inf_{j-1} + \inf_j$. Since we assume that $\inf_j < C^2/\pi \sup_j$ this implies that

$$\frac{\pi - C^2}{C^2}\inf_j < \left(1 - \frac{C^2}{\pi}\right)\sup_j < \inf_{j-1}.$$

The only thing we used above was that $I_j \in \mathcal{A}$. Since this implies that also $I_{j-1} \in \mathcal{A}$, we can iterate this argument until we reach I_1. This yields for $k < j$ that

$$\left(\frac{\pi - C^2}{C^2}\right)^{j-k}\inf_j < \left(\frac{\pi - C^2}{C^2}\right)^{j-k}\frac{C^2}{\pi}\sup_j < \inf_k. \tag{4.15}$$

Using that $|I_k|\inf_k \le 1 \le |I_k|\sup_k$ (for any k) we, for $k < j$, find that (4.15) implies

$$|I_j| \ge \frac{1}{\sup_j} \ge \left(\frac{\pi - C^2}{C^2}\right)^{j-k}\frac{C^2}{\pi}\frac{1}{\inf_k} \ge \left(\frac{\pi - C^2}{C^2}\right)^{j-k}\frac{C^2}{\pi}|I_k|,$$

where we used that, for $k \le j$, $\inf_k > 0$ since otherwise $\sup_j$ would be zero which cannot happen by the construction of the I_k's. Since C is small this proves the claim in the case of type A intervals.

For the case of type B intervals the proof is almost identical and one finds instead that, if $I_j \in \mathcal{B}$,

$$|I_j| \ge \left(\frac{\pi - C^2}{C^2}\right)^{k-j}\frac{C^2}{\pi}|I_k|, \qquad k > j.$$

We are now ready to complete the proof of the lemma. Begin by finding j such that $I_j \in \mathcal{A}$ and $I_{j+1} \in \mathcal{B}$ (if $\mathcal{A}$, alt. $\mathcal{B}$, is the empty set we take $j = 0$, alt. $j = m$). Then using the above estimates we obtain that

$$\begin{aligned}
\Big|J \cap \bigcup_k I_k\Big| &\le \sum_k |I_k| = \sum_{k\le j} |I_k| + \sum_{k>j} |I_k| \\
&\le |I_j|\Big(1 + \frac{\pi}{C^2}\sum_{k=1}^{j-1}\Big(\frac{C^2}{\pi - C^2}\Big)^{j-k}\Big) \\
&\quad + |I_{j+1}|\Big(1 + \frac{\pi}{C^2}\sum_{k=j+2}^{m}\Big(\frac{C^2}{\pi - C^2}\Big)^{k-j-1}\Big) \\
&< CR\Big(1 + \frac{\pi}{C^2}\sum_{l=1}^{\infty}\Big(\frac{C^2}{\pi - C^2}\Big)^{l}\Big) + CR\Big(1 + \frac{\pi}{C^2}\sum_{l=1}^{\infty}\Big(\frac{C^2}{\pi - C^2}\Big)^{l}\Big) \\
&= \frac{4C(\pi - C^2)}{\pi - 2C^2}R,
\end{aligned}$$

and dividing this quantity by $|J| = R/2$ completes the proof. □

4.5. *Proof of Theorem 4.1*

What we have found is that the Lebesgue measure of the subset of J where ρ is already large, or can be averaged to be large, is at least

$$\Big(\frac{1}{2} - \frac{4C(\pi - C^2)}{\pi - 2C^2}\Big)R.$$

Using this we can find a non-trivial uniform lower bound on $\bar{\rho}_J$ and therefore, using the local projection argument, we finally obtain that there exists a constant $c(\kappa) > 0$ such that

$$\int_R^L \Big(|u'|^2 + \frac{\rho}{r^2}|u|^2\Big) r\,\mathrm{d}r \ge \int_R^L \Big((1-\kappa)|u'|^2 + c(\kappa)^2 \frac{\alpha_N^2}{r^2}\mathbb{1}_{[3R, L-3R]}|u|^2\Big) r\,\mathrm{d}r.$$

We proceed as follows:

$$\begin{aligned}
\int_R^L \Big(|u'|^2 + \frac{\rho}{r^2}|u|^2\Big) r\,\mathrm{d}r &= \int_R^L (1-\kappa)|u'|^2 r\,\mathrm{d}r + \int_R^L \Big(\kappa|u'|^2 + \frac{\rho}{r^2}|u|^2\Big) r\,\mathrm{d}r \\
&\ge \int_R^L (1-\kappa)|u'|^2 r\,\mathrm{d}r + \int_R^L \Big(\frac{\kappa}{2}|u'|^2 + \frac{\hat{\rho}}{r^2}|u|^2\Big) r\,\mathrm{d}r,
\end{aligned}$$

where $\hat{\rho}$ denotes a new weight obtained by replacing $\rho(r)$ with $\frac{2\kappa}{1+2\kappa}\frac{\alpha^2 C^4}{24\pi^2}$ on all I_q covered by Lemma 4.12 that intersect $(3R, L-2R)$, by using Lemma 4.4 with $\kappa/2$. Thus the only remaining zeros of $\hat{\rho}$ on $(3R, L-2R)$ are those contained in intervals I_q which satisfy the assumptions of Lemma 4.14. Let $\mathcal{Q} \subset \mathbb{N}$ denote the set of integers q for which I_q is such an interval. We now cover $(3R, L-3R)$ by a collection of disjoint intervals $J \subset (3R, L-2R)$, each of length $|J| = R/2$. Specifically, we take the intervals $\left(3R + \frac{(n-1)R}{2}, 3R + \frac{nR}{2}\right)$ where n runs from 1 to $\left\lfloor \frac{2(L-5R)}{R} \right\rfloor$. On each such $J = (r_1, r_2)$ we then have that

$$\int_J \frac{\hat{\rho}}{r}\,\mathrm{d}r \geq \frac{1}{r_2}\int_J \hat{\rho}\,\mathrm{d}r \geq \frac{1}{r_2}\int_{J\cap(\cup_{q\in\mathcal{Q}} I_q)^c} \hat{\rho}\,\mathrm{d}r \geq \frac{2\kappa}{1+2\kappa}\frac{\alpha_N^2 C^4}{r_2 24\pi^2}\Big|J \cap \Big(\bigcup_{q\in\mathcal{Q}} I_q\Big)^c\Big|.$$

By Lemma 4.14 we then obtain for the weighted mean of $\hat{\rho}$ that

$$\int_J \frac{\hat{\rho}}{r}\,\mathrm{d}r \Big/ \int_J \frac{\mathrm{d}r}{r} \geq \frac{r_1}{r_2}\frac{2\kappa}{1+2\kappa}\frac{\alpha_N^2 C^4}{12\pi^2}\left(\frac{1}{2} - \frac{4C(\pi - C^2)}{\pi - 2C^2}\right), \quad \text{with} \quad \frac{r_1}{r_2} \geq \frac{6}{7}.$$

Thus for each J we can again apply Lemma 4.4 and obtain

$$\int_J \left(\frac{\kappa}{2}|u'|^2 + \frac{\hat{\rho}}{r^2}|u|^2\right) r\,\mathrm{d}r \geq \left(\frac{2\kappa}{1+2\kappa}\right)^2 \frac{C^4}{14\pi^2}\left(\frac{1}{2} - \frac{4C(\pi - C^2)}{\pi - 2C^2}\right)\int_J \frac{\alpha_N^2}{r}|u|^2\,\mathrm{d}r.$$

Applying this for each J we obtain the desired estimate with

$$c(\kappa)^2 = \left(\frac{2\kappa}{1+2\kappa}\right)^2 \frac{C^4}{14\pi^2}\left(\frac{1}{2} - \frac{4C(\pi - C^2)}{\pi - 2C^2}\right).$$

Maximizing this in $C \in (0, 1)$ we obtain for $C \approx 0.0996$ the extremely small (but positive) constant

$$c(\kappa) \geq 5.3 \cdot 10^{-4}\frac{\kappa}{1+2\kappa}.$$

This concludes the proof of Theorem 4.1 and hence the treatment of the long-range interaction of Theorem 1.3.

We note that with this choice of C we allow for approximately 80% of any (and all) $R/2$ long interval contained in $(R, L]$ to be covered by the intervals I_q satisfying (4.14). As we expect that this is rather far from the actual situation for most particle configurations there seems to be room for improvement in the above considerations. One such improvement could be to use that the effective potential must between every two I_q intervals go up to one and then back down again. Our current method does not take this into account and is blind to the fact that there must be helpful gaps between the I_q's.

Another way of improving this constant would be to refine the bounds in Lemma 4.12 by using the precise shape of the one-particle profile instead of the simpler lower bound provided by $f_\wedge$. One could also take into account that all intervals cannot be at the edge of a particle, that is make use of the observation that a large number of the particles are likely to cover more than one interval I_q.

5. Local Exclusion

We now formulate the obtained energy bounds for anyons in terms of local exclusion principles, following [45,46,49–51], with some refinements to take both the short- and the long-range magnetic interactions into account.

With a weight partition $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3) \in [0,1]^3$, $\kappa_1 + \kappa_2 + \kappa_3 = 1$, we can write for the total kinetic energy for N anyons in a normalized state $\Psi \in \mathscr{D}^N_{\alpha,R}$

$$\begin{aligned}\langle \Psi, \hat{T}_\alpha \Psi\rangle &= \kappa_1 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j\Psi|^2\,\mathrm{dx} + \kappa_2 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j\Psi|^2\,\mathrm{dx} + \kappa_3 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j\Psi|^2\,\mathrm{dx}\\ &\geq \int_{\mathbb{R}^{2N}} \sum_{j=1}^N \left(\kappa_1 \big|\nabla_j|\Psi|\big|^2 + \kappa_2 \sum_{\substack{k=1\\k\neq j}}^N 2\pi|\alpha| \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k)\,|\Psi|^2 + \kappa_3 |D_j\Psi|^2\right)\mathrm{dx},\end{aligned}$$

where we used Lemmas 1.4 and 1.1. We then make a partitioning of the plane $\mathbb{R}^2$ into disjoint squares Q's:

$$\langle \Psi, \hat{T}_\alpha \Psi\rangle \geq \sum_Q T_Q^{\boldsymbol{\kappa}}[\Psi],$$

where the expected local energy on each square Q is given by (the definitions extend to all $\boldsymbol{\kappa} \in \mathbb{R}^3$)

$$\begin{aligned}T_Q^{\boldsymbol{\kappa}}[\Psi] &:= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(\kappa_1 \big|\nabla_j|\Psi|\big|^2 + \kappa_2 \sum_{\substack{k=1\\k\neq j}}^N 2\pi|\alpha| \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k)\,|\Psi|^2 + \kappa_3 |D_j\Psi|^2\right)\mathbb{1}_Q(\mathbf{x}_j)\,\mathrm{dx}\\ &\geq \sum_{n=0}^N E_n^{\boldsymbol{\kappa}}(|Q|)\,p_n(\Psi; Q).\end{aligned} \tag{5.1}$$

Here the local n-particle energy (translation invariant and with Neumann b.c.) is given by

$$E_n^{\boldsymbol{\kappa}}(|Q|) := \inf_{\int_{Q^n}|\psi|^2=1} \sum_{j=1}^n \int_{Q^n} \left(\kappa_1 \big|\nabla_j|\psi|\big|^2 + \kappa_2 \sum_{\substack{k=1\\k\neq j}}^n 2\pi|\alpha| \frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k)\,|\psi|^2 + \kappa_3 |D_j\psi|^2\right)\mathrm{dx}, \tag{5.2}$$

and $p_n(\Psi; Q)$ denotes the n-particle probability distribution induced from Ψ,

$$p_n(\Psi; Q) := \sum_{A \subseteq \{1,\dots,N\}, |A|=n} \int_{(Q^c)^{N-n}} \int_{Q^n} |\Psi|^2 \prod_{k\in A} \mathrm{d}\mathbf{x}_k \prod_{l \notin A} \mathrm{d}\mathbf{x}_l,$$

having the normalizations $\sum_{n=0}^N p_n(\Psi; Q) = 1$ and $\sum_{n=0}^N n p_n(\Psi; Q) = \int_Q \varrho_\Psi$, the expected number of particles on Q. In (5.2) the operators D_j still depend on all N particles, with the first n on Q, and we take the infimum over the remaining $N-n$ positions in $\mathbb{R}^2 \setminus Q$.

The inequality (5.1) is obtained by simply partitioning the configuration space $\mathbb{R}^{2N}$, for example by inserting into the integrand the partition of unity $\mathbb{1} = \prod_{k=1}^N (\mathbb{1}_Q(\mathbf{x}_k) + \mathbb{1}_{Q^c}(\mathbf{x}_k))$ and expanding. This approach to bound the energy goes all the way back to Dyson and Lenard [16].

5.1. Short-Range Exclusion

We consider first the contribution to the local energy coming solely from the short-range part of the magnetic interaction.

Lemma 5.1. (Local exclusion—short range) *For any* $\alpha \in \mathbb{R}$, $R > 0$ *and* $Q \subseteq \mathbb{R}^2$ *a square, and with* $\gamma(Q) := R|Q|^{-\frac{1}{2}}$, *we have that*

$$E_n^{(1,1,0)}(|Q|) \geq \frac{e_{\mathrm{SR}}(\alpha, \gamma(Q), n)}{|Q|}(n-1)_+,$$

and

$$T_Q^{(1,1,0)}[\Psi] \geq \frac{e_{\mathrm{SR}}(\alpha, \gamma(Q), \int_Q \varrho_\Psi)}{|Q|}\left(\int_Q \varrho_\Psi\ -1\right)_+,$$

where

$$e_{\mathrm{SR}}(\alpha, \gamma, n) := \begin{cases} \dfrac{|\alpha| \min\left\{(1-\gamma^2/2)_+^{-1}, K_\alpha/2\right\}}{K_\alpha + 2|\alpha|\left(-\ln(\gamma/\sqrt{2})\right)_+} & \text{for } \gamma < \sqrt{2}, \\ 2|\alpha|\gamma^{-2} n & \text{for } \gamma \geq \sqrt{2}. \end{cases}$$

Here

$$K_\alpha := \sqrt{2|\alpha|}\frac{I_0(\sqrt{2|\alpha|})}{I_1(\sqrt{2|\alpha|})} \geq 2, \qquad K_0 := 2,$$

and I_ν *denotes the modified Bessel function of order* ν.

Proof of Lemma 5.1. We consider the local energy form in (5.2). In the case that $\gamma(Q) \geq \sqrt{2}$, the short-range potential in the second term covers the full domain Q for every particle, and hence

$$E_n^{(1,1,0)}(|Q|) \geq \frac{2\pi|\alpha|}{\pi R^2} n(n-1)_+ = \frac{2|\alpha|}{|Q|}\gamma(Q)^{-2} n(n-1)_+.$$

By convexity we then also have that

$$\sum_{n=0}^{N} E_n^{(1,1,0)}(|Q|) p_n(\Psi; Q) \geq \frac{2|\alpha|}{|Q|}\gamma(Q)^{-2}\left(\int_Q \varrho_\Psi\right)\left(\int_Q \varrho_\Psi\ -1\right)_+.$$

In the case that $\gamma(Q) < \sqrt{2}$, we use Dyson's lemma [14] in two dimensions (see [36, 41, 46]) to smear the potential to the full domain as done in [46, Proposition 19], keeping part of the potential intact and smearing the rest. For $n > 1$ and any $\kappa \in [0, 1]$ we can bound the energy form in $E_n^{(1,1,0)}(Q)$ from below by

$$n \int_{Q^2} \left((1-\kappa)\left(\left|\nabla_1|\psi|\right|^2 + \frac{2\pi|\alpha|}{\pi R^2} \mathbb{1}_{B_R(\mathbf{x}_2)}(\mathbf{x}_1)\,|\psi|^2 \right) + \kappa \frac{2\pi|\alpha|}{\pi R^2} \mathbb{1}_{B_R(\mathbf{x}_2)}(\mathbf{x}_1)\,|\psi|^2 \right) \mathrm{dx}$$

$$\geq (n-1)_+ \int_{Q^2} \left((1-\kappa) U(|\mathbf{x}_1 - \mathbf{x}_2|) \mathbb{1}_{B_R(\mathbf{x}_2)^c}(\mathbf{x}_1) + \kappa \frac{2\pi|\alpha|}{\pi R^2} \mathbb{1}_{B_R(\mathbf{x}_2)}(\mathbf{x}_1) \right) |\psi|^2 \,\mathrm{dx},$$

with

$$U(r) := |Q|^{-1}\left(1 - \frac{R^2}{2|Q|}\right)^{-1}\left(\frac{K_\alpha}{2|\alpha|} + \ln\frac{\sqrt{2}|Q|^{1/2}}{R}\right)^{-1}\mathbb{1}_{[R,\sqrt{2}|Q|^{1/2}]}(r).$$

This expression arises from the application of Dyson's lemma [36, Lemma 3.1] on the star-shaped domain $Q - \mathbf{x}_2$ with the requirement that

$$\int_R^{\sqrt{2}|Q|^{1/2}} U(r)\ln(r/a_R)\, r\mathrm{d}r \le 1, \qquad U(r) = 0 \qquad \text{for } r < R,$$

and where the considered pair potential is

$$W(\mathbf{x}) := \frac{W_0}{R^2}\mathbb{1}_{B_R(0)}(\mathbf{x}), \qquad W_0 = 4|\alpha|,$$

with scattering length (see for example [46, Appendix A.2.4])

$$a_R = R\exp\left(-\frac{1}{\sqrt{W_0/2}}\frac{I_0(\sqrt{W_0/2})}{I_1(\sqrt{W_0/2})}\right) = R\exp\left(-\frac{K_\alpha}{2|\alpha|}\right). \tag{5.3}$$

We now demand that κ be chosen such that the potentials match:

$$(1-\kappa)U(r) = \kappa\frac{2|\alpha|}{R^2},$$

that is,

$$\frac{\kappa}{1-\kappa} = \gamma(Q)^2\big(1 - \gamma(Q)^2/2\big)^{-1}\big(K_\alpha + 2|\alpha|\big(-\ln(\gamma(Q)/\sqrt{2})\big)\big)^{-1}.$$

However, note that the factor $(1-\gamma(Q)^2/2)^{-1}$ in U diverges as $\gamma(Q) \to \sqrt{2}$ while the other potential term stays bounded, implying $\kappa \to 1$. Hence, in order to be able to bound $1-\kappa$ uniformly we instead truncate the potential U by replacing the unbounded factor with

$$\min\big\{\big(1-\gamma(Q)^2/2\big)^{-1}, K_\alpha/2\big\} \in [1, K_\alpha/2],$$

also using that $K_\alpha \ge 2$ (see [59, Eqn. 10.33.1]). With this replacement in the above we then find that

$$\frac{\kappa}{1-\kappa} = \gamma(Q)^2\frac{\min\big\{\big(1-\gamma(Q)^2/2\big)^{-1}, K_\alpha/2\big\}}{K_\alpha + 2|\alpha|(-\ln(\gamma(Q)/\sqrt{2}))} \le \frac{\gamma(Q)^2}{2} \le 1,$$

and hence $\kappa \le 1/2$ and $1-\kappa \ge 1/2$. Summing up, we find for all $n \ge 0$ that

$$E_n(Q) \ge \frac{(n-1)_+}{|Q|}(1-\kappa)2|\alpha|\frac{\min\big\{\big(1-\gamma(Q)^2/2\big)_+^{-1}, K_\alpha/2\big\}}{K_\alpha + 2|\alpha|\big(-\ln(\gamma(Q)/\sqrt{2})\big)_+},$$

and may again use convexity in n to obtain the corresponding bound for $T_Q[\Psi]$. $\square$

Although not aiming to provide the sharpest possible bound, the above lemma has the advantage of being relatively simple and it captures the overall dependence of the pure short-range interaction on the parameters. In a certain regime however, referred to below as the *soft-core regime*, the following version (which could in some sense be viewed as a mix between the two and three-dimensional cases studied in [36,39–41]) will yield a comparatively good bound.

Lemma 5.2. (Soft-core exclusion) *For any $R \geq 0$ and $Q \subseteq \mathbb{R}^2$ a square, and with $\gamma(Q) := R|Q|^{-\frac{1}{2}}$, we have that*

$$E_n^{(\kappa,1-\kappa,0)}(Q) \geq 2\pi|\alpha|(1-\kappa)\big(1-2\gamma(Q)\big)_+^2 \frac{n(n-1)}{|Q|}\left(1-\frac{2|\alpha|\gamma(Q)^{-2}n(n-1)}{\pi^2\kappa/(1-\kappa)-2\pi|\alpha|n(n-1)}\right)_+,$$

for any $\kappa \in (0,1)$, $\alpha \in \mathbb{R}$ and $n \geq 2$ such that $\pi^2\kappa/(1-\kappa) > 2\pi|\alpha|n(n-1)$.

Proof. Following [36] we write for the operator of the left-hand side

$$H = \kappa\sum_{j=1}^{n}(-\Delta_{\mathbf{x}_j}) + (1-\kappa)W,$$

with (assuming $\alpha > 0$ for notational simplicity)

$$W = 2\pi\alpha\sum_{j\neq k}\frac{\mathbb{1}_{B_R(0)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k).$$

We apply the following result due to Temple [36,70]: If $H = H_0 + V$, for some Schrödinger operator $H_0 \geq 0$ and scalar potential $V \geq 0$, then the ground-state energy of H is bounded from below by

$$\lambda_0(H_0) + \langle V\rangle_{\psi_0} - \frac{\langle V^2\rangle_{\psi_0} - \langle V\rangle_{\psi_0}^2}{\lambda_1(H_0) - \langle V\rangle_{\psi_0}},$$

as long as $\lambda_1(H_0) - \langle V\rangle_{\psi_0}$ is positive. Here ψ_0 denotes the normalized ground state of H_0, $\langle V\rangle_{\psi_0} := \int V|\psi_0|^2$ is the expectation of V in the state ψ_0, and $\lambda_0(H_0)$ resp. $\lambda_1(H_0)$ is the first resp. second eigenvalue of H_0.

In our case, $H_0 = -\kappa\Delta_{Q^n}^{\mathcal{N}}$ (the Neumann Laplacian) and $\psi_0 \equiv |Q|^{-n/2}$, we have that

$$2\pi\alpha\frac{n(n-1)}{|Q|} \geq \langle W\rangle_{\psi_0} \geq 2\pi\alpha(1-2\gamma(Q))^2\frac{n(n-1)}{|Q|},$$

where for the lower bound one integrates the first particle of each pair on a smaller domain with margin R away from the boundary. Moreover, by Cauchy–Schwarz

$$\langle W^2\rangle_{\psi_0} \leq \frac{2\alpha}{R^2}n(n-1)\langle W\rangle_{\psi_0}.$$

Thus Temple's inequality yields that

$$\begin{aligned} H &\geq \langle (1-\kappa)W\rangle_{\psi_0} - \frac{\langle (1-\kappa)^2 W^2\rangle_{\psi_0} - \langle (1-\kappa)W\rangle_{\psi_0}^2}{\lambda_1(\kappa \sum_j (-\Delta_j)) - \langle (1-\kappa)W\rangle_{\psi_0}} \\ &\geq (1-\kappa)\langle W\rangle_{\psi_0}\left(1 - \frac{(1-\kappa)2\alpha R^{-2} n(n-1)}{\kappa\pi^2/|Q| - (1-\kappa)\langle W\rangle_{\psi_0}}\right) \\ &\geq 2\pi\alpha(1-\kappa)(1-2\gamma(Q))^2 \frac{n(n-1)}{|Q|}\left(1 - \frac{2\alpha\gamma(Q)^{-2} n(n-1)}{\pi^2\kappa/(1-\kappa) - 2\pi\alpha n(n-1)}\right), \end{aligned}$$

as claimed. □

5.2. *Long-Range Exclusion*

We now turn to local energy bounds for the pure long-range part of the magnetic interaction.

Lemma 5.3. (Local exclusion—long range) *For any $\alpha \in \mathbb{R}$, $R \geq 0$ and $Q \subseteq \mathbb{R}^2$ a square, and with $\gamma(Q) := R|Q|^{-\frac{1}{2}}$, we have that*

$$E_n^{(0,0,1)}(Q) \geq \frac{e_{\mathrm{LR}}(\alpha, \gamma(Q))}{|Q|}(n-1)_+,$$

and

$$T_Q^{(0,0,1)}[\Psi] \geq \frac{e_{\mathrm{LR}}(\alpha, \gamma(Q))}{|Q|}\left(\int_Q \varrho_\Psi - 1\right)_+,$$

with

$$e_{\mathrm{LR}}(\alpha, \gamma) := \frac{\pi}{24} g\big(c\alpha_N, 12\gamma\big)^2 (1-12\gamma)_+^3,$$

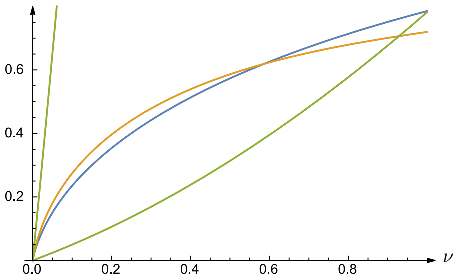
where $c = 5.3/\sqrt{8} \cdot 10^{-4}$.

For $R = 0$, the above bounds are valid with $e_{\mathrm{LR}}(\alpha, 0) = f((j'_{\alpha_N})^2)$ for all $\alpha \in \mathbb{R}$, where $f\colon [0, (j'_1)^2] \to \mathbb{R}$ is a function defined below satisfying

$$t/6 \leq f(t) \leq 2\pi t \quad \text{and} \quad f(t) = 2\pi t\big(1 - O(t^{1/3})\big) \tag{5.4}$$

(see Fig. 6 for both lower and upper bounds for f).

The tiny constant c stems from Theorem 1.3 and again we expect that it could be replaced with $c = 1/\sqrt{3}$ or just slightly smaller (recall Remark 4.2). Accordingly we have not aimed for the sharpest possible bounds in our proof for $R > 0$. Note however that for $R = 0$ and in the limit $\alpha \to 0$, the two-particle energy per particle is exactly the expected one from average-field theory, $\pi(j'_{\alpha_*})^2 \sim 2\pi\alpha_* \sim 2\pi|\alpha|$ for suitable α, however the bound is only linear (and not quadratic) in n and hence only good for small enough boxes Q, resulting in a worse constant (by a factor $1/2$) when applied below in the thermodynamic limit. Also note that the bounds involve α_N and not α_n or $\alpha_{\lceil \int_Q \varrho_\Psi \rceil}$ because there is a probability that more particles (in fact all the way up to N) can be found on Q.



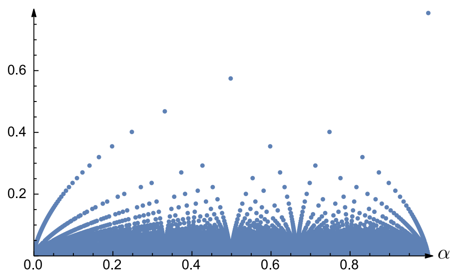


Fig. 6. *Left* A comparison between the optimized energy bounds for $f((j'_\nu)^2)$ on the unit square as a function of $\nu \in [0,1]$, obtained by means of the projection method (*blue*) and Temple (*yellow*), as well as the upper and lower bounds given in (5.4) (*green*). *Right* A numerical lower bound to the energy $e_{\mathrm{LR}}(\alpha, 0) = f((j'_{\alpha_*})^2)$ on the unit square as a function of α. The bound uses the projection method and the erratic behavior is due to the function $\alpha \mapsto \alpha_*$ being discontinuous at all odd-numerator rationals

Proof of Lemma 5.3. Ideal case. We begin with the more transparent case of $R = 0$, and note that we may set $|Q| = 1$ by scaling. Our starting point is the long-range magnetic interaction bound provided by Theorem 1.3. For the ideal case the theorem states that

$$\sum_{j=1}^{n} \int_{Q^n} |D_j \Psi|^2 \, \mathrm{dx} \geq \frac{1}{n} \sum_{j<k} \int_{Q^n} (j'_{\alpha_N})^2 \frac{\mathbb{1}_{B_{\delta(\mathbf{X}_{jk})}}(\mathbf{r}_{jk})}{\delta(\mathbf{X}_{jk})^2} |\Psi|^2 \, \mathrm{dx}.$$

In order to convert this non-uniform potential to a uniform bound for the energy we take part of the kinetic energy and then apply either Temple's inequality as in Lemma 5.2 or a projection argument as in [49, Lemma 7] or Lemma 4.4. To this end we take a fraction $\kappa \in [0,1]$ of the original kinetic energy for which we use the diamagnetic inequality and the identity

$$\sum_{j=1}^{n} |\mathbf{z}_j|^2 = \frac{1}{n-1} \sum_{j<k} \left(|\mathbf{z}_j|^2 + |\mathbf{z}_k|^2 \right), \qquad \mathbf{z}_j \in \mathbb{C},$$

and on the remaining fraction $1-\kappa$ we use Theorem 1.3. We then obtain that

$$\begin{aligned}
&\sum_{j=1}^{n} \int_{Q^n} |D_j \Psi|^2 \, \mathrm{dx} \\
&\quad \geq \frac{1}{n} \sum_{j<k} \int_{Q^n} \Bigg(\frac{\kappa n}{n-1} \left(\big|\nabla_j |\Psi|\big|^2 + \big|\nabla_k |\Psi|\big|^2 \right) \\
&\qquad + (1-\kappa)(j'_{\alpha_N})^2 \frac{\mathbb{1}_{B_{\delta(\mathbf{X}_{jk})}}(\mathbf{r}_{jk})}{\delta(\mathbf{X}_{jk})^2} |\Psi|^2 \Bigg) \mathrm{dx} \\
&\quad \geq \frac{1}{n} \sum_{j<k} \int_{Q^{n-2}} \int_{Q^2} \Bigg(\kappa \left(\big|\nabla_j |\Psi|\big|^2 + \big|\nabla_k |\Psi|\big|^2 \right) \\
&\qquad + (1-\kappa)(j'_{\alpha_N})^2 \frac{\mathbb{1}_{B_{\delta(\mathbf{X}_{jk})}}(\mathbf{r}_{jk})}{\delta(\mathbf{X}_{jk})^2} |\Psi|^2 \Bigg) \mathrm{d}\mathbf{x}_j \mathrm{d}\mathbf{x}_k \mathrm{dx}'
\end{aligned}$$

$$\geq (n-1)_+ \, e_{\mathrm{LR}}(\alpha, 0),$$

where $e_{\mathrm{LR}}(\alpha, 0) := f((j'_{\alpha_N})^2)$ and

$$f(t) := \frac{1}{2} \sup_{\kappa \in (0,1)} \inf_{\int_{Q^2} |\psi|^2 = 1} \int_{Q^2} \left(\kappa \left(\left|\nabla_1 |\psi|\right|^2 + \left|\nabla_2 |\psi|\right|^2 \right) + (1-\kappa) t \frac{\mathbb{1}_{B_{\delta(\mathbf{X})}}(\mathbf{r})}{\delta(\mathbf{X})^2} |\psi|^2 \right) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2.\text{[4]} \tag{5.5}$$

We then use the convexity in n to obtain the corresponding bound for $T_Q[\Psi]$ in terms of $e_{\mathrm{LR}}(\alpha, 0)$. The upper bound $f(t) \leq 2\pi t$ is found simply by taking the trial state $\psi = \psi_0 \equiv 1$ and then $\kappa = 0$, carrying out the integration as below (with $\hat{\delta} = 0$).

We now wish to find a lower bound for the integral in $f(t)$, which then is to be maximized in κ. This is equivalent to finding a lower bound for the ground-state energy of the Schrödinger operator

$$H := -\kappa \Delta^{\mathcal{N}}_{Q^2} + t(1-\kappa) V, \qquad V(\mathbf{x}_1, \mathbf{x}_2) := V(\mathbf{r}, \mathbf{X}) = \frac{\mathbb{1}_{B_{\delta(\mathbf{X})}}(\mathbf{r})}{\delta(\mathbf{X})^2}.$$

However, to apply a projection bound or use Temple's inequality requires that $V \in L^\infty(Q^2)$ and $V \in L^2(Q^2)$, respectively. As neither of these conditions are satisfied for our V we use the fact that $V \geq 0$ and thus truncating our potential will only lower the energy. Therefore we instead study the eigenvalue problem with V replaced by the truncated potential $\hat{V}$ defined in relative coordinates by

$$\hat{V}(\mathbf{r}, \mathbf{X}) := \begin{cases} \dfrac{\mathbb{1}_{B_{\delta(\mathbf{X})}}(\mathbf{r})}{\delta(\mathbf{X})^2}, & \delta(\mathbf{X}) \geq \hat{\delta} \\[2ex] \dfrac{\mathbb{1}_{B_{\delta(\mathbf{X})}}(\mathbf{r})}{\hat{\delta}^2}, & \delta(\mathbf{X}) < \hat{\delta} \end{cases}$$

(in slightly more compact notation, $\hat{V} = \min\{V, 1/\hat{\delta}^2\}$). As $\hat{V} \in L^\infty(Q^2)$, $\|\hat{V}\|_\infty = 1/\hat{\delta}^2$, it follows that also $\hat{V} \in L^2(Q^2)$.

We proceed by calculating the expectation of $\hat{V}$ and $\hat{V}^2$ in the ground state $\psi_0 \equiv 1$ of $-\Delta^{\mathcal{N}}_{Q^2}$, as needed for the bounds. Through a straightforward calculation one finds that

$$\begin{aligned} \langle \hat{V} \rangle_{\psi_0} &= 4 \int_Q \int_{Q_{\mathbf{X}}} \hat{V}(\mathbf{r}, \mathbf{X}) \, \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{X} \\ &= 4 \left(\int_{[\hat{\delta}, 1-\hat{\delta}]^2} \int_{Q_{\mathbf{X}}} \frac{1}{\delta(\mathbf{X})^2} \, \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{X} + \int_{Q \setminus [\hat{\delta}, 1-\hat{\delta}]^2} \int_{Q_{\mathbf{X}}} \frac{1}{\hat{\delta}^2} \, \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{X} \right) \\ &= 4\pi \left(1 + 2\hat{\delta}^2 - \frac{8\hat{\delta}}{3} \right), \end{aligned}$$

[4] It also turns out that we do not gain much by keeping the n-dependence in the first term if we are aiming for a bound which is convex in n.

and correspondingly for $\hat{V}^2$ we obtain

$$\begin{aligned}
\langle \hat{V}^2 \rangle_{\psi_0} &= 4 \int_Q \int_{Q_{\mathbf{X}}} \hat{V}(\mathbf{r}, \mathbf{X})^2 \, d\mathbf{r} d\mathbf{X} \\
&= 4 \Big(\int_{[\hat{\delta}, 1-\hat{\delta}]^2} \int_{Q_{\mathbf{X}}} \frac{1}{\delta(\mathbf{X})^4} \, d\mathbf{r} d\mathbf{X} + \int_{Q \setminus [\hat{\delta}, 1-\hat{\delta}]^2} \int_{Q_{\mathbf{X}}} \frac{1}{\hat{\delta}^4} \, d\mathbf{r} d\mathbf{X} \Big) \\
&= 8\pi \Big(\frac{8}{3\hat{\delta}} + 4 \ln(2\hat{\delta}) - 5 \Big).
\end{aligned}$$

Choosing $\hat{\delta} = \eta/2$ for some $\eta \in [0, 1]$ (this normalization is convenient) results in

$$\langle \hat{V} \rangle_{\psi_0} = 4\pi \Big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3} \Big), \quad \text{and} \quad \langle \hat{V}^2 \rangle_{\psi_0} = 8\pi \Big(\frac{16}{3} \eta^{-1} + 4 \ln \eta - 5 \Big).$$

Our considerations here have been for $\Omega = Q$ the unit square but also other domains Ω could be of interest. Similar calculations when Ω is the unit disk and $\hat{\delta} = \eta$ give instead

$$\langle \hat{V} \rangle_{\psi_0} = 4 \Big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3} \Big), \quad \text{and} \quad \langle \hat{V}^2 \rangle_{\psi_0} = 2 \Big(\frac{16}{3} \eta^{-1} + 4 \ln \eta - 5 \Big).$$

Let P denote the orthogonal projection onto the ground state $\psi_0 \equiv 1$, and let $P^\perp = 1 - P$. Then $(-\Delta_{Q^2}^{\mathcal{N}})P = 0$, and with $\lambda_1(-\Delta_{Q^2}^{\mathcal{N}})$ the first non-zero Neumann eigenvalue,

$$(-\Delta_{Q^2}^{\mathcal{N}}) P^\perp \geq \lambda_1(-\Delta_Q^{\mathcal{N}}) P^\perp = \pi^2 P^\perp.$$

Arguing as in Lemma 4.4, for any $\mu \in (0, 1)$ we obtain that

$$\hat{V} \geq (1 - \mu) P \hat{V} P + (1 - \mu^{-1}) P^\perp \hat{V} P^\perp,$$

the first of these operators is equal to $\langle \hat{V} \rangle_{\psi_0} P$, and we can control the second term by using that $\| P^\perp \hat{V} P^\perp \| \leq \| \hat{V} \|_\infty = 4/\eta^2$.

Thus, for any $\mu, \kappa, \eta \in (0, 1)$, we find that

$$\begin{aligned}
H &\geq (1 - \mu) 4\pi t (1 - \kappa) \Big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3} \Big) P + \Big(\kappa \pi^2 + (1 - \mu^{-1}) \frac{4t(1 - \kappa)}{\eta^2} \Big) P^\perp \\
&\geq \min \Big\{ (1 - \mu) 4\pi t (1 - \kappa) \Big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3} \Big), \kappa \pi^2 + (1 - \mu^{-1}) \frac{4t(1 - \kappa)}{\eta^2} \Big\} (P + P^\perp).
\end{aligned}$$

The last expression, seen as a function in t, is piecewise linear and concave. Thus to obtain the largest linear minorant of this function it suffices to find the largest value attained at the right endpoint of our range of values t, that is at $t = (j_1')^2 \approx 3.8996$.

By the μ dependence of each of the two terms in the minimum this quantity is seen to be maximal when the two terms are equal. Solving this quadratic equation in μ and choosing $\eta = \kappa = 0.68$ we find that

$$H \geq t/3 \quad \text{and hence} \quad f(t) \geq t/6.$$

To obtain that $f(t) = 2\pi t(1 - O(t^{1/3}))$ we apply Temple's inequality (as in Lemma 5.2). In our current setting it yields that

$$\begin{aligned} H &\geq \langle t(1-\kappa)\hat{V}\rangle_{\psi_0} - \frac{\langle t^2(1-\kappa)^2\hat{V}^2\rangle_{\psi_0} - \langle t(1-\kappa)\hat{V}\rangle_{\psi_0}^2}{\kappa\lambda_1(-\Delta_{Q^2}^{\mathcal{N}}) - \langle t(1-\kappa)\hat{V}\rangle_{\psi_0}} \\ &= 4\pi t(1-\kappa)\Bigg(1 + \frac{\eta^2}{2} - \frac{4\eta}{3} \\ &\quad - \frac{2t(1-\kappa)}{\pi}\frac{\frac{16}{3}\eta^{-1} + 4\ln\eta - 5 - 2\pi\big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3}\big)^2}{\kappa\pi - 4t(1-\kappa)\big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3}\big)}\Bigg), \end{aligned}$$

provided that $\kappa\pi - 4t(1-\kappa)\big(1 + \frac{\eta^2}{2} - \frac{4\eta}{3}\big) > 0$. We decrease the above quantity by throwing away positive terms and increasing the denominator of the last term yielding

$$H \geq 4\pi t(1-\kappa)\left(1 - \frac{4\eta}{3} - \frac{32}{3\pi}\frac{(1-\kappa)t\eta^{-1}}{\kappa\pi - 4(1-\kappa)t}\right).$$

The positivity of denominator is then ensured if $\kappa \geq \frac{4t}{\pi}$. We can thus, for t sufficiently small, choose $\kappa = t^{\beta}$ for some $0 < \beta < 1$ to be fixed later. Inserting this into our expression we find that

$$H \geq 4\pi t(1-t^{\beta})\left(1 - \frac{4\eta}{3} - \frac{32}{3\pi}\frac{(1-t^{\beta})t\eta^{-1}}{t^{\beta}\pi - 4(1-t^{\beta})t}\right).$$

Setting $\eta = t^{\gamma}$, $\gamma > 0$, we obtain that

$$H \geq 4\pi t(1 - O(t^{\beta}) - O(t^{\gamma}) - O(t^{1-\beta-\gamma})),$$

and choosing $\beta = \gamma = 1/3$ yields

$$H \geq 4\pi t(1 - O(t^{1/3})).$$

Inserting this into (5.5) we have

$$f(t) = 2\pi t(1 - O(t^{1/3})),$$

which completes the proof.

Extended case. Let, in the case that $R \geq 0$, γ denote the relative length scale of the interaction, $\gamma = \gamma(Q) = R|Q|^{-1/2}$, and note that we may again rescale everything so that $|Q| = 1$. We then proceed as above using projection, where the bound from Theorem 1.3 is replaced by

$$\sum_{j=1}^{n}\int_{Q^n}|D_j\Psi|^2\,\mathrm{dx} \geq (1-\kappa')\frac{1}{n}\sum_{j<k}\int_{Q^n} g\left(\nu, \frac{3\gamma}{\delta(\mathbf{X}_{jk}) - 3\gamma}\right)^2\frac{\mathbb{1}_A(\mathbf{x}_j, \mathbf{x}_k)}{\delta(\mathbf{X}_{jk})^2}|\Psi|^2\,\mathrm{dx},$$

where $\nu = c(\kappa')\alpha_N/\sqrt{1-\kappa'}$ and $\kappa' \in (0, 1)$ is an additional parameter that we may optimize over, however we will in order to simplify the analysis take $\kappa' = 1/2$. Since

$\delta(\mathbf{X}_{jk})$ maximally takes the value 1/2, the above expression is zero for $\gamma \geq 1/12$. For $0 \leq \gamma < 1/12$ we can proceed by truncating to the, in γ, uniformly bounded potential

$$\hat{V}(\mathbf{X}, \mathbf{r}) := \frac{1}{2} g\Big(\nu, \frac{3\gamma}{\delta(\mathbf{X}) - 3\gamma}\Big)^2 \frac{\mathbb{1}_{\hat{A}}(\mathbf{x}_1, \mathbf{x}_2)}{\delta(\mathbf{X})^2},$$

with the support (consisting of truncated relative annuli)

$$\hat{A} := \{(\mathbf{x}_1, \mathbf{x}_2) \in Q^2 : 3\gamma + 1/4 \leq \delta(\mathbf{X}) \leq 1/2 \text{ and } 3\gamma \leq |\mathbf{r}| \leq \delta(\mathbf{X}) - 3\gamma\},$$

and therefore, since $g(\nu, \gamma)$ is monotonically decreasing in γ,

$$\|\hat{V}\|_\infty \leq \frac{1}{2(3\gamma + 1/4)^2} g(\nu, 0)^2 \leq 8(j'_\nu)^2.$$

Also, using the coarea formula and that $|\nabla\delta| = 1$ almost everywhere, we obtain that

$$\begin{aligned}
\langle \hat{V} \rangle_{\psi_0} &= \frac{1}{2} \int_Q \int_{Q_\mathbf{X}} g\Big(\nu, \frac{3\gamma}{\delta(\mathbf{X}) - 3\gamma}\Big)^2 \frac{\mathbb{1}_{\hat{A}}(\mathbf{x}_1, \mathbf{x}_2)}{\delta(\mathbf{X})^2} 4\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{X} \\
&= 2\pi \int_Q g\Big(\nu, \frac{3\gamma}{\delta(\mathbf{X}) - 3\gamma}\Big)^2 \frac{\big((\delta(\mathbf{X}) - 3\gamma)^2 - (3\gamma)^2\big)_+}{\delta(\mathbf{X})^2} \,\mathrm{d}\mathbf{X} \\
&= 8\pi \int_{3\gamma+1/4}^{1/2} g\Big(\nu, \frac{3\gamma}{t - 3\gamma}\Big)^2 (1 - 6\gamma/t)(1 - 2t) \,\mathrm{d}t \\
&\geq \frac{\pi}{3} g(\nu, 12\gamma)^2 (1 - 12\gamma)^3,
\end{aligned}$$

where in the last step we again used the monotonicity of g, and

$$\begin{aligned}
\int_{3\gamma+1/4}^{1/2} (1 - 6\gamma/t)(1 - 2t) \,\mathrm{d}t &= \Big(\frac{1}{16} + \Big(\frac{3}{2} - 6 \ln \frac{2}{1 + 12\gamma}\Big)\gamma - 27\gamma^2\Big) \\
&\geq \frac{1}{24}(1 - 12\gamma)^3,
\end{aligned}$$

where the lower bound is found by Taylor expansion around $\gamma = 1/12$.

Thus, the corresponding projection bound for the operator $H = -\kappa\Delta_{Q^2}^{\mathcal{N}} + (1-\kappa)\hat{V}$ reads

$$H \geq \min\Big\{(1-\mu)(1-\kappa)\frac{\pi}{3} g(\nu, 12\gamma)^2 (1-12\gamma)_+^3,\ \kappa\pi^2 - (\mu^{-1}-1)8(1-\kappa)(j'_\nu)^2\Big\}.$$

We take, for simplicity, $\mu = 1/2$ and $\kappa = 1/2$, and use that $g(\nu, 12\gamma) \leq j'_\nu \ll \pi$, to obtain the claimed bound

$$\sum_{j=1}^n \int_{Q^n} |D_j \Psi|^2 \,\mathrm{d}x \geq (n-1)_+ \, e_{\mathrm{LR}}(\alpha, \gamma), \quad e_{\mathrm{LR}}(\alpha, \gamma) = \frac{\pi}{24} g(\nu, 12\gamma)^2 (1-12\gamma)_+^3,$$

with $\nu = c\alpha_N$ and $c = c(\kappa')/\sqrt{1 - \kappa'} = 5.3/\sqrt{8} \cdot 10^{-4}$. Again we may use the convexity in n to obtain the corresponding bound for $T_Q[\Psi]$. □

6. Application to the Homogeneous Anyon Gas

Let us finally consider the homogeneous gas in the thermodynamic limit, that is N particles confined to a large box (square) $Q_0 \subseteq \mathbb{R}^2$, where we shall take simultaneously $N \to \infty$ and $|Q_0| \to \infty$ while keeping the density $\bar{\varrho} := N/|Q_0|$ fixed. The only dimensionless parameters are then the magnetic interaction strength $\alpha \in \mathbb{R}$ and the relative interaction length scale (magnetic filling ratio) $\bar{\gamma} := R\bar{\varrho}^{\frac{1}{2}}$, also held fixed, so that in the limit the ground-state energy,

$$E_0(N, Q_0, \alpha, R) := \inf\left\{\langle \Psi, \hat{T}_\alpha \Psi\rangle \;:\; \Psi \in \mathscr{D}^N_{\alpha,R} \cap C^\infty_c(Q_0^N),\ \|\Psi\|_2 = 1\right\},$$

per particle must for dimensional reasons be given by

$$\frac{E_0(N, Q_0, \alpha, R)}{N} \to e(\alpha, \bar{\gamma})\bar{\varrho}, \tag{6.1}$$

where $e(\alpha, \bar{\gamma}) \geq 0$ is dimensionless. We have that $e(0, \bar{\gamma}) = 0$ for all $\bar{\gamma} \geq 0$, corresponding to non-interacting bosons, and $e(1, 0) = 2\pi$ for ideal fermions in two dimensions due to the Weyl asymptotics for the Laplacian eigenvalues. We also have a reflection-conjugation symmetry $e(-\alpha, \bar{\gamma}) = e(\alpha, \bar{\gamma})$ for all $\alpha, \bar{\gamma}$. Furthermore, in the dilute limit we should see a periodicity in the entire spectrum with respect to any shift in α by an even integer, and in particular

$$e(\alpha + 2n, 0) = e(\alpha, 0) \qquad \forall\, \alpha \in \mathbb{R},\ n \in \mathbb{Z},$$

due to the gauge equivalence (1.7). On the other hand, average-field theory (1.3) suggests a linear dependence $e(\alpha, \bar{\gamma}) = 2\pi|\alpha|$ for arbitrary α and large enough $\bar{\gamma}$. Hence there must be some non-trivial interpolation between these two regimes of low respectively high density.

Although the existence of the thermodynamic limit (6.1) might be expected on physical grounds, as is indeed the case for bosons and fermions with reasonable scalar interactions (see for example [6,35]), we are not aware of any proof of it for anyons, whose interaction is long-range and magnetic instead of scalar. Furthermore, there is for anyons also a subtlety in the choice of boundary conditions, partly since topology plays an important role in the whole problem and therefore periodic b.c. may seem problematic, and even in the case of a constant magnetic field we know that Neumann and Dirichlet b.c. differ substantially (cf. Section 2 and Proposition 4.3). We shall therefore replace the limit (6.1) with the lim inf and also stick to Dirichlet b.c. ('hard-wall' confined anyons) in all that follows.

Theorem 6.1. (Universal bounds for the homogeneous anyon gas) *Let $e(\alpha, \bar{\gamma})$, where $\bar{\gamma} = R\bar{\varrho}^{1/2}$, denote the ground-state energy per particle and unit density of the extended anyon gas in the thermodynamic limit at fixed $\alpha \in \mathbb{R}$, $R \geq 0$ and density $\bar{\varrho} > 0$ where Dirichlet boundary conditions have been imposed, that is*

$$e(\alpha, \bar{\gamma}) := \liminf_{\substack{N, |Q_0| \to \infty \\ N/|Q_0| = \bar{\varrho}}} \frac{E_0(N, Q_0, \alpha, R)}{\bar{\varrho} N}.$$

Then

$$e(\alpha,\bar{\gamma}) \geq C\biggl(2\pi \frac{|\alpha| \min\bigl\{2(1-\bar{\gamma}^2/4)^{-1}, K_\alpha\bigr\}}{K_\alpha + 2|\alpha|\bigl(-\ln(\bar{\gamma}/2)\bigr)} \mathbb{1}_{\bar{\gamma}<2} + 2\pi|\alpha|\mathbb{1}_{\bar{\gamma}\geq 2}$$
$$+ \pi g\bigl(c\alpha_*, 12\bar{\gamma}/\sqrt{2}\bigr)^2 (1-12\bar{\gamma}/\sqrt{2})_+^3\biggr), \tag{6.2}$$

for some universal constant $C > 0$, *with* K_α *given in Lemma 5.1, and* $c > 0$ *in Lemma 5.3. Furthermore, for any* $\alpha \in \mathbb{R}$ *and with* f *given in Lemma 5.3, we have for the ideal gas that*

$$e(\alpha, 0) \geq \frac{1}{4} f\bigl((j'_{\alpha_*})^2\bigr) = \frac{1}{2} 2\pi\alpha_*\bigl(1 - O\bigl(\alpha_*^{1/3}\bigr)\bigr). \tag{6.3}$$

Moreover, for any fixed $\alpha \in \mathbb{R}\backslash\{0\}$ *we obtain in the dilute limit that*

$$\liminf_{\bar{\gamma}\to 0} \frac{e\,(\alpha,\bar{\gamma})}{2\pi\,|\ln\bar{\gamma}|^{-1}} \geq 1, \qquad \textit{and} \qquad \liminf_{\bar{\gamma}\to 0} e\,(\alpha,\bar{\gamma}) \geq \frac{\pi}{81}\left(j'_{c\alpha_*}\right)^2 \geq \frac{c}{81} 2\pi\alpha_*, \tag{6.4}$$

while if $\bar{\gamma} > 0$ *is arbitrary but fixed, and*

$$|\alpha| \leq \varepsilon^5 \min\bigl\{\bar{\gamma}^2, \varepsilon^3\bar{\gamma}^{-4}\bigr\}, \quad 0 < \varepsilon < \sqrt{\pi}/8, \tag{6.5}$$

then

$$e(\alpha,\bar{\gamma}) \geq 2\pi|\alpha|(1 - O(\varepsilon)). \tag{6.6}$$

Note that for the short-range part of the interaction, one can view the height of the potential compared to the average density as a dimensionless interaction strength, and that in the dilute limit (6.4) with fixed $\alpha > 0$ we have that

$$\frac{\alpha}{R^2}/\bar{\varrho} = \alpha\bar{\gamma}^{-2} \to \infty,$$

corresponding to a hard-core interaction. On the other hand, under the conditions in (6.5),

$$\frac{\alpha}{R^2}/\bar{\varrho} = \alpha\bar{\gamma}^{-2} \leq \varepsilon^5 \ll 1,$$

and thus corresponding to a very weak soft-core interaction rather than a hard-core one in this regime.

We also note that the average-field description with its linear dependence on α has indeed been proved to be correct for the trapped anyon gas in a certain almost-bosonic regime; see [47]. In the present context this corresponds to taking Q_0 fixed, $\alpha \sim \beta/N$ and $R \sim N^{-\eta}$ with $0 < \eta < 1/4$, in which case we have that $\bar{\gamma} \sim N^{1/2-\eta} \to \infty$ and $\alpha\bar{\gamma}^{-2} \sim N^{2\eta-2} \to 0$ as $N \to \infty$, that is a combined high-density and weak soft-core limit. However, the sense in which average-field theory then holds is that all the anyons become identically distributed subject to a self-consistent magnetic field, and it should be remarked that the constant 2π that is predicted by the usual (constant-field) average-field approximation and which appears above does not take such self-interactions fully into account and may ultimately be replaced by a larger effective constant, at least in a particular limit [11].

Proof of Theorem 6.1. Let us begin with the universal bound (6.2) for all $\alpha, \bar{\gamma}$. We have a sequence of $N \geq 1$ and squares $Q_0 \subseteq \mathbb{R}^2$ with $N/|Q_0| = \bar{\varrho}$, and consider in each case an arbitrary function $\Psi \in \mathscr{D}^N_{\alpha,R}$ supported on Q_0^N. Let us again write

$$\begin{aligned} T[\Psi] &:= \langle \Psi, \hat{T}_\alpha \Psi \rangle = \kappa_1 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 \,\mathrm{dx} + \kappa_2 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 \,\mathrm{dx} \\ &\quad + \kappa_3 \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j \Psi|^2 \,\mathrm{dx} \\ &\geq \int_{\mathbb{R}^{2N}} \sum_{j=1}^N \Big(\kappa_1 \big|\nabla_j |\Psi|\big|^2 + \kappa_2 \sum_{k \neq j} 2\pi |\alpha| \frac{\mathbb{1}_{B_R(0)}}{\pi R^2} (\mathbf{x}_j - \mathbf{x}_k)\, |\Psi|^2 \\ &\quad + \kappa_3 |D_j \Psi|^2 \Big) \mathrm{dx}. \end{aligned} \tag{6.7}$$

Take $\kappa_1 = \kappa_2 = \kappa/2$ and $\kappa_3 = 1-\kappa$, and a partition of Q_0 into M^2 squares Q of equal size. Then, by the local exclusion principles of Lemmas 5.1 and 5.3,

$$\begin{aligned} N^{-1} T[\Psi] &\geq N^{-1} \sum_Q T_Q^{(\kappa/2, \kappa/2, 1-\kappa)}[\Psi] \qquad (6.8) \\ &\geq N^{-1} \sum_Q |Q|^{-1} \Big(\frac{\kappa}{2} e_{\mathrm{SR}}\big(\alpha, \gamma(Q), \textstyle\int_Q \varrho_\Psi\big) + (1-\kappa) e_{\mathrm{LR}}\big(\alpha, \gamma(Q)\big) \Big) \Big(\displaystyle\int_Q \varrho_\Psi - 1 \Big)_+ \\ &\geq N^{-1} |Q_0|^{-1} M^2 \sum_Q \Big(\int_Q \varrho_\Psi - 1 \Big)_+ \\ &\quad \times \begin{cases} \frac{\kappa}{2} |\alpha| \min\big\{(1-\gamma(Q)^2/2)^{-1}, K_\alpha/2\big\} \big(K_\alpha + 2|\alpha| \big(-\ln(\gamma(Q)/\sqrt{2})\big)\big)^{-1} & \\ \quad + (1-\kappa) \frac{\pi}{24} g\big(c\alpha_N, 12\gamma(Q)\big)^2 (1 - 12\gamma(Q))_+^3, & \text{for } \gamma(Q) < \sqrt{2} \\ \kappa |\alpha| \gamma(Q)^{-2} \int_Q \varrho_\Psi, & \text{for} \gamma(Q) \geq \sqrt{2}. \end{cases} \end{aligned}$$

Note that $\gamma(Q) = \bar{\gamma} M N^{-1/2}$ and we are free to choose $\kappa \in [0,1]$ and the integer $M \geq 1$ as we like. We choose $M := \mu N^{1/2}$ for suitable $\mu > 0$, so that $\gamma(Q) = \mu\bar{\gamma}$. Then for $\mu < \min\{\sqrt{2}/\bar{\gamma}, 1\}$ we have, using $\sum_Q (\int_Q \varrho_\Psi - 1)_+ \geq (N - M^2)_+$, that

$$\begin{aligned} N^{-1} T[\Psi] \geq \bar{\varrho} \mu^2 (1-\mu^2)_+ \Big(&\frac{\kappa}{2} |\alpha| \frac{\min\big\{(1-\mu^2\bar{\gamma}^2/2)_+^{-1}, K_\alpha/2\big\}}{K_\alpha + 2|\alpha| \big(-\ln(\mu\bar{\gamma}/\sqrt{2})\big)} \\ &+ (1-\kappa) \frac{\pi}{24} g(c\alpha_N, 12\mu\bar{\gamma})^2 (1 - 12\mu\bar{\gamma})_+^3 \Big). \end{aligned} \tag{6.9}$$

On the other hand for $\sqrt{2}/\bar{\gamma} \leq \mu \leq 1$, we may use

$$\frac{1}{M^2} \sum_Q \int_Q \varrho_\Psi \Big(\int_Q \varrho_\Psi - 1 \Big)_+ \geq \frac{N}{M^2} \Big(\frac{N}{M^2} - 1 \Big)_+,$$

which follows from convexity, to obtain that

$$N^{-1}T[\Psi] \geq \kappa|\alpha|\bar{\varrho}\bar{\gamma}^{-2}(\mu^{-2}-1)_+.$$

Hence, in the case $\bar{\gamma} \geq 2 > \sqrt{2}$ we can in the thermodynamic limit choose $\kappa = 1$ and $\mu = \sqrt{2}/\bar{\gamma}$ in order to obtain that

$$e(\alpha,\bar{\gamma}) \geq \frac{1}{2}|\alpha|(1-2/\bar{\gamma}^2) \geq \frac{1}{4}|\alpha|,$$

while for $\bar{\gamma} < 2$ we choose, for simplicity, $\kappa = 2/3$ and $\mu = 1/\sqrt{2}$ obtaining that

$$\begin{aligned} e(\alpha,\bar{\gamma}) \geq \frac{1}{288}\Bigg(& 12|\alpha|\frac{\min\{2(1-\bar{\gamma}^2/4)_+^{-1}, K_\alpha\}}{K_\alpha + 2|\alpha|(-\ln(\bar{\gamma}/2))} \\ & + \pi g(c\alpha_N, 12\bar{\gamma}/\sqrt{2})^2(1-12\bar{\gamma}/\sqrt{2})_+^3\Bigg). \end{aligned}$$

This proves the first part of the theorem with $C = 1/288$.

In the ideal case $R = 0$, and hence $\bar{\gamma} = 0$, we take $\kappa = 0$ and $M \sim \sqrt{N/2}$ in (6.8) (which means approximately 2 particles in each box) to obtain (6.3) from (5.4) of Lemma 5.3.

The second bound in (6.4) follows immediately from (6.9) and the properties of g, by setting $\kappa = 0$ and $\mu = 1/\sqrt{2}$. For the first bound we set $\kappa_1 = 1-\kappa$, $\kappa_2 = \kappa$ and $\kappa_3 = 0$ in (6.7) and use the result [41] of Lieb and Yngvason for the dilute repulsive Bose gas in two dimensions. We find for the (bosonic, and therefore positive; see [35, Corollary 3.1]) ground state Ψ_0 of this expression, with fixed $\kappa \in (0,1)$ and $\alpha > 0$, that

$$\begin{aligned} \frac{T[\Psi]}{N\bar{\varrho}} &\geq \frac{1-\kappa}{N\bar{\varrho}}\int_{\mathbb{R}^{2N}}\left(\sum_{j=1}^{N}\left|\nabla_j\Psi_0\right|^2 + \sum_{j<k}W(\mathbf{x}_j-\mathbf{x}_k)\,|\Psi_0|^2\right)\mathrm{dx} \\ &= \frac{4\pi(1-\kappa)}{|\ln a_R^2\bar{\varrho}|}\left(1+O\left(|\ln a_R^2\bar{\varrho}|^{-1/5}\right)\right) \\ &= \frac{2\pi(1-\kappa)}{K'_{\alpha,\kappa}-\ln\bar{\gamma}}\left(1+O\left((K'_{\alpha,\kappa}-\ln\bar{\gamma})^{-1/5}\right)\right), \end{aligned}$$

where we used that the pair potential

$$W(\mathbf{x}) := \frac{W_0}{R^2}\mathbb{1}_{B_R(0)}(\mathbf{x}), \qquad W_0 = 4\alpha\kappa/(1-\kappa),$$

has scattering length (cf. (5.3))

$$a_R = R\exp\left(-\frac{1}{\sqrt{W_0/2}}\frac{I_0(\sqrt{W_0/2})}{I_1(\sqrt{W_0/2})}\right) = R\exp(-K'_{\alpha,\kappa}),$$

with

$$K'_{\alpha,\kappa} := \frac{1}{\sqrt{2\alpha\kappa/(1-\kappa)}}\frac{I_0(\sqrt{2\alpha\kappa/(1-\kappa)})}{I_1(\sqrt{2\alpha\kappa/(1-\kappa)})} = \frac{K_{\alpha\kappa/(1-\kappa)}}{2\alpha\kappa/(1-\kappa)}.$$

Hence for any $\alpha > 0$ and $0 < \varepsilon \ll 1$ we, by setting $\kappa = \varepsilon$ and then taking the limit $\bar{\gamma} \to 0$, obtain that

$$\begin{aligned} \frac{|\ln \bar{\gamma}|}{2\pi} e(\alpha, \bar{\gamma}) &\geq (1-\varepsilon)\big(1 + K'_{\alpha,\varepsilon}|\ln \bar{\gamma}|^{-1}\big)^{-1}\big(1 + O\big((K'_{\alpha,\varepsilon} + |\ln \bar{\gamma}|)^{-1/5}\big)\big) \\ &\to 1 - \varepsilon. \end{aligned}$$

So for each fixed $\alpha \in \mathbb{R}\backslash\{0\}$

$$\liminf_{\bar{\gamma}\to 0} \frac{e(\alpha, \bar{\gamma})}{2\pi |\ln \bar{\gamma}|^{-1}} \geq 1.$$

To obtain the bound (6.6) for the soft-core regime we follow [36, 39–41]. Again we partition Q_0 into M^2 squares Q of equal size, and let $\ell = |Q|^{1/2}$. With $\kappa \in [0,1]$ we then have that

$$N^{-1}T[\Psi] \geq N^{-1}\sum_Q T_Q^{(\kappa,1-\kappa,0)}[\Psi] \geq N^{-1}\sum_Q \sum_{n\geq 0} E_n^{(\kappa,1-\kappa,0)}(|Q|)p_n(\Psi; Q).$$

Set $c_n = \sum_Q p_n(\Psi; Q)|Q|/|Q_0|$, that is c_n is the fraction of cells Q containing precisely n particles, then

$$\sum_{n\geq 0} c_n = 1 \quad \text{and} \quad \sum_{n\geq 0} c_n n = \bar{\varrho}\ell^2.$$

Rearranging the sum and from now on suppressing the weight $\boldsymbol{\kappa} = (\kappa, 1-\kappa, 0)$ we find that

$$N^{-1}T[\Psi] \geq \frac{1}{\bar{\varrho}\ell^2}\sum_{n\geq 0} E_n(|Q|)c_n, \tag{6.10}$$

which is precisely the starting point of the argument in [39–41].

Fix $p \in \mathbb{N}$. Since the energy is superadditive, $E_{n+n'} \geq E_n + E_{n'}$, we for all $n \geq p$ have that

$$E_n(|Q|) \geq \lfloor n/p \rfloor E_p(|Q|) \geq \frac{n}{2p}E_p(|Q|).$$

Applying Lemma 5.2 yields

$$E_n(|Q|) \geq \pi|\alpha|\frac{n(p-1)}{\ell^2}K(p,\ell),$$

where

$$K(n,\ell) := (1-\kappa)\left(1 - \frac{2R}{\ell}\right)_+^2\left(1 - \frac{2|\alpha|\ell^2R^{-2}n(n-1)}{\pi^2\kappa/(1-\kappa) - 2\pi|\alpha|n(n-1)}\right)_+,$$

if the expression in the last denominator is positive and $K(n,\ell) := 0$ otherwise.

If instead $n < p$ we use that $K(n,\ell)$ is decreasing in n to find

$$E_n(|Q|) \geq 2\pi|\alpha|\frac{n(n-1)}{\ell^2}K(p,\ell).$$

Splitting the sum (6.10) into two we thus find that

$$\begin{aligned}\sum_{n\geq 0} E_n(|Q|)c_n &= \sum_{n<p} E_n(|Q|)c_n + \sum_{n\geq p} E_n(|Q|)c_n \\ &\geq \frac{2\pi|\alpha|}{\ell^2} K(p,\ell)\Bigg(\sum_{n<p} n(n-1)c_n + \frac{1}{2}\sum_{n\geq p} n(p-1)c_n\Bigg).\end{aligned}$$

We wish to minimize

$$\sum_{n<p} n(n-1)c_n + \frac{1}{2}\sum_{n\geq p} n(p-1)c_n. \tag{6.11}$$

Set

$$k := \bar{\varrho}\ell^2 \quad \text{and} \quad t := \sum_{n<p} c_n n \leq k,$$

by convexity (6.11) is then larger than

$$t(t-1) + \frac{1}{2}(k-t)(p-1).$$

If $p \geq 4k-1$ and $t \leq k$ this is minimized at $t = k$, where it is equal to $k(k-1)$. Thus by choosing $p = \lfloor 4\bar{\varrho}\ell^2 \rfloor$ we have shown that

$$N^{-1}T[\Psi] \geq 2\pi|\alpha|\bar{\varrho}\left(1 - \frac{1}{\bar{\varrho}\ell^2}\right)_+ K(4\bar{\varrho}\ell^2, \ell),$$

and hence, upon taking the thermodynamic limit $N, |Q_0| \to \infty$ with all the other parameters kept fixed,

$$e(\alpha,\bar{\gamma}) \geq 2\pi|\alpha|(1-\kappa)\left(1-\frac{1}{\bar{\varrho}\ell^2}\right)_+\left(1-2\frac{\bar{\gamma}}{\bar{\varrho}^{1/2}\ell}\right)_+^2\left(1-\frac{32|\alpha|\bar{\gamma}^{-2}\bar{\varrho}^3\ell^6}{\pi^2\kappa/(1-\kappa)-32\pi|\alpha|\bar{\varrho}^2\ell^4}\right)_+, \tag{6.12}$$

as long as $32|\alpha|\bar{\varrho}^2\ell^4 < \pi\kappa/(1-\kappa)$.

Given $\varepsilon > 0$, let us choose $\kappa = \varepsilon$ and also demand that $(\bar{\varrho}\ell^2)^{-1} \leq \varepsilon$, $\bar{\gamma}(\bar{\varrho}^{1/2}\ell)^{-1} \leq \varepsilon$, $|\alpha|\bar{\gamma}^{-2}\bar{\varrho}^3\ell^6 \leq \varepsilon^2$, and $|\alpha|\bar{\varrho}^2\ell^4 \leq \varepsilon\pi/64$. We therefore choose

$$\ell = (\varepsilon\bar{\varrho})^{-1/2}\max\left\{1, \varepsilon^{-1/2}\bar{\gamma}\right\}$$

and then find that, together with the requirement (6.5) on α and ε which implies $|\alpha|\bar{\varrho}^2\ell^4 \leq \varepsilon^3 < \varepsilon\pi/64$, all conditions above are satisfied, and the error terms in (6.12) are of order ε or higher. □

Appendix A. Some Properties of Bessel Functions

Proposition A.1. *For $\nu > 0$ we let j'_ν denote the first positive zero of the derivative of the Bessel function J_ν. Then we have that*

$$\sqrt{2\nu} \le j'_\nu \le \sqrt{2\nu(1+\nu)}.$$

A proof of the above proposition and much more refined bounds for the zeros of Bessel functions and their derivatives can be found in [26]. For completeness we provide an elementary proof which covers our needs.

Proof. By a standard variational argument it can be shown that

$$\inf_u \frac{\int_0^1 \big(|u'|^2 + \nu^2 r^{-2}|u|^2\big) r \,\mathrm{d}r}{\int_0^1 |u|^2 r \,\mathrm{d}r} = (j'_\nu)^2,$$

where the infimum is taken over all $u \in W^{1,2}([0,1], r\mathrm{d}r)$ and is attained by $u(r) = J_\nu(j'_\nu r)$.

For $\nu > 0$ and $u \in W^{1,2}([0,1], r\mathrm{d}r)$ with $u(0) = 0$ we obtain using Hölder's inequality that

$$\begin{aligned}
|u(t)|^2 &= 2\Re\bigg[\int_0^t \bar{u}(r)u'(r)\,\mathrm{d}r\bigg] \\
&\le 2\bigg(\int_0^t |u'(r)|^2 r\,\mathrm{d}r\bigg)^{1/2}\bigg(\int_0^t |u(r)|^2 r^{-1}\,\mathrm{d}r\bigg)^{1/2} \\
&= \frac{2}{\nu}\bigg(\int_0^t |u'(r)|^2 r\,\mathrm{d}r\bigg)^{1/2}\bigg(\nu^2\int_0^t |u(r)|^2 r^{-1}\,\mathrm{d}r\bigg)^{1/2}.
\end{aligned}$$

Through an application of Young's inequality we then find

$$|u(t)|^2 \le \frac{1}{\nu}\int_0^t \Big(|u'(r)|^2 + \frac{\nu^2}{r^2}|u(r)|^2\Big) r\,\mathrm{d}r \le \frac{1}{\nu}\int_0^1 \Big(|u'(r)|^2 + \frac{\nu^2}{r^2}|u(r)|^2\Big) r\,\mathrm{d}r,$$

and integrating both sides in t over $(0,1)$ against $t\,\mathrm{d}t$ yields

$$\begin{aligned}
\int_0^1 |u(t)|^2 t\,\mathrm{d}t &\le \frac{1}{\nu}\bigg(\int_0^1 t\,\mathrm{d}t\bigg)\bigg(\int_0^1 \Big(|u'(r)|^2 + \frac{\nu^2}{r^2}|u(r)|^2\Big) r\,\mathrm{d}r\bigg) \\
&= \frac{1}{2\nu}\int_0^1 \Big(|u'(r)|^2 + \frac{\nu^2}{r^2}|u(r)|^2\Big) r\,\mathrm{d}r,
\end{aligned}$$

which implies that

$$\frac{\int_0^1 \big(|u'|^2 + \nu^2 r^{-2}|u|^2\big) r\,\mathrm{d}r}{\int_0^1 |u|^2 r\,\mathrm{d}r} \ge 2\nu.$$

Taking the infimum over all functions $u \in W^{1,2}([0,1], r\mathrm{d}r)$ such that $u(0) = 0$, in particular this includes J_ν, we see that

$$(j'_\nu)^2 \ge 2\nu,$$

which completes the proof of the lower bound. To obtain the upper bound, simply take $u(r) = r^\nu$ in the variational quotient above. □

In the case of R-extended anyons our bounds result in studying the behavior of solutions to a Bessel-type eigenvalue equation of order ν with Neumann boundary conditions on the interval $(\gamma, 1)$, for some $0 < \gamma < 1$. Thus it is of interest for us to understand the behavior of the lowest eigenvalue of such an equation in both parameters γ and ν.

Proposition A.2. *Given $\nu > 0$ and $0 < \gamma < 1$, let $g(\nu, \gamma) := \sqrt{\lambda}$, where λ denotes the first positive solution to the eigenvalue equation*

$$-u''(r) - \frac{u'(r)}{r} + \Big(\frac{\nu^2}{r^2} - \lambda\Big)u(r) = 0, \tag{A.1}$$

with the Neumann boundary conditions $u'(\gamma) = u'(1) = 0$. Then, for fixed γ, $g(\nu, \gamma)$ is a monotonically increasing function in ν. Also, for fixed ν, $g(\nu, \gamma)$ is a monotonically decreasing function of γ, and satisfies

$$\nu < g(\nu, \gamma) < \min\{j'_\nu, \nu/\gamma\}.$$

Moreover, we have that $\lim_{\gamma\to 0} g(\nu, \gamma) = j'_\nu$ and $\lim_{\gamma\to 1} g(\nu, \gamma) = \nu$.

Proof. That $g(\nu, \gamma)$ is monotonically increasing in ν is clear from the variational characterization of λ,

$$\lambda = \inf_u \frac{\int_\gamma^1 \big(|u'|^2 + \nu^2 r^{-2}|u|^2\big) r \, \mathrm{d}r}{\int_\gamma^1 |u|^2 r \, \mathrm{d}r}.$$

It is well known that the solution of the above differential equation is given by a linear combination of the Bessel functions $J_\nu(\sqrt{\lambda} r)$ and $Y_\nu(\sqrt{\lambda} r)$. Only if γ were zero could we exclude the Bessel function of the second kind since it fails to be in $W^{1,2}([0, 1], r\mathrm{d}r)$ and thus cannot be a solution. Thus the problem reduces to finding the smallest $\lambda > 0$ such that the system

$$\alpha J'_\nu(\sqrt{\lambda}\gamma) + \beta Y'_\nu(\sqrt{\lambda}\gamma) = 0$$
$$\alpha J'_\nu(\sqrt{\lambda}) + \beta Y'_\nu(\sqrt{\lambda}) = 0$$

admits a non-trivial solution, which is equivalent to the determinant equation

$$J'_\nu(\sqrt{\lambda}\gamma) Y'_\nu(\sqrt{\lambda}) - Y'_\nu(\sqrt{\lambda}\gamma) J'_\nu(\sqrt{\lambda}) = 0.$$

Assuming that $\sqrt{\lambda}$ is smaller than the first zero of Y'_ν (this will be seen to be true once we find our solution) we can equivalently solve the equation

$$\frac{J'_\nu(\sqrt{\lambda})}{Y'_\nu(\sqrt{\lambda})} = \frac{J'_\nu(\sqrt{\lambda}\gamma)}{Y'_\nu(\sqrt{\lambda}\gamma)}.$$

Letting $G_\nu(x) := J'_\nu(x)/Y'_\nu(x)$ we find that

$$G'_\nu(x) = \frac{2(\nu^2 - x^2)}{\pi x^3 Y'_\nu(x)^2},$$

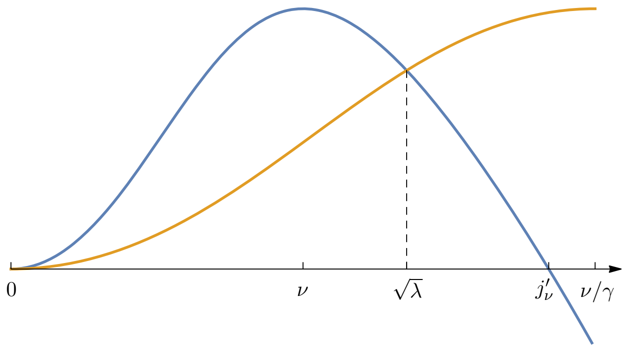


Fig. 7. The function $G_\nu(x)$ (*blue*) and its dilation $G_\nu(\gamma x)$ (*yellow*) plotted for $\nu = 1$ and $\gamma = 1/2$

where we used that J_ν and Y_ν satisfy the Bessel Equation (A.1) and the well-known identity $J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x) = 2/(\pi x)$; see for example [59, Eqn. 10.5.2].

Thus $G_\nu(x)$ is strictly increasing on $(0, \nu)$ and decreasing after that. We also know that $G_\nu(0) = G_\nu(j'_\nu) = 0$. But then it is clear that the graph of $G_\nu(x)$ and that of its dilation $G_\nu(\gamma x)$ must intersect between $x = \nu$ and the minimum of $x = \nu/\gamma$ and $x = j'_\nu$ (compare Fig. 7), and as this solution is less than the first zero of Y'_ν the assumption above is seen to be true. Moreover, as $\gamma \to 0$ we see that the solution $x = \sqrt{\lambda}$ tends to the zero j'_ν and if instead $\gamma \to 1$ it tends to the maximum point ν.

By the above geometric considerations we can conclude that for $0 < \gamma < 1$ and $\nu > 0$ we have that λ, the smallest positive eigenvalue of (A.1), satisfies

$$\lambda \in [\nu^2, \min\{j'_\nu, \nu/\gamma\}^2],$$

and is monotonically decreasing in γ. □

Appendix B. Concavity of the One-Particle Profile

We have several times used concavity properties of the one-particle profile $f(d, \cdot)$, which however may fail if d is small. More precisely we have the following:

Proposition B.1. *For any $d \geq 0$ the function $f(d, \cdot)$ given by (4.12) is concave on its support intersected with $[R, \infty)$. If in addition $d \geq R$ the function is concave on its full support $[d - R, d + R]$.*

Proof. Without loss of generality we may, and do, assume that $R = 1$. The proof is then a straightforward computation. We begin with assuming that $d < R = 1$. For such d the function $f(d, \cdot)$ is C^2 on $[1, d + 1]$ (and zero on $(d + 1, \infty)$) which reduces the statement to proving that $\partial_r^2 f(d, r) \leq 0$ in this region. Calculating this derivative one finds

$$\partial_r^2 f(d, r) = -\frac{2((d^2 - 1)^3 - 3(d^2 - 1)^2 r^2 + (5 + 3d^2)r^4 - r^6)}{\pi r((r + 1 - d)(1 + d - r)(d + r - 1)(1 + d + r))^{3/2}},$$

and clearly the overall sign is determined by that of the polynomial in the denominator

$$p(d,r) := (d^2-1)^3 - 3(d^2-1)^2r^2 + (5+3d^2)r^4 - r^6.$$

We need to prove that $p \geq 0$ for (r,d) in the triangular region given by $1 \leq r \leq d+1$ where $0 \leq d \leq 1$.

We first check the statement on the boundary of the region:

$$\begin{aligned} p(1,r) &= r^4(8-r^2) > 0 \\ p(d,1) &= d^2(12-6d^2+d^4) > 0 \\ p(d,d+1) &= (1+r)(r-1)(4r^2+1-r^4) > 0. \end{aligned}$$

Thus all that remains is to check that we have no stationary points for p in the interior of the region. Calculating the derivative in r one finds that

$$\partial_r p(d,r) = 6d(d^2-1)^2 - 12d(d^2-1)r^2 + 6dr^4.$$

As this is a quadratic polynomial in r^2 we can solve the equation $p_r(d,r) = 0$ and find that there are no solutions in our region. This completes the proof of the claim in the case $d < R = 1$.

In the case $d \geq R = 1$ we wish to prove that $f(d,\cdot)$ is concave on $[d-1, d+1]$. It is here convenient to study the problem in the variables d and $\eta = r - d$, and letting

$$g(d,\eta) := f(d, d+\eta) = \frac{2(d+\eta)}{\pi} \arccos\left(\frac{d^2+(d+\eta)^2-1}{2d(d+\eta)}\right),$$
$$d \geq 1,\ \eta \in [-1,1].$$

Differentiating twice in η we find that

$$\partial_\eta^2 g(d,\eta) = \frac{2P(d,\eta)}{\pi(d+\eta)((1-\eta^2)(2d+\eta-1)(2d+\eta+1))^{3/2}}, \tag{B.1}$$

where

$$\begin{aligned} P(d,\eta) := &-8d^4 + 8d^3(\eta^3-4\eta) + 12d^2(\eta^4-3\eta) + d(6\eta^5-20\eta^3+6\eta) \\ &+ \eta^6 - 5\eta^4 + 3\eta^2 + 1. \end{aligned}$$

As before the sign of (B.1) is determined by that of the polynomial P. If we can prove that $P(d,\eta) \leq 0$ for all $d \geq 1$ and $-1 \leq \eta \leq 1$ the claim follows. To this end we proceed as above. The values of P on the boundaries of this region are (in the same manner as before) readily checked to be negative:

$$\begin{aligned} P(d,1) &= -8d(d+1)^3 < 0, \\ P(d,-1) &= -8d(d-1)^3 \leq 0, \\ P(1,\eta) &= (1+\eta)^4(\eta^2+2\eta-7) \leq 0. \end{aligned}$$

What remains is to check stationary points in the interior. For this polynomial solving either of the equations $\partial_\eta P(d,\eta) = 0$ or $\partial_d P(d,\eta) = 0$ is slightly harder. However, since certain terms cancel one can instead solve the equation

$$\partial_\eta P(d,\eta) = \partial_d P(d,\eta),$$

and the solutions are $d = 0$, $\eta = -d - \sqrt{d^2-1}$ and $\eta = -d + \sqrt{d^2-1}$. The third solution is the only one contained within our region. Evaluating the derivative at this solution we obtain

$$\partial_\eta P(d, -d + \sqrt{\mathrm{d}^2 - 1}) = -32(\mathrm{d}^2 - 1)^{3/2},$$

and since this is non-zero in the interior of our domain, the proof is complete. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. AROVAS, D., SCHRIEFFER, J.R., WILCZEK, F.: Fractional statistics and the quantum Hall effect. *Phys. Rev. Lett.* **53**, 722–723, 1984
2. AROVAS, D.P., SCHRIEFFER, R., WILCZEK, F., ZEE, A.: Statistical mechanics of anyons. *Nucl. Phys. B* **251**, 117–126, 1985
3. BAKER, G.A., CANRIGHT, G.S., MULAY, S.B., SUNDBERG, C.: On the spectral problem for anyons. *Commun. Math. Phys.* **153**, 277–295, 1993
4. BALINSKY, A.A., EVANS, W.D., LEWIS, R.T.: *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer International Publishing, 2015
5. BLOCH, I., DALIBARD, J., ZWERGER, W.: Many-body physics with ultracold gases. *Rev. Mod. Phys.* **80**, 885–964, 2008
6. CATTO, I., LE BRIS, C., LIONS, P.-L.: *The mathematical theory of thermodynamic limits: Thomas–Fermi type models*, Oxford Mathematical Monographs, The Clarendon Press Oxford Univ. Press, New York 1998
7. CHEN, Y.H., WILCZEK, F., WITTEN, E., HALPERIN, B.I.: On anyon superconductivity. *Int. J. Mod. Phys. B* **3**, 1001–1067, 1989
8. CHITRA, R., SEN, D.: Ground state of many anyons in a harmonic potential. *Phys. Rev. B* **46**, 10923–10930, 1992
9. CHOI, M.Y., LEE, C., LEE, J.: Soluble many-body systems with flux-tube interactions in an arbitrary external magnetic field. *Phys. Rev. B* **46**, 1489–1497, 1992

10. Cooper, N.R.: Rapidly rotating atomic gases. *Advances in Physics* **57**, 539–616, 2008
11. Correggi, M., Lundholm, D., Rougerie, N.: Local density approximation for the almost-bosonic anyon gas. *Analysis & PDE* **10**(5), 1169–1200, 2017
12. Dell'Antonio, G., Figari, R., Teta, A.: Statistics in space dimension two. *Lett. Math. Phys.* **40**, 235–256 (1997)
13. Doebner, H.-D., Groth, W., Hennig, J.D.: On quantum mechanics of n-particle systems on 2-manifolds—a case study in topology. *J. Geom. Phys.* **31**, 35–50, 1999
14. Dyson, F.J.: Ground-state energy of a hard-sphere gas. *Phys. Rev.* **106**, 20–26, 1957
15. Dyson, F.J.: *Stability of matter*, in Statistical Physics, Phase Transitions and Superfluidity, Brandeis University Summer Institute in Theoretical Physics 1966, Gordon and Breach, New York, 1968, pp. 179–239
16. Dyson, F.J., Lenard, A.: Stability of matter. I. *J. Math. Phys.* **8**, 423–434, 1967
17. Fetter, A.L., Hanna, C.B., Laughlin, R.B.: Random-phase approximation in the fractional-statistics gas. *Phys. Rev. B* **39**, 9679–9681, 1989
18. Fournais, S., Helffer, B.: *Spectral Methods in Surface Superconductivity*, Progress in Nonlinear Differential Equations and their Applications, 77, Birkhäuser Boston, Inc., Boston, MA, 2010
19. Frank R.L., Seiringer, R.: Lieb–Thirring inequality for a model of particles with point interactions. *J. Math. Phys.* **53**(9), 095201, 2012
20. Fröhlich, J.: *Quantum statistics and locality*, in Proceedings of the Gibbs Symposium (New Haven, CT, 1989), Amer. Math. Soc., Providence, RI, 1990, pp. 89–142
21. Girvin, S.: Introduction to the fractional quantum Hall effect. *Sémin. Poincaré* **2**, 54–74, 2004
22. Goerbig, M.O.: *Quantum Hall effects*. Lecture notes, arXiv:0909.1998, 2009
23. Goldin, G.A., Menikoff, R., Sharp, D.H.: Representations of a local current algebra in nonsimply connected space and the Aharonov–Bohm effect. *J. Math. Phys.* **22**, 1664–1668, 1981
24. Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., Laptev, A., Tidblom, J.: Many-particle Hardy Inequalities. *J. Lond. Math. Soc.* **77**, 99–114, 2008
25. Iengo, R., Lechner, K.: Anyon quantum mechanics and Chern–Simons theory. *Phys. Rep.* **213**, 179–269, 1992
26. Ismail, M.E.H., Muldoon, M.E.: Bounds for the small real and purely imaginary zeros of Bessel and related functions. *Methods Appl. Anal.* **2**, 1–21, 1995
27. Jain, J.K.: *Composite fermions*, Cambridge University Press, Cambridge, 2007
28. Khare, A.: *Fractional Statistics and Quantum Theory*, World Scientific, Singapore, 2nd ed., 2005
29. Laptev, A., Weidl, T.: *Hardy inequalities for magnetic Dirichlet forms*, in Mathematical results in quantum mechanics (Prague, 1998), vol. 108 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 1999, pp. 299–305
30. Laughlin, R.B.: Nobel lecture: Fractional quantization. *Rev. Mod. Phys.* **71**, 863–874, 1999
31. Leinaas, J.M., Myrheim, J.: On the theory of identical particles. *Nuovo Cimento B* **37**, 1–23, 1977
32. Lenard, A.: *Lectures on the Coulomb stability problem*, in Statistical Mechanics and Mathematical Problems, Battelle Rencontres, Seattle, WA, 1971, vol. 20 of Lecture Notes in Phys., Springer, Berlin, 1973, pp. 114–135
33. Lerda, A.: *Anyons*, Springer, Berlin, 1992
34. Lieb, E.H., Loss, M.: *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2nd ed., 2001
35. Lieb, E.H., Seiringer, R.: *The Stability of Matter in Quantum Mechanics*, Cambridge University Press, Cambridge, 2010
36. Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J.: *The mathematics of the Bose gas and its condensation*, Oberwolfach Seminars, Birkhäuser, 2005
37. Lieb, E.H., Thirring, W.E.: Bound on kinetic energy of fermions which proves stability of matter. *Phys. Rev. Lett.* **35**, 687–689, 1975

38. Lieb, E.H., Thirring, W.E.: *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, Princeton Univ. Press, 1976, pp. 269–303
39. Lieb, E.H., Yngvason, J.: Ground state energy of the low density Bose gas. *Phys. Rev. Lett.* **80**, 2504–2507, 1998
40. Lieb, E.H., Yngvason, J.: *The ground state energy of a dilute Bose gas*, in Differential equations and mathematical physics (Birmingham, AL, 1999), vol. 16 of AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 2000, pp. 295–306
41. Lieb, E.H., Yngvason, J.: The ground state energy of a dilute two-dimensional Bose gas. *J. Statist. Phys.* **103**, 509–526, 2001. Special issue dedicated to the memory of Joaquin M. Luttinger
42. Liguori, A., Mintchev, M.: Fock representations of quantum fields with generalized statistics. *Commun. Math. Phys.* **169**, 635–652, 1995
43. Lundholm, D.: Many-anyon trial states. *Phys. Rev. A* **96**, 012116, 2017
44. Lundholm, D.: *Anyon wave functions and probability distributions*. IHÉS preprint, IHES/P/13/25, 2013
45. Lundholm, D., Nam, P.T., Portmann, F.: Fractional Hardy–Lieb–Thirring and related inequalities for interacting systems. *Arch. Ration. Mech. Anal.* **219**, 1343–1382 (2016)
46. Lundholm, D., Portmann, F., Solovej, J.P.: Lieb-Thirring bounds for interacting Bose gases. *Commun. Math. Phys.* **335**, 1019–1056, 2015
47. Lundholm, D., Rougerie, N.: The average field approximation for almost bosonic extended anyons. *J. Stat. Phys.* **161**, 1236–1267, 2015
48. Lundholm, D., Rougerie, N.: Emergence of fractional statistics for tracer particles in a Laughlin liquid. *Phys. Rev. Lett.* **116**, 170401, 2016
49. Lundholm, D., Solovej, J.P.: Hardy and Lieb–Thirring inequalities for anyons. *Commun. Math. Phys.* **322**, 883–908 (2013)
50. Lundholm, D., Solovej, J.P.: Local exclusion principle for identical particles obeying intermediate and fractional statistics. *Phys. Rev. A* **88**, 062106, 2013
51. Lundholm, D., Solovej, J.P.: Local exclusion and Lieb–Thirring inequalities for intermediate and fractional statistics. *Ann. Henri Poincaré* **15**, 1061–1107, 2014
52. Mashkevich, S.: Finite-size anyons and perturbation theory. *Phys. Rev. D* **54**, 6537–6543, 1996
53. Melgaard, M., Ouhabaz, E.-M., Rozenblum, G.: Negative Discrete Spectrum of Perturbed Multivortex Aharonov–Bohm Hamiltonians. *Ann. Henri Poincaré* **5**, 979–1012, 2004
54. Minor, W.R.: Ground-state energy of a dilute anyon gas. *Phys. Rev. B* **47**, 12716–12721, 1993
55. Morris, A., Feder, D.: Gaussian potentials facilitate access to quantum Hall states in rotating Bose gases. *Phys. Rev. Lett.* **99**, 240401, 2007
56. Murthy, M.V.N., Law, J., Brack, M., Bhaduri, R.K.: Quantum spectrum of three anyons in an oscillator potential. *Phys. Rev. Lett.* **67**, 1817–1820, 1991
57. Myrheim, J.: *Anyons*, in Topological aspects of low dimensional systems, A. Comtet, T. Jolicœur, S. Ouvry, and F. David, eds., vol. 69 of Les Houches-Ecole d'Ete de Physique Theorique, (Springer, Berlin), 1999, pp. 265–413
58. Nayak, C., Simon, S.H., Stern, A., Freedman, M., Das Sarma, S.: Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.* **80**, 1083–1159, 2008
59. Olver, F.W.J., Maximon, L.C.: *Bessel functions*, in NIST handbook of mathematical functions, U.S. Dept. Commerce, Washington, DC, 2010, pp. 215–286
60. Ouvry, S.: *Anyons and lowest Landau level anyons*, *Sémin. Poincaré* **11**, 77–107, 2007
61. Pauli, W.: *Nobel Lecture: Exclusion principle and quantum mechanics*, Editions du Griffon, Neuchatel, 1947
62. Roncaglia, M., Rizzi, M., Dalibard, J.: *From rotating atomic rings to quantum Hall states*, www.nature.com, Scientific Reports, 1, 2011
63. Schick, M.: Two-dimensional system of hard-core bosons. *Phys. Rev. A* **3**, 1067–1073, 1971

64. Sen, D., Chitra, R.: Anyons as perturbed bosons. *Phys. Rev. B* **45**, 881–894, 1992
65. Souriau, J.-M.: *Structure des systèmes dynamiques*, Maîtrises de mathématiques, Dunod, Paris, 1970. English translation by R. H. Cushman and G. M. Tuynman, Progress in Mathematics, 149, Birkhäuser Boston Inc., Boston, MA, 1997
66. Sporre, M., Verbaarschot, J.J.M., Zahed, I.: Numerical solution of the three-anyon problem. *Phys. Rev. Lett.* **67** 1813–1816, 1991
67. Sporre, M., Verbaarschot, J.J.M., Zahed, I.: Four anyons in a harmonic well. *Phys. Rev. B* **46**, 5738–5741 (1992)
68. Stern, A.: Anyons and the quantum Hall effect—a pedagogical review. *Ann. Phys.* **323**, 204–249, 2008. January Special Issue 2008
69. Störmer, H., Tsui, D., Gossard, A.: The fractional quantum Hall effect. *Rev. Mod. Phys.* **71**, S298–S305, 1999
70. Temple, G.: The theory of Rayleigh's principle as applied to continuous systems. *Proc. R. Soc. Lond. A* **119**, 276–293, 1928
71. Trugenberger, C.: Ground state and collective excitations of extended anyons. *Phys. Lett. B* **288**, 121–128, 1992
72. Trugenberger, C.: The anyon fluid in the Bogoliubov approximation. *Phys. Rev. D* **45**, 3807–3817, 1992
73. Viefers, S.: Quantum Hall physics in rotating Bose–Einstein condensates. *J. Phys. C* **12**, 123202, 2008
74. Wen, X.G., Zee, A.: Compressibility and superfluidity in the fractional-statistics liquid. *Phys. Rev. B* **41**, 240–253, 1990
75. Westerberg, E.: Mean field approximation for anyons in a magnetic field. *Int. J. Mod. Phys. B* **7**, 2177–2199 (1993)
76. Wilczek, F.: Magnetic flux, angular momentum, and statistics. *Phys. Rev. Lett.* **48**, 1144–1146, 1982
77. Wilczek, F.: Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.* **49**, 957–959, 1982
78. Wilczek, F.: *Fractional Statistics and Anyon Superconductivity*, World Scientific, Singapore, 1990
79. Wu, Y.-S.: Multiparticle quantum mechanics obeying fractional statistics. *Phys. Rev. Lett.* **53**, 111–114, 1984

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Paper H

Lieb–Thirring inequalities for wave functions vanishing on the diagonal set
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LIEB–THIRRING INEQUALITIES FOR WAVE FUNCTIONS VANISHING ON THE DIAGONAL SET

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ABSTRACT. We propose a general strategy to derive Lieb–Thirring inequalities for scale-covariant quantum many-body systems. As an application, we obtain a generalization of the Lieb–Thirring inequality to wave functions vanishing on the diagonal set of the configuration space, without any statistical assumption on the particles.

CONTENTS

1. INTRODUCTION

The celebrated Lieb–Thirring inequality states that the expected kinetic energy of a free Fermi gas is bounded from below by its semiclassical approximation up to a universal factor, namely

$$\left\langle \Psi_N, \sum_{i=1}^{N} (-\Delta_{\mathbf{x}_i})^s \Psi_N \right\rangle \geq K \int_{\mathbb{R}^d} \varrho_{\Psi_N}(\mathbf{x})^{1+2s/d} d\mathbf{x}. \tag{1.1}$$

Here Ψ_N is an N-particle wave function in $L^2((\mathbb{R}^d)^N)$, normalized so that $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})} = 1$ and thus encoding in its squared amplitude a probability distribution for particle positions $\mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, $\mathbf{x}_j \in \mathbb{R}^d$, with one-body density

$$\varrho_{\Psi_N}(\mathbf{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{i \neq j} d\mathbf{x}_i,$$

and, crucially, subject to the anti-symmetry

$$\Psi_N(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N) = -\Psi_N(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_N), \quad \forall i \neq j. \tag{1.2}$$

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This is Pauli's exclusion principle for fermions[(1)]. Replacing the minus sign in (1.2) by a plus sign defines bosonic particles, while if the particles are non-identical, i.e. distinguishable, no exchange symmetry may be imposed.

The inequality (1.1) was first proved by Lieb and Thirring in 1975 for the case $s = 1$ relevant to non-relativistic particles [13, 14], and extended by Daubechies in 1983 to general $s > 0$, thus also including the relativistic case $s = 1/2$ [2]. The constant $K = K(d, s) > 0$ is independent of N and Ψ_N (see [5] for the best known value of K).

The Lieb–Thirring inequality is a beautiful combination of the uncertainty and exclusion principles of quantum mechanics, and has also been very actively studied in the mathematical literature from the dual perspective of estimation of eigenvalues of one-body Schrödinger operators (see e.g. [12, 9] for reviews). Historically, the Lieb–Thirring inequality was invented to give a short, elegant proof of the stability of ordinary non-relativistic matter with Coulomb forces [13]. In that context it is well known that stability of the *first* kind, i.e. that the ground state energy of the Coulomb system is finite, follows easily from some sort of the uncertainty principle (e.g. Sobolev's inequality). On the other hand, the stability of the *second* kind, that the ground state energy does not diverge faster than the number of particles, is much more subtle: for this the fermionic nature of particles is crucial. In fact, the stability of the second kind fails for bosonic (or distinguishable) charged systems [3].

Without the anti-symmetry condition (1.2), the Lieb–Thirring inequality (1.1) fails and the best one can get is the Gagliardo-Nirenberg-Sobolev inequality

$$\left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s \Psi_N \right\rangle \geq K N^{-2s/d} \int_{\mathbb{R}^d} \varrho_{\Psi_N}(\mathbf{x})^{1+2s/d} d\mathbf{x} \tag{1.3}$$

(see e.g. [17]). The emergence of the factor $N^{-2s/d}$ can be seen by considering the bosonic trial state $\Psi_N = u^{\otimes N}$ (whose density is $\varrho_{\Psi_N}(x) = N|u(x)|^2$). This factor is small when N becomes large, making (1.3) not very useful in applications.

Note that Pauli's exclusion principle (1.2) implies that the wave function Ψ_N vanishes on the diagonal set

$$\triangle := \{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^d)^N : \mathbf{x}_i = \mathbf{x}_j \text{ for some } i \neq j\}, \tag{1.4}$$

namely there is zero probability for two quantum particles to occupy a common single position in the configuration space.

In this paper, we want to address the following

Question: Does the Lieb–Thirring inequality (1.1) remain valid if the anti-symmetry assumption (1.2) is replaced by the weaker condition $\Psi_N|_{\triangle} = 0$?

We will show that the answer is **yes** if and only if $2s > d$. In fact, $2s > d$ is the optimal condition for the vanishing assumption $\Psi_N|_{\triangle} = 0$ to be non-trivial (heuristically this follows from Sobolev's embedding $H^s(\mathbb{R}^d) \subseteq C(\mathbb{R}^d)$ for $2s > d$). The precise statement of our result and its consequences will be presented in the next section.

(1) Here we ignore the spin of particles for simplicity (in our analysis the effect of the spin is mathematically trivial).

2. Main results

Recall that for every $s > 0$ (not necessarily an integer) the operator $(-\Delta)^s$ on $L^2(\mathbb{R}^d)$ is defined as the multiplication operator $|\mathbf{p}|^{2s}$ in Fourier space, namely

$$\left[(-\Delta)^s f\right]^\wedge(\mathbf{p}) = |\mathbf{p}|^{2s}\widehat{f}(\mathbf{p}), \qquad \widehat{f}(\mathbf{p}) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}\,d\mathbf{x}.$$

The associated space $H^s(\mathbb{R}^d)$ is a Hilbert space with norm

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{\dot{H}^s(\mathbb{R}^d)}, \qquad \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} := \langle u, (-\Delta)^s u\rangle.$$

The N-particle space $H^s(\mathbb{R}^{dN})$ is defined in the same way. Let us denote the subspace of functions vanishing on the diagonal set $\triangle$ in (1.4) by

$$\mathcal{H}^{s,N}(\mathbb{R}^d) := \overline{\left\{\Psi_N \in C_c^\infty(\mathbb{R}^{dN}) : \Psi_N|_\triangle = 0\right\}}^{H^s(\mathbb{R}^{dN})}.$$

Our main result is

Theorem 2.1 (Lieb–Thirring inequality for wave functions vanishing on diagonals)**.** *Let $2s > d \geq 1$. Then for every $N \geq 1$ and $\Psi_N \in \mathcal{H}^{s,N}(\mathbb{R}^d)$, with $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})} = 1$, we have*

$$\left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s \Psi_N \right\rangle \geq C \int_{\mathbb{R}^d} \varrho_{\Psi_N}(\mathbf{x})^{1+2s/d}\,d\mathbf{x}. \tag{2.1}$$

Here $C = C(d,s) > 0$ is a universal constant independent of N and Ψ_N.

We have some immediate remarks.

1. The condition $2s > d$ in Theorem 2.1 is optimal. If $2s \leq d$, then

$$\mathcal{H}^{s,N}(\mathbb{R}^d) = H^s(\mathbb{R}^{dN})$$

by the relatively small size, i.e. the large codimensionality, of the diagonal set (see Appendix B) and thus the Lieb–Thirring inequality fails.

2. For $d = 1$ and $s = 1$, it is well known that a *symmetric* wave function which vanishes on the diagonal set is equal to an anti-symmetric wave function up to multiplication by an appropriate sign function [7], and hence (2.1) reduces to the usual Lieb–Thirring inequality [14] in this case. However, when $d > 1$ this boson-fermion correspondence is no longer available and our result is new. Furthermore, one may consider *hard-core* bosons defined by the higher-order vanishing around diagonals

$$\mathcal{H}_0^{s,N}(\mathbb{R}^d) := \overline{\left\{\Psi_N \in C_c^\infty(\mathbb{R}^{dN} \setminus \triangle)\right\}}^{H^s(\mathbb{R}^{dN})}, \tag{2.2}$$

and subject to symmetry. For large enough order $2s > d$ there is even for $d = 1$ a non-trivial difference between these spaces, and our result assumes only the weaker vanishing conditions imposed by $\mathcal{H}^{s,N}(\mathbb{R}^d)$ (see Appendix B for some further remarks).

3. Theorem 2.1 verifies a conjecture in [17, page 1362] that the Lieb–Thirring inequality (2.1) holds for all wave functions in the form domain of the interaction potential

$$W_s(\mathrm{x}) := \sum_{1\leq i<j\leq N} |\mathbf{x}_i - \mathbf{x}_j|^{-2s}, \qquad \mathrm{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^d)^N.$$

In fact, we have (again, see Appendix B for details)

$$\left\{\Psi_N \in H^s(\mathbb{R}^{dN}) : \int_{\mathbb{R}^{dN}} W_s(\mathrm{x})|\Psi_N(\mathrm{x})|^2\, d\mathrm{x} < \infty\right\} \subseteq \mathcal{H}_0^{s,N}(\mathbb{R}^d) \subseteq \mathcal{H}^{s,N}(\mathbb{R}^d), \tag{2.3}$$

by the singular nature of the potential at the diagonals. We may think of the potential W_s as defining (by Friedrichs extension) a one-parameter family of non-negative and scale-covariant (scaling homogeneously to degree $-2s$) interacting N-body Hamiltonian operators

$$H_\beta := \sum_{j=1}^N (-\Delta_{\mathbf{x}_j})^s + \beta W_s, \qquad \beta \geq 0.$$

The case $\beta > 0$ was treated in [18, 17], while our setting here concerns the limit $\beta \to 0$ of zero-range/contact interaction. A crucial difference is the strength of the interaction term, which is of order N^2 and thus provides a large repulsive energy for fixed $\beta > 0$, while for $\beta \ll 1/N$ it ought to be much weaker than the kinetic term. Nevertheless, for $2s > d$ the potential W_s is singular enough to impose the vanishing condition at $\triangle$, and Theorem 2.1 yields a non-trivial bound (a generalized uncertainty principle) for $H_{\beta=0}$.

4. The original proof of the Lieb–Thirring inequality [13, 14] is based on the following operator bound

$$0 \leq \gamma_{\Psi_N}^{(1)} \leq \mathbb{1} \tag{2.4}$$

which is a consequence of Pauli's exclusion principle (1.2). Here $\gamma_{\Psi_N}^{(1)}$ is the one-body density matrix of Ψ_N, a trace-class operator on $L^2(\mathbb{R}^d)$ with kernel

$$\gamma_{\Psi_N}^{(1)}(\mathbf{x};\mathbf{x}') = \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_j = \mathbf{x},\ldots,\mathbf{x}_N)\overline{\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_j = \mathbf{x}',\ldots,\mathbf{x}_N)} \prod_{k\neq j} d\mathbf{x}_k.$$

However, unlike the full anti-symmetry condition (1.2), the vanishing condition $\Psi_N|_\triangle = 0$ alone is not known to be sufficient to ensure the operator inequality (2.4), and therefore the original proof in [13, 14] as well as subsequent proofs based on (2.4) (e.g. Rumin's method [25]) do not apply.

Our result is in fact more general than as previously formulated. More precisely, define for any $k \geq 2$ the diagonal set of k-particle coincidences

$$\triangle_k := \left\{(\mathbf{x}_1,\ldots,\mathbf{x}_N) \in (\mathbb{R}^d)^N : \mathbf{x}_{j_1} = \ldots = \mathbf{x}_{j_k} \text{ for distinct indices } j_1,\ldots,j_k\right\}, \tag{2.5}$$

and the corresponding space of N-particle wave functions with a vanishing condition on $\triangle_k$

$$\mathcal{H}_k^{s,N}(\mathbb{R}^d) := \overline{\left\{\Psi_N \in C_c^\infty(\mathbb{R}^{dN}) : \Psi_N|_{\triangle_k} = 0\right\}}^{H^s(\mathbb{R}^{dN})}.$$

We have

Theorem 2.2 (Lieb–Thirring inequality for wave functions vanishing on k-diagonals)**.** *Let $d \geq 1$, $k \geq 2$ and $2s > d(k-1)$. Then for every $N \geq 1$ and every $\Psi_N \in \mathcal{H}_k^{s,N}(\mathbb{R}^d)$, with $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})} = 1$, we have*

$$\left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s \Psi_N \right\rangle \geq C \int_{\mathbb{R}^d} \varrho_{\Psi_N}(\mathbf{x})^{1+2s/d}\, d\mathbf{x}. \tag{2.6}$$

Here $C = C(d,s,k) > 0$ is a universal constant independent of N and Ψ_N.

The proof of Theorem 2.2 occupies the rest of the paper. Our proof is based on a general strategy of deriving Lieb–Thirring inequalities for wave functions satisfying some partial exclusion properties, which was proposed by Lundholm and Solovej in [20] and developed further in [6, 21, 22, 18, 17, 10, 23, 19, 16]. We will quickly review this strategy in Section 3 for the reader's convenience, following the simplification by Lundholm, Nam and Portmann [17].

The main new ingredient is a local version of the exclusion principle using the vanishing condition on the diagonal set. In Section 4, we will discuss a very useful reduction of the desired local exclusion to simply the positivity of a local energy using the scale-covariance of the kinetic operator $(-\Delta)^s$. This step refines and generalizes a recent bootstrap argument for the energy of ideal anyons by Lundholm and Seiringer [19]. In Section 5, the remaining crucial fact that the local energy eventually becomes positive with increasing particle number will be settled by means of a new many-particle Poincaré inequality. Some standard and non-standard results on relevant function spaces are collected in the appendices for completeness.

We stress that our method will also work for any other deformations of the Laplacian which retain similar positivity and scale-covariance properties, including other types of point interactions as well as particles subject to intermediate statistics (ideal anyons) in one and two dimensions.

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3. General strategy of deriving Lieb–Thirring inequalities

In the following we will summarize the proof of the usual Lieb–Thirring inequality (1.1) for fermionic wave functions, mainly following the simplified representation in [17]. The starting point is the following obvious localization formula: if $\{\Omega\}$ is a collection of disjoint subsets of $\mathbb{R}^d$, then

$$(-\Delta)^s_{|\mathbb{R}^d} \geq \sum_{\Omega} (-\Delta)^s_{|\Omega}, \tag{3.1}$$

where the Neumann localization $(-\Delta)^s_{|\Omega}$ is defined via the quadratic form (Sobolev semi-norm)

$$\langle u, (-\Delta)^s_{|\Omega} u\rangle = \|u\|^2_{\dot{H}^s(\Omega)} := \begin{cases} \sum\limits_{|\alpha|=m} \dfrac{m!}{\alpha!} \int_Q |D^\alpha u|^2 & \text{if } s = m, \\ c_{d,\sigma} \sum\limits_{|\alpha|=m} \dfrac{m!}{\alpha!} \iint_{\Omega\times\Omega} \dfrac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}} d\mathbf{x}d\mathbf{y} & \text{if } s = m+\sigma, \end{cases}$$

for all $u \in H^s(\mathbb{R}^d)$, with $m \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d$ multi-indices, D^α corresponding derivatives, and

$$0 < \sigma < 1, \quad c_{d,\sigma} := \frac{2^{2\sigma-1}}{\pi^{d/2}} \frac{\Gamma((d+2\sigma)/2)}{|\Gamma(-\sigma)|}.$$

Consequently, for any N-body wave function $\Psi_N \in H^s(\mathbb{R}^{dN})$ we have

$$\mathcal{E}_{\mathbb{R}^d}[\Psi_N] \geq \sum_{\Omega} \mathcal{E}_\Omega[\Psi_N], \tag{3.2}$$

where the expected local energy on Ω is

$$\mathcal{E}_\Omega[\Psi_N] := \left\langle \Psi_N, \sum_{j=1}^N (-\Delta_{\mathbf{x}_j})^s_{|\Omega} \Psi_N \right\rangle = \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \|\Psi_N\|^2_{\dot{H}^s_{\mathbf{x}_j}(\Omega)} \prod_{\ell \neq j} d\mathbf{x}_\ell \tag{3.3}$$

Next, we have the following three key tools [17, Lemmas 8, 11, 12].

Lemma 3.1 (Local uncertainty). *Let $d \geq 1$ and $s > 0$. Let Ψ_N be a wave function in $H^s(\mathbb{R}^{dN})$ for arbitrary $N \geq 1$ and let Q be an arbitrary cube in $\mathbb{R}^d$. Then*

$$\mathcal{E}_Q[\Psi_N] \geq \frac{1}{C} \frac{\int_Q \varrho_{\Psi_N}^{1+2s/d}}{\left(\int_Q \varrho_{\Psi_N}\right)^{2s/d}} - \frac{C}{|Q|^{2s/d}} \int_Q \varrho_{\Psi_N}. \tag{3.4}$$

Hereafter, $C = C(d,s) > 0$ denotes a universal constant (independent of N, Ψ_N and Q).

Lemma 3.1 can be interpreted as a local version of the lower bound (1.3) (the negative term appears due to the lack of Dirichlet boundary condition).

Lemma 3.2 (Local exclusion for fermions). *Let $d \geq 1$ and $s > 0$. Let Ψ_N be a fermionic wave function in $H^s(\mathbb{R}^{dN})$ satisfying (1.2) for $N \geq 2$ and let Q be an arbitrary cube in $\mathbb{R}^d$. Then*

$$\mathcal{E}_Q[\Psi_N] \geq C|Q|^{-2s/d} \left[\int_Q \varrho_{\Psi_N}(\mathbf{x})\, d\mathbf{x} - q \right]_+, \tag{3.5}$$

where $q := \#\{\text{multi-indices } \alpha \in \mathbb{N}_0^d : 0 \leq |\alpha| < s\}$.

In the non-relativistic case $s = 1$, Lemma 3.2 simply states that as soon as there is more than one particle on Q the energy must be strictly positive, and furthermore that it grows at least linearly with the number of particles. Such a weak formulation of the exclusion principle was used by Dyson and Lenard in their first proof of the stability of matter [4], while its general applicability in the above format was noted by Lundholm and Solovej in [20, 21].

Lemma 3.3 (Covering lemma). *Let $0 \leq f \in L^1(\mathbb{R}^d)$ be a function with compact support such that $\int_{\mathbb{R}^d} f \geq \Lambda > 0$. Then the support of f can be covered by a collection of disjoint cubes $\{Q\}$ in $\mathbb{R}^d$ such that*

$$\int_Q f \leq \Lambda, \quad \forall Q \tag{3.6}$$

and

$$\sum_Q \frac{1}{|Q|^\alpha} \left(\left[\int_Q f - q \right]_+ - b \int_Q f \right) \geq 0 \tag{3.7}$$

for all $\alpha > 0$ and $0 \leq q < \Lambda 2^{-d}$, where

$$b := \left(1 - \frac{2^d q}{\Lambda}\right) \frac{2^{d\alpha} - 1}{2^{d\alpha} + 2^d - 2} > 0.$$

Conclusion of (1.1)**.** Let q be as in Lemma 3.2 and let $\Lambda = 2^d q + 1$. If $N \leq \Lambda$, then (1.1) follows immediately from (1.3), whose proof is similar to (indeed simpler than) that of Lemma 3.1. If $N > \Lambda$, then we can apply Lemma 3.3 with $f = \varrho_{\Psi_N}$ (by standard approximation we may reduce to compact support), $\alpha = 2s/d$, and obtain a collection of disjoint cubes $\{Q\}$. Combining with (3.2), (3.4) and (3.5) we obtain

$$\begin{aligned}(\varepsilon+1)\mathcal{E}_{\mathbb{R}^d}[\Psi_N] &\geq \varepsilon \sum_Q \left[\frac{1}{C_1} \frac{\int_Q \varrho_{\Psi_N}^{1+2s/d}}{\left(\int_Q \varrho_{\Psi_N}\right)^{2s/d}} - \frac{C_1}{|Q|^{2s/d}} \int_Q \varrho_{\Psi_N} \right] \\ &\quad + \sum_Q C_2 |Q|^{-2s/d} \left[\int_Q \varrho_\Psi(x)\, dx - q \right]_+ \\ &\geq \frac{\varepsilon}{C_1} \frac{\int_{\mathbb{R}^d} \varrho_{\Psi_N}^{1+2s/d}}{\Lambda^{2s/d}}\end{aligned}$$

for any fixed constant $\varepsilon > 0$ satisfying $\varepsilon C_1 \leq C_2 b$. Thus (1.1) holds true.

As we can see from the above strategy, the only place where the anti-symmetry (1.2) plays a role is the local exclusion bound in Lemma 3.2. Extending this result to the weaker condition $\Psi_N|_\triangle = 0$ is the main task of our proof below.

4. Reduction of local exclusion

In this section, we prove a very useful observation, that allows to reduce the local exclusion (3.5) to the positivity of the local energy, using the scale-covariance of the kinetic energy. This step is inspired by the recent work of Lundholm and Seiringer [19] on the energy of ideal anyons. We formulate it abstractly as follows:

Lemma 4.1 (Covariant energy bound)**.** *Assume that to any $n \in \mathbb{N}_0$ and any cube $Q \subset \mathbb{R}^d$ there is associated a non-negative number ('energy') $E_n(Q)$ satisfying the following properties, for some constant $s > 0$:*

- *(scale-covariance) $E_n(\lambda Q) = \lambda^{-2s} E_n(Q)$ for all $\lambda > 0$;*
- *(translation-invariance) $E_n(Q + \mathbf{x}) = E_n(Q)$ for all $\mathbf{x} \in \mathbb{R}^d$;*
- *(superadditivity) For any collection of disjoint cubes $\{Q_j\}_{j=1}^J$ such that their union is a cube,*
$$E_n\Big(\bigcup_{j=1}^J Q_j\Big) \geq \min_{\{n_j\} \in \mathbb{N}_0^J \text{ s.t. } \sum_j n_j = n} \sum_{j=1}^J E_{n_j}(Q_j);$$
- *(a priori positivity) There exists $q \geq 0$ such that $E_n(Q) > 0$ for all $n \geq q$.*

Then there exists a constant $C > 0$ independent of n and Q such that

$$E_n(Q) \geq C|Q|^{-2s/d} n^{1+2s/d}, \quad \forall n \geq q. \tag{4.1}$$

Proof. Note that for $q \leq n \leq N$, (4.1) holds for some $C = C_N > 0$ by the a priori positivity. The main point here is to remove the N-dependence of the constant.

Denote $E_n := E_n(Q_0)$ with $Q_0 = [0,1]^d$. Assume by induction in N that

$$E_n \geq C n^{1+2s/d}, \quad \forall q \leq n \leq N-1 \tag{4.2}$$

with a uniform constant $C > 0$ and consider $n = N$. Split Q_0 into 2^d subcubes of half side length and obtain by the superadditivity, translation-invariance and scale-covariance

$$E_N \geq 2^{2s} \min_{\{n_j\}\, s.t.\, \sum_j n_j = N} \sum_{j=1}^{2^d} E_{n_j}. \tag{4.3}$$

Consider a configuration $\{n_j\} \subset \mathbb{N}_0^{2^d}$ such that the minimum in (4.3) is attained. The a priori positivity $E_N > 0$ ensures that none of the n_j can be N (in the same way we deduce that $E_0 = 0$). Assume that there exist exactly M numbers $n_j < q$ with $0 \leq M \leq 2^d$. Then

$$\sum_{n_j \geq q} 1 = 2^d - M \quad \text{and} \quad \sum_{n_j \geq q} n_j = N - \sum_{n_j < q} n_j \geq N - qM.$$

Therefore, from (4.3), (4.2) and Hölder's inequality we deduce that

$$E_N \geq C 2^{2s} \sum_{n_j \geq q} n_j^{1+2s/d} \geq C 2^{2s} \frac{\left(\sum_{n_j \geq q} n_j\right)^{1+2s/d}}{\left(\sum_{n_j \geq q} 1\right)^{2s/d}} \geq C N^{1+2s/d} \frac{(1 - qMN^{-1})^{1+2s/d}}{(1 - M2^{-d})^{2s/d}} \tag{4.4}$$

with the same constant C as in (4.2). If we take

$$N \geq q 2^d \Big(1 + \frac{d}{2s}\Big),$$

so that also $qMN^{-1} \leq 1$, then by Bernoulli's inequality

$$(1 - qMN^{-1})^{1+d/(2s)} \geq 1 - qMN^{-1}\Big(1 + \frac{d}{2s}\Big) \geq 1 - M2^{-d},$$

and hence (4.4) reduces to

$$E_N \geq C N^{1+2s/d} \tag{4.5}$$

with the same constant C as in (4.2).

By induction we obtain (4.5) for all $N \geq q$, with a constant C independent of N. This is the desired bound (4.1) for the unit cube Q_0. The result for the general cube follows from scale-covariance and translation-invariance. □

Remark 4.2. It is in fact also possible to allow for $E_n < 0$ for finitely many $n > 0$ in Lemma 4.1, under a small refinement of the assumption of a priori positivity. It is sufficient that there exists $q > 0$ and $c > 1$ such that for all $n \geq q$

$$E_n > c \frac{d}{2s} 2^{d+2s} E_-, \qquad E_- := \max_{0 \leq n < q} (-E_n). \tag{4.6}$$

Namely, with this assumption, the bound in (4.4) may again be used for all $n_j \geq q$, and one obtains $E_N \geq C N^{1+2s/d} f(M/2^d)$, $C = \min_{q \leq n \leq N-1} E_n / n^{1+2s/d}$, where the function

$$f(x) := \frac{(1 - q2^d N^{-1} x)^{1+2s/d}}{(1-x)^{2s/d}} - \frac{E_- 2^{d+2s}}{C N^{1+2s/d}} x, \qquad x \in [0, 1),$$

is strictly increasing if N is large enough.

We will apply the above general bound to the local ground-state energy among wave functions satisfying the vanishing condition on k-particle diagonals

$$E_N(\Omega) := \inf\Big\{\|\Psi_N\|^2_{\dot H^{s,N}(\Omega)} : \Psi_N \in \mathcal{H}^{s,N}_k(\mathbb{R}^d), \|\Psi_N\|_{L^2(\Omega^N)} = 1\Big\}, \tag{4.7}$$

where we have introduced the 'completely localized' kinetic functional

$$\|\Psi_N\|^2_{\dot H^{s,N}(\Omega)} := \Big\langle \mathbb{1}_{\Omega^N}\Psi_N, \sum_{j=1}^N (-\Delta_{\mathbf{x}_j})^s_{|\Omega}\mathbb{1}_{\Omega^N}\Psi_N\Big\rangle = \sum_{j=1}^N \int_{\Omega^{N-1}} \|\Psi_N\|^2_{\dot H^s_{\mathbf{x}_j}(\Omega)} \prod_{\ell\neq j} d\mathbf{x}_\ell. \tag{4.8}$$

Note that $\|\Psi_N\|^2_{\dot H^{s,N}(\Omega)}$ is *different* from the functional $\mathcal{E}_\Omega[\Psi_N]$ in (3.3), and its properties will be crucial to deduce the desired local exclusion for $\mathcal{E}_\Omega[\Psi_N]$. The seminorm $\|\cdot\|_{\dot H^{s,N}(\Omega)}$ in general contains only some of the terms of the standard homogeneous Sobolev seminorm $\|\cdot\|_{\dot H^s(\Omega^N)}$; however, the corresponding norms (i.e. the seminorms plus the L^2-norm) are actually equivalent modulo N-dependent constants, not only globally on $\mathbb{R}^{dN}$ but also locally on Q^N (see Appendix A).

The superadditivity of the energy $E_N(\Omega)$ follows from the partitioning of the many-body space and by locality respectively non-negativity of any non-local part of the kinetic energy, i.e. (3.1). The method was also used in [19, Lemma 4.2] for anyons.

Lemma 4.3 (Superadditivity of $E_n(\Omega)$). *Let $\{\Omega_j\}_{j=1}^J$ be a collection of disjoint subsets of $\mathbb{R}^d$ and $\Omega = \cup_j\Omega_j$. Then*

$$E_N(\Omega) \geq \min_{\{n_j\}\in\mathbb{N}_0^J \text{ s.t. } \sum_j n_j = N} \sum_{j=1}^J E_{n_j}(\Omega_j). \tag{4.9}$$

Proof. For any partition $A = \{A_j\}_{j=1}^J$ of $\{1, 2, \ldots, N\}$ (i.e. the A_j are disjoint subsets of $\{1, 2, \ldots, N\}$ such that $\sum_j |A_j| = N$), we denote by $\mathbb{1}_A$ the characteristic function of the set

$$\Big\{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^d)^N : \mathbf{x}_i \in \Omega_j \Leftrightarrow i \in A_j, \text{ for all } i, j\Big\}.$$

Using the operator bound similar to (3.1)

$$(-\Delta_{\mathbf{x}_i})^s_{|\Omega} \geq \sum_{j=1}^J (-\Delta_{\mathbf{x}_i})^s_{|\Omega_j},$$

the partition of unity

$$\mathbb{1}_{\Omega^N} = \sum_A \mathbb{1}_A, \tag{4.10}$$

and the fact that $\mathbb{1}_A$ commutes with $(-\Delta_{\mathbf{x}_i})^s_{|\Omega_j}$, we can write for any $\Psi \in \mathcal{H}^{s,N}_k(\mathbb{R}^d)$

$$\begin{aligned}
\|\Psi\|^2_{\dot{H}^{s,N}(\Omega)} &= \left\langle \mathbb{1}_{\Omega^N}\Psi, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s_{|\Omega}\mathbb{1}_{\Omega^N}\Psi \right\rangle \\
&\geq \sum_{j=1}^J \left\langle \mathbb{1}_{\Omega^N}\Psi, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s_{|\Omega_j}\mathbb{1}_{\Omega^N}\Psi \right\rangle \\
&= \sum_{j=1}^J \sum_A \left\langle \mathbb{1}_A\Psi, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s_{|\Omega_j}\mathbb{1}_A\Psi \right\rangle \\
&= \sum_{j=1}^J \sum_A \int_{\mathbb{R}^{d(N-|A_j|)}} \|\Psi(\,\cdot\,; \mathrm{x}_{A_j^c})\|^2_{\dot{H}^{s,|A_j|}(\Omega_j)} \prod_{\ell\neq j}\Big[\mathbb{1}_{\Omega_\ell^{|A_\ell|}}(\mathrm{x}_{A_\ell})d\mathrm{x}_{A_\ell}\Big].
\end{aligned}$$

Here we have introduced the shorthand notation

$$(\mathbf{x}_1, \ldots, \mathbf{x}_N) = (\mathrm{x}_{A_j}; \mathrm{x}_{A_j^c}), \quad \mathrm{x}_{A_j} = (\mathbf{x}_\ell)_{\ell\in A_j} \in (\mathbb{R}^d)^{|A_j|}.$$

Since $\Psi \in \mathcal{H}^{s,N}_k(\mathbb{R}^d)$, for a.e. $\mathrm{x}_{A_j^c} \in \mathbb{R}^{d(N-|A_j|)}$ the function $\Psi(\,\cdot\,; \mathrm{x}_{A_j^c})$ is in $\mathcal{H}^{s,|A_j|}_k(\mathbb{R}^d)$, and hence

$$\begin{aligned}
\|\Psi(\,\cdot\,; \mathrm{x}_{A_j^c})\|^2_{\dot{H}^{s,|A_j|}(\Omega_j)} &\geq E_{|A_j|}(\Omega_j)\int_{\Omega_j^{|A_j|}} |\Psi(\mathrm{x}_{A_j}; \mathrm{x}_{A_j^c})|^2 d\mathrm{x}_{A_j} \\
&= E_{|A_j|}(\Omega_j)\int_{\mathbb{R}^{d|A_j|}} |\Psi(\mathrm{x}_{A_j}; \mathrm{x}_{A_j^c})|^2 \mathbb{1}_{\Omega_j^{|A_j|}}(\mathrm{x}_{A_j}) d\mathrm{x}_{A_j}.
\end{aligned}$$

Thus in summary

$$\begin{aligned}
\|\Psi\|^2_{\dot{H}^{s,N}(\Omega)} &\geq \sum_{j=1}^J \sum_A E_{|A_j|}(\Omega_j)\int_{\mathbb{R}^{dN}} |\Psi|^2 \prod_{\ell=1}^J \Big[\mathbb{1}_{\Omega_\ell^{|A_\ell|}}(\mathrm{x}_{A_\ell})d\mathrm{x}_{A_\ell}\Big] \\
&= \sum_{j=1}^J \sum_A E_{|A_j|}(\Omega_j)\langle \Psi, \mathbb{1}_A\Psi\rangle = \sum_A \Big[\sum_{j=1}^J E_{|A_j|}(\Omega_j)\Big]\langle \Psi, \mathbb{1}_A\Psi\rangle \\
&\geq \Big[\min_{\{n_j\}\in\mathbb{N}_0^J \ s.t.\ \sum_j n_j = N} \ \sum_{j=1}^J E_{n_j}(\Omega_j)\Big] \sum_A \langle \Psi, \mathbb{1}_A\Psi\rangle \\
&= \Big[\min_{\{n_j\}\in\mathbb{N}_0^J \ s.t.\ \sum_j n_j = N} \ \sum_{j=1}^J E_{n_j}(\Omega_j)\Big] \|\Psi\|^2_{L^2(\Omega^N)}.
\end{aligned}$$

Here in the last identity we have used the partition of unity (4.10) again. This implies the desired estimate (4.9). $\square$

Now we are ready to prove the reduction of the local exclusion.

Lemma 4.4 (Energy positivity implies local exclusion)**.** *Assume that there exists a constant $q > 0$ such that for any cube $Q \subset \mathbb{R}^d$,*

$$E_N(Q) > 0, \quad \forall N \geq q. \tag{4.11}$$

Then for all $N \geq 1$ and for all wave functions $\Psi_N \in \mathcal{H}_k^{s,N}(\mathbb{R}^d)$, $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})} = 1$, we have

$$\mathcal{E}_Q[\Psi_N] \geq C|Q|^{-2s/d}\left[\int_Q \varrho_{\Psi_N}(\mathbf{x})\, d\mathbf{x} - q\right]_+. \tag{4.12}$$

Here $C > 0$ is a constant independent of N, Ψ_N and Q.

Proof. Given (4.11), the energy functional $E_n(Q)$ defined in (4.7) verifies all conditions in Lemma 4.1. Therefore, there exists a constant $C > 0$ independent of n and Q such that

$$E_n(Q) \geq C|Q|^{-2s/d} n^{1+2s/d} \mathbb{1}_{\{n \geq q\}} \geq C|Q|^{-2s/d}[n-q]_+, \quad \forall n \geq 0. \tag{4.13}$$

Now we adapt the localization method in the proof of Lemma 4.3 to treat the functional $\mathcal{E}_Q[\Psi_N]$ (instead of $\|\Psi_N\|^2_{\dot{H}^{s,N}(Q)}$). To be precise, for any subset B of $\{1,\dots,N\}$ we denote by $\mathbb{1}_B$ the characteristic function of the set

$$\Big\{(\mathbf{x}_1,\dots,\mathbf{x}_N) \in (\mathbb{R}^d)^N : \mathbf{x}_i \in Q \Leftrightarrow i \in B, \text{ for all } i\Big\}.$$

For any $\Psi_N \in \mathcal{H}_k^{s,N}(\mathbb{R}^d)$, $\|\Psi_N\|_{L^2(\mathbb{R}^{dN})} = 1$, by inserting the partition of unity

$$\mathbb{1}_{\mathbb{R}^{dN}} = \sum_B \mathbb{1}_B \tag{4.14}$$

we can write

$$\begin{aligned}
\mathcal{E}_Q[\Psi_N] &= \left\langle \Psi_N, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s_{|Q} \Psi_N \right\rangle = \sum_B \left\langle \mathbb{1}_B \Psi_N, \sum_{i=1}^N (-\Delta_{\mathbf{x}_i})^s_{|Q} \mathbb{1}_B \Psi_N \right\rangle \\
&= \sum_B \int_{(\mathbb{R}^d \setminus Q)^{N-|B|}} \|\Psi_N(\,\cdot\,; \mathrm{x}_{B^c})\|^2_{\dot{H}^{s,|B|}(Q)} \mathbb{1}_{(\mathbb{R}^d \setminus Q)^d} d\mathrm{x}_{B^c} \\
&\geq \sum_B \int_{(\mathbb{R}^d \setminus Q)^{N-|B|}} E_{|B|}(Q) \|\Psi_N(\,\cdot\,; \mathrm{x}_{B^c})\|^2_{L^2(Q^{|B|})} d\mathrm{x}_{B^c} \\
&= \sum_B E_{|B|}(Q) \langle \Psi_N, \mathbb{1}_B \Psi_N \rangle.
\end{aligned} \tag{4.15}$$

Here we have used the fact that $\mathbb{1}_B$ commutes with $(-\Delta_{\mathbf{x}_i})^s_{|Q}$ and the shorthand notation

$$(\mathbf{x}_1,\dots,\mathbf{x}_N) = (\mathrm{x}_B; \mathrm{x}_{B^c}), \quad \mathrm{x}_B = (\mathbf{x}_\ell)_{\ell \in B} \in (\mathbb{R}^d)^{|B|}.$$

On the other hand, the partition of unity (4.14) implies that

$$\sum_B \langle \Psi_N, \mathbb{1}_B \Psi_N \rangle = \langle \Psi_N, \Psi_N \rangle = 1$$

and

$$\sum_B |B| \langle \Psi_N, \mathbb{1}_B \Psi_N \rangle = \sum_B \left\langle \mathbb{1}_B \Psi_N, \sum_{i=1}^N \mathbb{1}_Q(\mathbf{x}_i) \mathbb{1}_B \Psi_N \right\rangle = \left\langle \Psi_N, \sum_{i=1}^N \mathbb{1}_Q(\mathbf{x}_i) \Psi_N \right\rangle = \int_Q \varrho_{\Psi_N}.$$

Thus from (4.15) and (4.13) we conclude that

$$\begin{aligned}\mathcal{E}_Q[\Psi_N] &\geq \frac{C}{|Q|^{2s/d}}\sum_B \Big[|B|-q\Big]_+\langle\Psi_N, \mathbb{1}_B\Psi_N\rangle \\ &\geq \frac{C}{|Q|^{2s/d}}\Big[\sum_B(|B|-q)\langle\Psi_N, \mathbb{1}_B\Psi_N\rangle\Big]_+ \\ &= \frac{C}{|Q|^{2s/d}}\Big[\int_Q \varrho_{\Psi_N} - q\Big]_+\end{aligned}$$

by Jensen's inequality and the convexity of the function $t \mapsto [t]_+$. □

5. Many-body Poincaré inequality

The crucial fact that the local energy $E_n(\Omega)$ in (4.7) eventually becomes positive with increasing particle number is the content of the following Poincaré inequality:

Theorem 5.1 (Poincaré inequality for functions vanishing on diagonals)**.** *Fix an integer $k \geq 2$ and a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^d$. Assume that $2s > d(k-1)$. For $N \in \mathbb{N}$ large enough ($N \geq \lceil s\rceil^d k$ is sufficient) there exists a positive constant C depending only on s, k, N, Ω so that*

$$\|u\|_{\dot{H}^{s,N}(\Omega)} \geq C\|u\|_{L^2(\Omega^N)} \tag{5.1}$$

for all $u \in C^\infty(\Omega^N)$ whose restriction to $\triangle_k$ is zero.

Since Theorem 5.1 is of independent interest, we state the result for more general domains although the result for cubes is sufficient for our application.

Conclusion of Theorem 2.2. From Theorem 5.1 and Lemma 4.4 we obtain the local exclusion bound (4.12). Theorem 2.2 then immediately follows from the proof strategy in Section 3. □

It remains to prove Theorem 5.1. The central fact used in the proof is that a function minimizing (5.1) must be a polynomial, and that if a polynomial vanishes on too many diagonals it must be zero.

Lemma 5.2 (Low-degree polynomials vanishing on diagonals are trivial)**.** *Given $d, k, S \in \mathbb{N}_1$ and $N \geq (S+1)^d k$. Let the dN-variable polynomial $f(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, with $\mathbf{x}_i \in \mathbb{R}^d$, satisfy*

- $\deg_{\mathbf{x}_j} f \leq S$ *for all* $j \in \{1, \ldots, N\}$,
- $f(\mathbf{x}_1, \ldots, \mathbf{x}_N) = 0$ *on* $\triangle_k$.

Then $f \equiv 0$.

Proof. The case $k = 1$ ($\triangle_1 = \mathbb{R}^d$) is trivial. We prove the other cases by induction.

Step 1: Consider $d = 1$ and $k = 2$. Then $f(x_1, \ldots, x_N) = 0$ if $x_i = x_j$ for some $i \neq j$. Consequently, when $x_2, \ldots, x_N$ are mutually different, the one variable polynomial $g(x_1) = f(x_1, x_2, \ldots, x_N)$ has $\deg g \leq S$ but it has $N-1$ different roots $x_1 = x_2, \ldots, x_1 = x_N$. Therefore, if

$$N - 1 > S$$

(which holds if $N \geq (S+1)k$) then $g(x_1) \equiv 0$. Thus

$$f(x_1, \ldots, x_N) = 0$$

for all $x_1, \dots, x_N \in \mathbb{R}$ satisfying that $x_2, \dots, x_N$ are mutually different. By continuity, we conclude that $f \equiv 0$.

Step 2: Consider $d = 1$ and $k > 2$. Then $f(x_1, \dots, x_N) = 0$ if at least k points x_i's coincide. Then if $x_k, \dots, x_N$ are mutually different, the one-variable polynomial

$$g(x_1) = f(x_1, \dots, x_1, x_k, \dots, x_N)$$

has $\deg g \le S(k-1)$ but it has $N - k + 1$ different roots $x_1 = x_k, \dots, x_1 = x_N$. Therefore, if

$$N - k + 1 > S(k-1)$$

(which holds if $N \ge (S+1)k$) then $g \equiv 0$. Thus

$$f(x_1, \dots, x_1, x_k, \dots, x_N) = 0$$

if $x_k, \dots, x_N$ are mutually different. By continuity, we conclude that

$$f(x_1, \dots, x_1, x_k, \dots, x_N) = 0$$

for all $x_1, \dots, x_N$. Similarly, by a renumbering, we can show that

$$f(x_1, x_2, \dots, x_N) = 0$$

if at least $(k-1)$ points x_i's coincide. By induction in k, we conclude that $f \equiv 0$.

Step 3: Now consider $d > 1$ and $k \ge 2$. Let us denote

$$\mathbf{x}_i = (y_i, \mathbf{z}_i) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Take

$$n = (S+1)k, \quad N \ge (S+1)^d k = (S+1)^{d-1} n.$$

Then for any $\mathbf{z} \in \mathbb{R}^{d-1}$ and $\mathbf{x}_{n+1}, \dots, \mathbf{x}_N \in \mathbb{R}^d$, the polynomial

$$g(y_1, \dots, y_n) = f((y_1, \mathbf{z}), \dots, (y_n, \mathbf{z}), \mathbf{x}_{n+1}, \dots, \mathbf{x}_N)$$

satisfies that $\deg_{y_i} g \le S$ and $g = 0$ if (at least) k points y_i's coincide. By the result in the 1D case (with the choice $n = (S+1)k$) we conclude that $g \equiv 0$. Similarly, we obtain that

$$f(\mathbf{x}_1, \dots, \mathbf{x}_N) = f((y_1, \mathbf{z}_1), \dots, (y_N, \mathbf{z}_N)) = 0$$

if at least n points $\mathbf{z}_i$'s coincide. By induction in d (i.e. using the induction hypothesis with $d-1$ and $k = n$, $N \ge (S+1)^{d-1} n$) we conclude that $f \equiv 0$. $\square$

We will also need the following technical lemma, which essentially states that if a multivariable function is a polynomial in each variable separately, then it is a multivariable polynomial. The proof of this seemingly obvious fact is indeed non-trivial; see Carroll [1] for an elegant proof in the two variables case. Here we provide an alternative proof for n variables.

Lemma 5.3. *Let $f(x_1, \dots, x_n) \in L^1_{\rm loc}(\mathbb{R}^n)$ satisfy that for any $j = 1, 2, \dots, n$ and for a.e. $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ the mapping $x_j \mapsto f(x_1, \dots, x_j, \dots, x_n)$ is a polynomial of degree at most M_j. Then f is a polynomial of n variables $(x_1, \dots, x_n)$ of degree at most $M = \sum_{j=1}^n M_j$.*

From the proof below, it is clear that we can replace $\mathbb{R}^n$ by a subdomain (e.g. a cube).

Proof. **Step 1.** We use the notation $\mathrm{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, and for every $j = 1, 2, \ldots, n$ we write

$$\mathrm{x} = (x_j; \mathrm{x}_j'), \quad \alpha = (\alpha_j; \alpha_j').$$

By assumption, for a.e. $\mathrm{x}_j' \in \mathbb{R}^{n-1}$, the mapping $x_j \mapsto f(x_j; \mathrm{x}_j')$ is a polynomial of degree at most M_j. Therefore, for any $\alpha_j > M_j$, $D^{\alpha_j} f(\,\cdot\,; \mathrm{x}_j') = 0$ as distribution on $\mathbb{R}$, namely

$$\int_{\mathbb{R}} f(x_j; \mathrm{x}_j') D^{\alpha_j} h(x_j)\, dx_j = 0, \quad \forall h \in C_c^\infty(\mathbb{R}). \tag{5.2}$$

Consequently, $D^\alpha f = 0$ as distribution in $\mathbb{R}^n$ if $|\alpha| > M$. Indeed, since $|\alpha| > M$ we have $\alpha_j > M_j$ for some j, and hence for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ using Fubini's theorem and (5.2) we can write

$$\int_{\mathbb{R}^n} f D^\alpha \varphi \, d\mathrm{x} = \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} f(x_j; \mathrm{x}_j') D^{\alpha_j} \Big(D^{\alpha_j'} \varphi(x_j; \mathrm{x}_j') \Big) dx_j \right] d\mathrm{x}_j' = 0.$$

Step 2. Thus it remains to prove that if $D^\alpha f = 0$ as distribution in $\mathbb{R}^n$ for any $|\alpha| > M$, then f is a polynomial of n variables. We prove this statement by induction in M.

If $M = 0$, then $D_{x_j} f = 0$ as distribution for any $j = 1, 2, \ldots, n$, and hence f is constant by [11, Theorem 6.1].

Now we prove the statement for $M \geq 1$ using the induction hypothesis for $M - 1$. From

$$D^\alpha f = 0, \quad \forall |\alpha| > M$$

we have for any $j = 1, 2, \ldots, n$,

$$D^\alpha (D_{x_j} f) = 0, \quad \forall |\alpha| > M - 1.$$

Thus by the induction hypothesis for $M - 1$, $D_{x_j} f$ is a polynomial of n variables for any $j = 1, 2, \ldots, n$. Since $D_{x_j} f \in C(\mathbb{R}^n)$ for all $j = 1, 2, \ldots, n$, we obtain that $f \in C^1(\mathbb{R}^n)$ by [11, Theorem 6.10] and we have the formula [11, Theorem 6.9]

$$f(\mathrm{x}) = f(0) + \int_0^1 \mathrm{x} \cdot (\nabla f)(t\mathrm{x})\, dt, \quad \forall \mathrm{x} \in \mathbb{R}^n.$$

The latter formula and the fact that $D_{x_j} f$ is a polynomial of n variables for any $j = 1, 2, \ldots, n$ imply that f is a polynomial of n variables. This ends the proof. $\square$

Proof of Theorem 5.1. We argue by contradiction. Assume that (5.1) is false, then there exists a sequence $u_n \in C^\infty(\Omega^N)$ satisfying $\|u_n\|_{L^2} = 1$, $u_n\big|_{\triangle_k} \equiv 0$, and

$$\|u_n\|_{\dot{H}^{s,N}(\Omega)} \to 0, \quad \text{as } n \to \infty. \tag{5.3}$$

In particular, u_n is bounded in the Sobolev space $H^\nu(\Omega^N)$ with $\nu = \min\{s, 1\}$. Indeed, for $d = 1$ this follows from Lemma A.1 and Sobolev's embedding theorem. If $d \geq 2$ then $s > 1$ and the claim follows from Sobolev's embedding theorem combined with that for any Ω the $\dot{H}^1(\Omega^N)$ and $\dot{H}^{1,N}(\Omega)$ seminorms are equivalent. By compactness of the embedding $H^\nu(\Omega^N) \subset L^2(\Omega^N)$, up to a subsequence, u_n converges strongly to a function P in $L^2(\Omega^N)$. Since $\|u_n\|_{L^2(\Omega^N)} = 1$ we have that $\|P\|_{L^2(\Omega^N)} = 1$.

On the other hand, by Poincaré's inequality for $\dot{H}^s(\Omega)$ (combining [11, Theorem 8.11] and [8, Lemma 2.2])

$$\begin{aligned}\|u_n\|^2_{\dot{H}^{s,N}(\Omega)} &= \sum_{j=1}^N \int_{\Omega^{N-1}} \|u_n(x_j;\mathrm{x}')\|^2_{\dot{H}^s_{\mathbf{x}_j}(\Omega)} d\mathrm{x}' \\ &\geq C \sum_{j=1}^N \int_{\Omega^N} \left|u_n(\mathrm{x}) - P_j^{(n)}(\mathrm{x})\right|^2 d\mathrm{x},\end{aligned}$$

where $P_j^{(n)}(\mathrm{x})$ is a polynomial in $\mathbf{x}_j$ of degree $\leq \lceil s-1 \rceil$. In fact, the polynomial can be written explicitly as

$$P_j^{(n)}(\mathrm{x}) = \sum_{|\beta| \leq \lceil s-1 \rceil} \mathbf{x}_j^\beta \langle \varphi_\beta(\mathbf{x}_j), u_n(\mathbf{x}_j;\mathrm{x}')\rangle_{L^2_{\mathbf{x}_j}(\Omega)} \tag{5.4}$$

for universal functions $\varphi_\beta \in C^\infty(\Omega)$. Since u_n converges strongly in $L^2(\Omega^N)$, we can conclude that $P_j^{(n)}(\mathrm{x}) \to P_j(\mathrm{x})$ strongly in $L^2(\Omega^N)$ and the limit is again a polynomial in $\mathbf{x}_j$ of degree $\leq \lceil s-1 \rceil$. The assumption (5.3) allows us to identify the limiting functions and we find that

$$P(\mathrm{x}) = P_j(\mathrm{x}) \text{ in } L^2(\Omega^N), \quad \forall j.$$

Thus the function $P(x)$ is a polynomial in each variable $\mathbf{x}_j$ (of degree $\leq \lceil s-1 \rceil$). By Lemma 5.3, $P(x)$ is a multivariate polynomial whose degree in each $\mathbf{x}_j$ is $\leq \lceil s-1 \rceil$.

We now want to use that $u_n = 0$ on $\triangle_k$ to prove that $P = 0$ on $\triangle_k$. Once this is done, then Lemma 5.2 implies that $P \equiv 0$ if $N \geq \lceil s \rceil^d k$. This contradicts that $\|P\|_{L^2(\Omega^N)} = 1$ and hence completes our proof. Note that if we can prove that $P \equiv 0$ in some open subset this is sufficient, in particular we can find some open cube $Q \subseteq \Omega$ and consider instead u_n and P restricted to Q^N.

We consider the diagonal $\mathbf{x}_1 = \mathbf{x}_2 = \ldots = \mathbf{x}_k$; the other cases are treated identically. By Lebesgue's differentiation theorem it suffices to prove that

$$\lim_{\delta\to 0} \frac{1}{\delta^{d(k-1)}} \int_{Q^{N-k+1}} \int_{\max_{j\leq k}|\mathbf{x}_1-\mathbf{x}_j|<\delta} |P(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathrm{x}')|\, d\mathrm{x} = 0. \tag{5.5}$$

By Fatou's lemma we have for any $\delta > 0$ that

$$\begin{aligned}&\int_{Q^{N-k+1}} \int_{\max_{j\leq k}|\mathbf{x}_1-\mathbf{x}_j|<\delta} |P(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathrm{x}')|\, d\mathrm{x} \\ &\qquad \leq \lim_{n\to\infty} \int_{Q^{N-k+1}} \int_{\max_{j\leq k}|\mathbf{x}_1-\mathbf{x}_j|<\delta} |u_n(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathrm{x}')|\, d\mathrm{x}.\end{aligned} \tag{5.6}$$

Since $u_n = 0$ on $\triangle_k$ it holds that

$$\begin{aligned}&\int_{Q^{N-k+1}} \int_{\max_{j\leq k}|\mathbf{x}_1-\mathbf{x}_j|<\delta} |u_n(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathrm{x}')|\, d\mathrm{x} \\ &\quad = \int_{Q^{N-k+1}} \int_{\max_{j\leq k}|\mathbf{x}_1-\mathbf{x}_j|<\delta} |u_n(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathrm{x}') - u_n(\mathbf{x}_1,\ldots,\mathbf{x}_1;\mathrm{x}')|\, d\mathrm{x}.\end{aligned} \tag{5.7}$$

By Lemma A.1, any $u \in L^2(Q^l)$ with $\|u\|_{\dot{H}^{s,l}(Q)} < \infty$ satisfies that $u \in H^s(Q^l)$ and moreover there is a constant C depending only on Q, l, s such that

$$\|u\|_{H^s(Q^l)} \le C\big(\|u\|_{L^2(Q^l)} + \|u\|_{\dot{H}^{s,l}(Q)}\big). \tag{5.8}$$

If $2s > dl$, by Sobolev's embedding theorem (see for instance [24, Theorem 8.2]), there is for any $\gamma \in \big(0, \min\{1, \frac{2s-dl}{2}\}\big)$ a constant C so that

$$\|u\|_{C^{0,\gamma}(Q^l)} \le C\|u\|_{H^s(Q^l)}, \quad \text{for all } u \in H^s(Q^l).$$

By assumption $2s > d(k-1)$, and hence we can apply this result to the function

$$(\mathbf{x}_2, \dots, \mathbf{x}_k) \mapsto u_n(\mathbf{x}_1, \dots, \mathbf{x}_k; \mathrm{x}')$$

(whose $\dot{H}^{s,k-1}(Q)$-seminorm is bounded for a.e. $(\mathbf{x}_1, \mathrm{x}')$). Equation (5.7) then implies that

$$\begin{aligned}\int_{Q^{N-k+1}} \int_{\max_{j\le k} |\mathbf{x}_1 - \mathbf{x}_j| < \delta} &|u_n(\mathbf{x}_1, \dots, \mathbf{x}_k; \mathrm{x}')|\, d\mathrm{x} \\ &\le C \int_{Q^{N-k+1}} \int_{\max_{j\le k} |\mathbf{x}_1 - \mathbf{x}_j| < \delta} \|u_n(\mathbf{x}_1, \mathrm{x}''; \mathrm{x}')\|_{H^s_{\mathrm{x}''}(Q^{k-1})} |\mathrm{x}'' - \mathrm{x}''_1|^\gamma d\mathrm{x}'' d\mathbf{x}_1 d\mathrm{x}',\end{aligned}$$

where we set $\mathrm{x}'' = (\mathbf{x}_2, \dots, \mathbf{x}_k)$ and $\mathrm{x}''_1 = (\mathbf{x}_1, \dots, \mathbf{x}_1)$. Applying (5.8) and Hölder's inequality yields

$$\int_{Q^{N-k+1}} \int_{\max_{j\le k} |\mathbf{x}_1 - \mathbf{x}_j| < \delta} |u_n(\mathbf{x}_1, \dots, \mathbf{x}_k; \mathrm{x}')|\, d\mathrm{x} \le C\delta^{d(k-1)+\gamma}\big(\|u_n\|_{L^2(Q^N)} + \|u_n\|_{\dot{H}^{s,N}(Q)}\big).$$

Since $\|u_n\|_{L^2(Q^N)} + \|u_n\|_{\dot{H}^{s,N}(Q)} \le C$ and $\gamma > 0$, we arrive at (5.5) which completes the proof of Theorem 5.1. □

We finally note that the many-body nature of the wave functions is crucial for Theorem 5.1 to hold. The following example shows that the requirement that the particle number N is large, in fact typically strictly larger than k, is necessary.

Proposition 5.4 (Counterexample to the k-body case)**.** *Theorem 5.1 cannot hold for $N < k$, or for $N = k$ if s is integer and*

$$\max\{d, 2\}(k-1) < 2s < (d+k)(k-1).$$

Replacing the condition $u|_{\triangle_k} = 0$ by the stronger condition

$$u \in \mathcal{H}^{s,N}_{0,k}(\mathbb{R}^d) := \overline{\{\Psi \in C_c^\infty(\mathbb{R}^{dN} \setminus \triangle_k)\}}^{H^s(\mathbb{R}^{dN})},$$

or

$$u \in \mathcal{H}^{s,N}_{W,k}(\mathbb{R}^d) := \Big\{\Psi \in H^s(\mathbb{R}^{dN}) : \int_{\mathbb{R}^{dN}} W_{s,k}|\Psi|^2 < \infty\Big\},$$

with the k-particle generalization of W_s,

$$W_{s,k}(\mathrm{x}) := \sum_{\substack{A \subseteq \{1,\dots,N\} \\ |A| = k}} \Big(\sum_{\substack{j,l \in A \\ j < l}} |\mathbf{x}_j - \mathbf{x}_l|^2\Big)^{-2s},$$

does not help.

Proof. If $N < k$ there is no diagonal set $\triangle_k$ and we may take the constant function as a counterexample. For $N = k$ we consider the polynomial

$$u(\mathbf{x}_1, \dots, \mathbf{x}_k) := \prod_{1 \le j < l \le k} (x_{j,1} - y_{l,1}),$$

for which, by the arithmetic mean-geometric mean inequality and the triangle inequality,

$$|u(\mathrm{x})|^2 \lesssim \left(\sum_{j<l} |x_{j,1} - x_{l,1}|^2 \right)^{\binom{k}{2}} \le \left(\sum_{j<l} |\mathbf{x}_j - \mathbf{x}_l|^2 \right)^{\binom{k}{2}} \lesssim \left(\sum_{l \ge 2} |\mathbf{x}_1 - \mathbf{x}_l|^2 \right)^{\binom{k}{2}} =: R^{2\binom{k}{2}},$$

where $R \ge 0$ may serve as a radial coordinate on $\mathbb{R}^{d(k-1)}$ relative to $\mathbf{x}_1$. Hence, we have that

$$\int_{Q^k} W_{s,k} |u|^2 \lesssim \int_Q \int_{Q^{k-1}} \frac{R^{2\binom{k}{2}}}{R^{2s}} d\mathbf{x}_2 \dots d\mathbf{x}_k d\mathbf{x}_1 \lesssim \int_0^C R^{k(k-1)-2s+d(k-1)-1} dR < \infty,$$

if $d(k-1) < 2s < (d+k)(k-1)$. Thus (analogously to Lemma B.2, and by extension)

$$u \in \mathcal{H}^{s,N}_{W,k}(\mathbb{R}^d) \subseteq \mathcal{H}^{s,N}_{0,k}(\mathbb{R}^d).$$

On the other hand

$$\|u\|^2_{\dot{H}^{s,k}(\Omega)} = \sum_{j=1}^k \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{Q^k} |D^\alpha_{\mathbf{x}_j} u|^2 = 0,$$

if $s = m > k - 1$. □

A particular case included in the above is $d = 3$, $s = 2$, $k = 2$, with the function $u(\mathbf{x}, \mathbf{y}) := x_1 - y_1$.

Appendix A. Equivalence of Sobolev spaces

In this appendix we discuss the N-particle space

$$H^{s,N}(\Omega) := \left\{ u \in L^2(\Omega^N) : \|u\|_{\dot{H}^{s,N}(\Omega)} < \infty \right\}$$

and its relation to the standard Sobolev space $H^s(\Omega^N)$.

If $\Omega = \mathbb{R}^d$ the equivalence of the seminorms (and consequently the spaces)

$$c_{s,N} \|u\|_{\dot{H}^s(\mathbb{R}^{dN})} \le \|u\|_{\dot{H}^{s,N}(\mathbb{R}^d)} \le C_{s,N} \|u\|_{\dot{H}^s(\mathbb{R}^{dN})} \tag{A.1}$$

can be seen via the Fourier transform. However, the constants in the equivalence depend on N and s. In particular, if $s \ne 1$ the equivalence degenerates as N tends to infinity; either $c_{s,N} \to 0$ or $C_{s,N} \to \infty$. Specifically, the sharp constants in (A.1) are given by

$$c_{s,N} = \min\{1, N^{(1-s)/2}\} \quad \text{and} \quad C_{s,N} = \max\{1, N^{(1-s)/2}\}.$$

Thus it is a slightly subtle question of what happens to these spaces in the many-body limit. An even more subtle question is what happens to the local versions of these spaces, i.e. when $\mathbb{R}^d$ is replaced by $\Omega \subsetneq \mathbb{R}^d$. For us, the following equivalence of the spaces in the case of cubes will suffice:

Lemma A.1. *Let $u \in L^2(Q^N)$, $Q = [0,1]^d$. There exist positive constants c, C depending only on d, s, N so that*

$$c\big(\|u\|_{L^2(Q^N)} + \|u\|_{\dot{H}^{s,N}(Q)}\big) \le \|u\|_{H^s(Q^N)} \le C\big(\|u\|_{L^2(Q^N)} + \|u\|_{\dot{H}^{s,N}(Q)}\big).$$

Lemma A.1 is an immediate consequence of the equivalence (A.1) of the two seminorms on $\mathbb{R}^{dN}$ and the following extension lemma:

Lemma A.2. *Let $u \in L^2(Q^N)$, $Q = [0,1]^d$, and assume that $\|u\|_{L^2(Q^N)} + \|u\|_{\dot H^{s,N}(Q)} < \infty$. There exists a function $\tilde u \in L^2(\mathbb{R}^{dN})$ with compact support satisfying*

$$\tilde u\big|_{Q^N} = u, \quad \text{and} \quad \|\tilde u\|_{L^2(\mathbb{R}^{dN})} + \|\tilde u\|_{\dot H^{s,N}(\mathbb{R}^d)} \le C\big(\|u\|_{L^2(Q^N)} + \|u\|_{\dot H^{s,N}(Q)}\big),$$

where C is a constant depending only on s, d and N.

Proof. We shall prove the lemma by using higher-order reflection through one side of the hypercube Q at a time. To this end we recall that if $v \in C^n([0,1])$, for some $n \ge 0$, we can construct an explicit extension $\tilde v \in C^n((-\infty,1])$ satisfying $\tilde v(x) = 0$ when $x < -\delta$. Namely, set

$$\tilde v(x) = \begin{cases} v(x), & \text{if } x \in [0,1], \\ \varphi(x)\sum_{j=1}^{n+1}\lambda_j v(-x/j), & \text{if } x < 0, \end{cases}$$

where $\varphi \in C^\infty((-\infty,0])$ such that $\varphi(x) \equiv 0$ for $x < -\delta$ and $\varphi(x) \equiv 1$ in $[-\delta/2,0]$. What remains is to verify that we can choose the λ_j's so that $\tilde v \in C^n$. But if we differentiate $\tilde v$ for x away from zero we see that the system of equations that we need the λ_j to satisfy to get continuity of the derivatives across $x = 0$ is

$$\Big[(-j)^{1-i}\Big]_{i,j=1}^{n+1}\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_{n+1}\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

But the determinant of this matrix is non-zero (it is a Vandermonde matrix) and hence there exists a unique solution $(\lambda_1,\dots,\lambda_{n+1})$.

We shall now prove that we can use this one-dimensional extension repeatedly to construct an extension of u to $\mathbb{R}^{dN}$. The idea is to use the one-dimensional result one coordinate at a time and show that the new function in each step has the quantity corresponding to the $\dot H^{s,N}$-seminorm controlled by that of u.

Without loss we can assume that $u \in C^n(Q^N)$ (the construction is stable under approximation), where we take $n = \lceil s\rceil$. Consider $u(x_1;\mathrm{x}')$, $x_1 \in [0,1]$ and $\mathrm{x}' \in [0,1]^{dN-1}$. And apply the above lemma for each fixed x', that is, we define v_1 by

$$v_1(x_1;\mathrm{x}') = \begin{cases} u(x_1;\mathrm{x}'), & \text{if } x_1 \in [0,1], \\ \varphi(x_1)\sum_{j=1}^{n+1}\lambda_j u(-x_1/j;\mathrm{x}'), & \text{if } x_1 \in [-1,0). \end{cases}$$

It is a simple calculation to use Sobolev's embedding theorem to prove that we can bound the L^p-norm of l-th order derivatives of v_1 by the corresponding one for u if $l \le n$. We need to prove that also the fractional order seminorm is preserved. That is, we wish to show that, with $s = m + \sigma$ and $Q' = [-1,1]\times[0,1]^{d-1}$,

$$\begin{aligned}&\int_{Q^{N-1}}\iint_{Q'\times Q'}\frac{|D^\alpha_{\mathbf{x}_1}v_1(\mathbf{x}_1;\mathrm{x}') - D^\alpha_{\mathbf{y}_1}v_1(\mathbf{y}_1;\mathrm{x}')|^2}{|\mathbf{x}_1-\mathbf{y}_1|^{d+2\sigma}}d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\ &\quad + \sum_{i=2}^N\int_{Q'\times Q^{N-2}}\iint_{Q\times Q}\frac{|D^\alpha_{\mathbf{x}_i}v_1(\mathbf{x}_i;\mathrm{x}') - D^\alpha_{\mathbf{y}_i}v_1(\mathbf{y}_i;\mathrm{x}')|^2}{|\mathbf{x}_i-\mathbf{y}_i|^{d+2\sigma}}d\mathbf{x}_i d\mathbf{y}_i d\mathrm{x}' \\ &\le C\big(\|u\|^2_{\dot H^{s,N}(Q)} + \|u\|^2_{L^2(Q^N)}\big),\end{aligned} \tag{A.2}$$

for all multi-indices $|\alpha| = m$. If we can prove this inequality, then by repeating the procedure to extend v_1 to $x_1 > 1$ the same proof gives that we can bound the corresponding $\dot{H}^{s,N}$ quantity in terms of that of v_1, and hence u. By repeating the procedure for each coordinate at a time we, after $2dN$ reflections, find a function $\tilde{u} \in L^2(\mathbb{R}^{dN})$ satisfying the claims of the lemma. Thus all that remains is to prove (A.2).

We start with the first term which is also the most difficult:

$$
\begin{aligned}
&\int_{Q^{N-1}} \iint_{Q' \times Q'} \frac{|D^\alpha_{\mathbf{x}_1} v_1(\mathbf{x}_1; \mathrm{x}') - D^\alpha_{\mathbf{y}_1} v_1(\mathbf{y}_1; \mathrm{x}')|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\
&= \int_{Q^{N-1}} \Bigg[\iint_{Q \times Q} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - D^\alpha_{\mathbf{y}_1} u(\mathbf{y}_1; \mathrm{x}')|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 \\
&+ 2 \iint_{Q \times (Q' \setminus Q)} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j D^\alpha_{\mathbf{y}_1} (\varphi(y_{1,1}) u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}'))|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 \qquad (A.3) \\
&+ \iint_{(Q' \setminus Q) \times (Q' \setminus Q)} |\mathbf{x}_1 - \mathbf{y}_1|^{-d-2\sigma} \Bigg| \sum_{j=1}^{n+1} \lambda_j \Big(D^\alpha_{\mathbf{x}_1} \big[\varphi(x_{1,1}) u(-x_{1,1}/j, \mathbf{x}_1'; \mathrm{x}')\big] \\
&- D^\alpha_{\mathbf{y}_1} \big[\varphi(y_{1,1}) u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}')\big] \Big) \Bigg|^2 d\mathbf{x}_1 d\mathbf{y}_1 \Bigg] d\mathrm{x}'.
\end{aligned}
$$

Clearly the integral over $Q \times Q$ is bounded by $\|u\|_{\dot{H}^{s,N}(Q)}$. We treat the two remaining terms separately. In order to bound the integral over $Q \times (Q' \setminus Q)$ we write

$$
\begin{aligned}
Q_1 &= \{\mathbf{x} \in Q' \setminus Q : x_1 > -\delta/2\}, \\
Q_2 &= \{\mathbf{x} \in Q' \setminus Q : x_1 \leq -\delta/2\}.
\end{aligned}
$$

Thus we can bound the second integral in (A.3) as follows:

$$
\begin{aligned}
&\int_{Q^{N-1}} \iint_{Q \times (Q' \setminus Q)} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j D^\alpha_{\mathbf{y}_1} (\varphi(y_{1,1}) u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}'))|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\
&= \int_{Q^{N-1}} \iint_{Q \times Q_1} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j D^\alpha_{\mathbf{y}_1} (u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}'))|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\
&+ \int_{Q^{N-1}} \iint_{Q \times Q_2} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j D^\alpha_{\mathbf{y}_1} (\varphi(y_{1,1}) u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}'))|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\
&\leq \int_{Q^{N-1}} \iint_{Q \times Q_1} \frac{|D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j (-j)^{-\alpha_1} D^\alpha_{\mathbf{y}_1} u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}')|^2}{|\mathbf{x}_1 - \mathbf{y}_1|^{d+2\sigma}} d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}' \\
&+ \frac{C}{\delta^{d+2\sigma}} \int_{Q^{N-1}} \iint_{Q \times Q_2} |D^\alpha_{\mathbf{x}_1} u(\mathbf{x}_1; \mathrm{x}') - \sum_{j=1}^{n+1} \lambda_j D^\alpha_{\mathbf{y}_1} (\varphi(y_{1,1}) u(-y_{1,1}/j, \mathbf{y}_1'; \mathrm{x}'))|^2 d\mathbf{x}_1 d\mathbf{y}_1 d\mathrm{x}'.
\end{aligned}
$$

Using the triangle inequality and Sobolev's embedding theorem one finds that the second term is $\lesssim \|u\|^2_{L^2(Q^N)} + \|u\|^2_{\dot{H}^{s,N}(Q)}$. Since $\sum_j \lambda_j (-j)^{-\alpha_1} = 1$ for any $\alpha_1 \leq m+1$, one obtains for the first integral

$$
\begin{aligned}
&\int_{Q^{N-1}}\iint_{Q\times Q_1}\frac{|D^\alpha_{\mathbf{x}_1}u(\mathbf{x}_1;\mathrm{x}')-\sum_{j=1}^{n+1}\lambda_j(-j)^{-\alpha_1}D^\alpha_{\mathbf{y}_1}u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')|^2}{|\mathbf{x}_1-\mathbf{y}_1|^{d+2\sigma}}d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad=\int_{Q^{N-1}}\iint_{Q\times Q_1}\frac{|\sum_{j=1}^{n+1}\lambda_j(-j)^{-\alpha_1}\big(D^\alpha_{\mathbf{x}_1}u(\mathbf{x}_1;\mathrm{x}')-D^\alpha_{\mathbf{y}_1}u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')\big)|^2}{|\mathbf{x}_1-\mathbf{y}_1|^{d+2\sigma}}d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\sum_{j=1}^{n+1}\int_{Q^{N-1}}\iint_{Q\times Q_1}\frac{|D^\alpha_{\mathbf{x}_1}u(\mathbf{x}_1;\mathrm{x}')-D^\alpha_{\mathbf{y}_1}u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')|^2}{|\mathbf{x}_1-\mathbf{y}_1|^{d+2\sigma}}d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\sum_{j=1}^{n+1}\int_{Q^{N-1}}\iint_{Q\times Q}\frac{|D^\alpha_{\mathbf{x}_1}u(\mathbf{x}_1;\mathrm{x}')-D^\alpha_{\mathbf{y}_1}u(\mathbf{y}_1;\mathrm{x}')|^2}{(|\mathbf{x}_1'-\mathbf{y}_1'|^2+(x_{1,1}+jy_{1,1})^2)^{d/2+\sigma}}d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\|u\|^2_{\dot H^{s,N}(Q)}.
\end{aligned}
$$

In the last step we used the inequality $(x+jy)^2\ge(x-y)^2$ for $x,y\ge 0$ and $j\ge 1$.

For the last integral in (A.3) we have

$$
\begin{aligned}
&\int_{Q^{N-1}}\iint_{(Q'\setminus Q)\times(Q'\setminus Q)}|\mathbf{x}_1-\mathbf{y}_1|^{-d-2\sigma}\Big|\sum_{j=1}^{n+1}\lambda_j\Big(D^\alpha_{\mathbf{x}_1}\big[\varphi(x_{1,1})u(-x_{1,1}/j,\mathbf{x}_1';\mathrm{x}')\big]\\
&\qquad-D^\alpha_{\mathbf{y}_1}\big[\varphi(y_{1,1})u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')\big]\Big)\Big|^2d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad=\int_{Q^{N-1}}\iint_{(Q'\setminus Q)\times(Q'\setminus Q)}|\mathbf{x}_1-\mathbf{y}_1|^{-d-2\sigma}\Big|\sum_{j=1}^{n+1}\sum_{\gamma+\beta=\alpha_1}\lambda_j(-j)^{-\beta}\\
&\qquad\times\Big(\varphi^{(\gamma)}(x_{1,1})D^{\alpha'}_{\mathbf{x}_1}u(-x_{1,1}/j,\mathbf{x}_1';\mathrm{x}')-\varphi^{(\gamma)}(y_{1,1})D^{\alpha'}_{\mathbf{y}_1}u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')\Big)\Big|^2d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}',
\end{aligned}
$$

where we set α' as the multi-index α but with α_1 exchanged for β. By the triangle inequality and the fact that $\sum_j\lambda_j(-j)^{-\beta}=1$ the integral is smaller than

$$
\begin{aligned}
&C\sum_{j=1}^{n+1}\int_{Q^{N-1}}\iint_{(Q'\setminus Q)\times(Q'\setminus Q)}|\mathbf{x}_1-\mathbf{y}_1|^{-d-2\sigma}\Big|\sum_{\gamma+\beta=\alpha_1}\Big(\varphi^{(\gamma)}(x_{1,1})D^{\alpha'}_{\mathbf{x}_1}u(-x_{1,1}/j,\mathbf{x}_1';\mathrm{x}')\\
&\qquad-\varphi^{(\gamma)}(y_{1,1})D^{\alpha'}_{\mathbf{y}_1}u(-y_{1,1}/j,\mathbf{y}_1';\mathrm{x}')\Big)\Big|^2d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\sum_{j=1}^{n+1}\int_{Q^{N-1}}\iint_{Q\times Q}(|\mathbf{x}_1'-\mathbf{y}_1'|^2+j^2(x_{1,1}-y_{1,1})^2)^{-d/2-\sigma}\\
&\qquad\times\Big|\sum_{\gamma+\beta=\alpha_1}\Big(\varphi^{(\gamma)}(-jx_{1,1})D^{\alpha'}_{\mathbf{x}_1}u(\mathbf{x}_1;\mathrm{x}')-\varphi^{(\gamma)}(-jy_{1,1})D^{\alpha'}_{\mathbf{y}_1}u(\mathbf{y}_1;\mathrm{x}')\Big)\Big|^2d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\sum_{j=1}^{n+1}\int_{Q^{N-1}}\iint_{Q\times Q}\frac{\big|D^\alpha_{\mathbf{x}_1}\big[\varphi(-jx_{1,1})u(\mathbf{x}_1;\mathrm{x}')\big]-D^\alpha_{\mathbf{y}_1}\big[\varphi(-jy_{1,1})u(\mathbf{y}_1;\mathrm{x}')\big]\big|^2}{|\mathbf{x}_1-\mathbf{y}_1|^{d+2\sigma}}d\mathbf{x}_1d\mathbf{y}_1d\mathrm{x}'\\
&\quad\le C\|u\|^2_{\dot H^{s,N}(Q)},
\end{aligned}
$$

where we used that $\|\psi u\|_{\dot{H}^s(Q)} \leq C_\psi \|u\|_{\dot{H}^s(Q)}$ for any $\psi \in C^\infty(Q)$.

To show that the remaining terms in (A.2) are $\lesssim \|u\|_{L^2}^2 + \|u\|_{\dot{H}^{s,N}}^2$ one can proceed in an almost identical manner. The main difference is that in these terms the differentiation is with respect other variables than the variable in which the extension has been made, and the splitting of the integrals is slightly different. However, in the end this only simplifies each step of the proof. □

Appendix B. Spaces of contact interaction

We consider in the following only 2-particle diagonals $\triangle$, for simplicity, however analogous statements can be made for the case of k-particle diagonals.

Define for $N \geq 2$ the restricted N-particle spaces

$$\mathcal{H}_W^{s,N}(\mathbb{R}^d) := \left\{ \Psi \in H^s(\mathbb{R}^{dN}) : \int_{\mathbb{R}^{dN}} W_s |\Psi|^2 < \infty \right\},$$

$$\mathcal{H}_0^{s,N}(\mathbb{R}^d) := \overline{\left\{ \Psi \in C_c^\infty(\mathbb{R}^{dN} \setminus \triangle) \right\}}^{H^s(\mathbb{R}^{dN})},$$

$$\mathcal{H}^{s,N}(\mathbb{R}^d) := \overline{\left\{ \Psi \in C_c^\infty(\mathbb{R}^{dN}) : \Psi|_\triangle = 0 \right\}}^{H^s(\mathbb{R}^{dN})}.$$

Then we have for all $s > 0$ the chain of inclusions

$$\mathcal{H}_W^{s,N}(\mathbb{R}^d) \subseteq \mathcal{H}_0^{s,N}(\mathbb{R}^d) \subseteq \mathcal{H}^{s,N}(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^{dN}).$$

The latter two inclusions are trivial while the first one will be proved below. Moreover, for $2s < d$ all four spaces are equal by the Hardy–Rellich inequality (see e.g. [27]):

$$\int_{\mathbb{R}^{dN}} |\mathbf{x}_1 - \mathbf{x}_2|^{-2s} |\Psi(\mathbf{x}_1; \mathrm{x}')|^2 \, d\mathbf{x}_1 d\mathrm{x}' \leq C \int_{\mathbb{R}^{d(N-1)}} \|\Psi\|_{\dot{H}^s_{\mathbf{x}_1}(\mathbb{R}^d)}^2 d\mathrm{x}' \leq C \|\Psi\|_{H^s(\mathbb{R}^{dN})}^2.$$

In the critical case $2s = d$ we still have $\mathcal{H}_0^{s,N}(\mathbb{R}^d) = \mathcal{H}^{s,N}(\mathbb{R}^d) = H^s(\mathbb{R}^{dN})$, as is also shown below, but a strict inclusion $\mathcal{H}_W^{s,N}(\mathbb{R}^d) \subsetneq \mathcal{H}_0^{s,N}(\mathbb{R}^d)$, as illustrated by

$$\Psi(\mathrm{x}) = e^{-|\mathrm{x}|^2}$$

which is in $H^s(\mathbb{R}^{dN})$ but not in $\mathcal{H}_W^{s,N}(\mathbb{R}^d)$ due to the non-integrability of W_s. For $2s > d$ and $s - d/2 \notin \mathbb{Z}$ it again holds by the Hardy–Rellich inequality that $\mathcal{H}_W^{s,N}(\mathbb{R}^d) = \mathcal{H}_0^{s,N}(\mathbb{R}^d)$, while not necessarily $\mathcal{H}_0^{s,N}(\mathbb{R}^d) = \mathcal{H}^{s,N}(\mathbb{R}^d)$, as with the example

$$\Psi(x_1, x_2) = (x_1 - x_2) e^{-|\mathrm{x}|^2}$$

which is in $\mathcal{H}^{s,2}(\mathbb{R}^d)$ but not in $\mathcal{H}_W^{s,2}(\mathbb{R}^d)$ for $s = 2$ and $d = 1$.

Let $\chi_\varepsilon^{(*)}(\mathrm{x}) := \prod_{1 \leq j < k \leq N} \varphi_\varepsilon^{(*)}(\mathbf{x}_j - \mathbf{x}_k)$ where $\varphi_\varepsilon(\mathbf{x}) = \varphi(|\mathbf{x}|/\varepsilon)$ and $\varphi_\varepsilon^*(\mathbf{x}) = \varphi^*(\varepsilon \ln |\mathbf{x}|)$. We take $\varphi^{(*)}$ as smooth functions from $\mathbb{R}$ to $[0,1]$ such that $\varphi(x) = 0$ for $x \leq 1$, $\varphi(x) = 1$ for $x \geq 2$, and $\varphi^*(x) = 0$ for $x \leq -2$, $\varphi^*(x) = 1$ for $x \geq -1$.

Lemma B.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. For all $s = m + \sigma > 0$, $d \geq 1$ and $N \geq 1$ it holds as $\varepsilon \to 0$ that*

$$\|\chi_\varepsilon\|_{\dot{H}^{s,N}(\Omega)} \leq C \varepsilon^{d/2 - s}, \qquad \|D^\alpha \chi_\varepsilon\|_{\dot{H}^{\sigma,N}(\Omega)} \leq C \varepsilon^{d/2 - |\alpha| - \sigma},$$

while for $2s = d$

$$\|\chi_\varepsilon^*\|_{\dot H^{s,N}(\Omega)} \le C\varepsilon^{1/2}, \qquad \|D^\alpha \chi_\varepsilon^*\|_{\dot H^{\sigma,N}(\Omega)} \le C\varepsilon^{1/2}$$

for $|\alpha| \le d/2 - \sigma$.

Proof. For $\alpha \ne 0$ there are in $D^\alpha_{\mathbf{x}_j}\chi_\varepsilon$ a total of $|\alpha|$ derivatives of functions $\varphi_\varepsilon(\mathbf{x}_j - \mathbf{x}_k)$, $k \ne j$, and remaining factors involving the other particles. These factors are uniformly bounded while each derivative yields an additional factor $1/\varepsilon$, while reducing the support in $\mathbf{x}_j$ to $B_{2\varepsilon}(\mathbf{x}_k) \setminus B_\varepsilon(\mathbf{x}_k)$. Furthermore, we thus have

$$|D^\alpha \varphi_\varepsilon(\mathbf{x})| \le C\varepsilon^{-|\alpha|}\mathbb{1}_{B_{2\varepsilon}(0)\setminus B_\varepsilon(0)},$$

$$|D^\alpha \varphi_\varepsilon(\mathbf{x}) - D^\alpha \varphi_\varepsilon(\mathbf{y})| \le C\varepsilon^{-|\alpha|-1}|\mathbf{x}-\mathbf{y}|\mathbb{1}_{\mathbf{x},\mathbf{y}\in B_{2\varepsilon}(0)\setminus B_\varepsilon(0)},$$

and for $B(j,\varepsilon) = \cup_{k\ne j} B_{2\varepsilon}(\mathbf{x}_k) \setminus \cup_{k \ne j} B_\varepsilon(\mathbf{x}_k)$,

$$|\chi_\varepsilon(\mathbf{x}_j; \mathrm{x}') - \chi_\varepsilon(\mathbf{y}_j;\mathrm{x}')| \le C\varepsilon^{-1}|\mathbf{x}_j - \mathbf{y}_j|\mathbb{1}_{\mathbf{x}_j,\mathbf{y}_j\in B(j,\varepsilon)},$$

and

$$|D^\alpha\chi_\varepsilon(\mathbf{x}_j; \mathrm{x}') - D^\alpha\chi_\varepsilon(\mathbf{y}_j;\mathrm{x}')| \le C\varepsilon^{-|\alpha|-1}|\mathbf{x}_j - \mathbf{y}_j|\mathbb{1}_{\mathbf{x}_j,\mathbf{y}_j\in B(j,\varepsilon)}.$$

Hence, $\|D^\alpha\chi_\varepsilon\|^2_{L^2_{\mathbf{x}_j}(\Omega)} \lesssim \varepsilon^{-2|\alpha|+d}$, and for any $0<\sigma<1$

$$\begin{aligned}\|D^\alpha\chi_\varepsilon\|^2_{\dot H^\sigma_{\mathbf{x}_j}(\Omega)} &= \iint_{\Omega\times\Omega} \frac{|D^\alpha\chi_\varepsilon(\mathbf{x};\mathrm{x}') - D^\alpha\chi_\varepsilon(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}}\,d\mathbf{x}d\mathbf{y} \\ &\lesssim \varepsilon^{-2|\alpha|-2}\sum_{k\ne j}\iint_{B_{2\varepsilon}(\mathbf{x}_k)\times B_{2\varepsilon}(\mathbf{x}_k)} |\mathbf{x}-\mathbf{y}|^{-d-2\sigma+2}d\mathbf{x}d\mathbf{y} \lesssim \varepsilon^{-2|\alpha|-2\sigma+d},\end{aligned}$$

so that $\|\chi_\varepsilon\|^2_{\dot H^{s,N}(\Omega)} \lesssim \varepsilon^{-2s+d}$.

Similarly, for χ_ε^* we consider $B(j,\varepsilon) = \cup_{k\ne j}B_{e^{-1/\varepsilon}}(\mathbf{x}_k) \setminus \cup_{k\ne j}B_{e^{-2/\varepsilon}}(\mathbf{x}_k)$ and

$$|D^\alpha\varphi^*_\varepsilon(\mathbf{x})| = |D^\alpha_{\mathbf{x}}\varphi^*(\varepsilon \ln|\mathbf{x}|)| \le C\varepsilon|\mathbf{x}|^{-|\alpha|}\mathbb{1}_{B_{e^{-1/\varepsilon}}(0)\setminus B_{e^{-2/\varepsilon}}(0)}. \tag{B.1}$$

In χ_ε^* this could involve different points $\mathbf{x}_k$ but the worst case is if they are the same,

$$\|D^\alpha\chi^*_\varepsilon\|^2_{L^2_{\mathbf{x}_j}(\Omega)} \lesssim \varepsilon^2\int_{B(j,\varepsilon)}|\mathbf{x}_j - \mathbf{x}_k|^{-2|\alpha|}\,d\mathbf{x}_j \lesssim \begin{cases}\varepsilon^2 & \text{for } 0<2|\alpha|<d,\\ \varepsilon^2\int_{-2\varepsilon^{-1}}^{-\varepsilon^{-1}}ds = \varepsilon & \text{for } 2|\alpha| = d.\end{cases}$$

This covers the even-dimensional critical case $d = 2m$, $m\in\mathbb{N}_1$.

In the odd-dimensional critical case $d = 2m+2\sigma$, $\sigma = 1/2$, we observe that

$$\begin{aligned}\|D^\alpha\chi_\varepsilon\|^2_{\dot H^\sigma_{\mathbf{x}_j}(\Omega)} &= \iint_{\Omega\times\Omega} \frac{|D^\alpha\chi_\varepsilon(\mathbf{x};\mathrm{x}') - D^\alpha\chi_\varepsilon(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}\,d\mathbf{x}d\mathbf{y} \\ &\lesssim \varepsilon^{-2|\alpha|-2}\sum_{k\ne j}\iint_{B_{2\varepsilon}(\mathbf{x}_k)\times B_{2\varepsilon}(\mathbf{x}_k)} |\mathbf{x}-\mathbf{y}|^{-d+1}d\mathbf{x}d\mathbf{y} \lesssim \varepsilon^{-2|\alpha|-1+d},\end{aligned}$$

which is not enough for $2|\alpha| = d-1$. Instead we shall use χ_ε^*.

For the case $2|\alpha| = d-1$ things are a bit less straightforward. We start with the case $d=1$ which is the easiest. Here our approach differs slightly due to the fact that in this case $|\alpha| = 0$.

Let $U_1 = \cap_{k\ne j}B_{e^{-1/\varepsilon}}(\mathbf{x}_k)^c$, $U_2 = \cup_{k\ne j}B_{e^{-2/\varepsilon}}(\mathbf{x}_k)$ and $U = \Omega\setminus(U_1\cup U_2)$.

We estimate the seminorm $\|\chi_\varepsilon^*\|_{\dot H^s_{\mathbf{x}_j}(\Omega)}$. By construction of χ_ε^* we have that

$$|\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - \chi_\varepsilon^*(\mathbf{y};\mathrm{x}')| \le 1, \quad \forall \mathbf{x},\mathbf{y} \in \Omega.$$

Moreover, the difference is zero whenever $(\mathbf{x},\mathbf{y}) \in U_1^2 \cup U_2^2$.

For $\mathbf{x}$ and $\mathbf{y}$ close we need to estimate this quantity more precisely. By Taylor's theorem we can estimate

$$\begin{aligned}
&|\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - \chi_\varepsilon^*(\mathbf{y};\mathrm{x}')| \\
&\quad = \bigg| \int_0^1 \sum_{k\neq j}^N \Big(\prod_{i\notin\{k,j\}} \varphi_\varepsilon^*(\mathbf{x}-\mathbf{x}_i) \Big) (\varphi^*)'(\varepsilon(\ln|\mathbf{x}-\mathbf{x}_k| + t(\ln|\mathbf{y}-\mathbf{x}_k| - \ln|\mathbf{x}-\mathbf{x}_k|))) \\
&\qquad\quad \times \varepsilon(\ln|\mathbf{y}-\mathbf{x}_k| - \ln|\mathbf{x}-\mathbf{x}_k|)\, dt \bigg| \\
&\quad \le C\varepsilon \sum_{k\neq j} \big|\ln|\mathbf{x}-\mathbf{x}_k| - \ln|\mathbf{y}-\mathbf{x}_k|\big|.
\end{aligned}$$

By symmetry in $\mathbf{x},\mathbf{y}$ we find

$$\begin{aligned}
\|\chi_\varepsilon^*\|^2_{\dot H^s_{\mathbf{x}_j}(\Omega)} &= \iint_{\Omega\times\Omega} \frac{|\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - \chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} \\
&\le 2\iint_{U\times\Omega} \frac{|\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - \chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} + 2\iint_{U_1\times U_2} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y}.
\end{aligned} \tag{B.2}$$

The latter term is fairly easy to estimate:

$$\iint_{U_1\times U_2} \frac{1}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} \le 2|U_1|e^{-2/\varepsilon} \int_{-e^{-2/\varepsilon}}^{e^{-2/\varepsilon}} \frac{1}{(e^{-1/\varepsilon}+r)^2} dr \le Ce^{-2/\varepsilon}.$$

We return to the remaining term of (B.2):

$$\begin{aligned}
\iint_{U\times\Omega} \frac{|\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - \chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} &\le C\varepsilon^2 \sum_{k\neq j} \iint_{U\times\Omega} \frac{(\ln|\mathbf{x}-\mathbf{x}_k| - \ln|\mathbf{y}-\mathbf{x}_k|)^2}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y} \\
&= C\varepsilon^2 \sum_{k\neq j} \iint_{(U-\mathbf{x}_k)\times(\Omega-\mathbf{x}_k)} \frac{1}{|\mathbf{x}|^2} \frac{\ln^2\big|\frac{\mathbf{y}}{\mathbf{x}}\big|}{\big(1-\big|\frac{\mathbf{y}}{\mathbf{x}}\big|\big)^2} d\mathbf{y}d\mathbf{x} \\
&\le C\varepsilon^2 \sum_{k\neq j} \int_{U-\mathbf{x}_k} \frac{1}{|\mathbf{x}|} \int_0^\infty \frac{\ln^2 z}{(1-z)^2} dz d\mathbf{x}.
\end{aligned}$$

The inner integral is convergent and hence we are left with

$$\varepsilon^2 \sum_{k\neq j} \int_{U-\mathbf{x}_k} \frac{1}{|\mathbf{x}|} d\mathbf{x} \le C\varepsilon^2 \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} z^{-1} dz = C\varepsilon.$$

When $2|\alpha| = d-1$ and $d>1$ the estimates for the difference quotient are a bit more technical. Similarly to above, Taylor's theorem combined with (B.1) yields

$$\begin{aligned}|D^\alpha_{\mathbf{x}_j}\chi^*_\varepsilon(\mathbf{x};\mathrm{x}') - D^\alpha_{\mathbf{x}_j}\chi^*_\varepsilon(\mathbf{y};\mathrm{x}')| &= \left|\sum_{|\beta|=1}\int_0^1 D^{\alpha+\beta}_{\mathbf{x}_j}\chi^*_\varepsilon(\mathbf{x}+t(\mathbf{y}-\mathbf{x});\mathrm{x}')(\mathbf{y}-\mathbf{x})^\beta dt\right|\\ &\le C\varepsilon|\mathbf{x}-\mathbf{y}|\sum_{k\neq j}\int_0^1 \frac{\mathbb{1}_{B^c_{e^{-2/\varepsilon}}(\mathbf{x}_k)}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))}{|\mathbf{x}-t(\mathbf{y}-\mathbf{x})-\mathbf{x}_k|^{|\alpha|+1}}dt.\end{aligned}$$

We estimate the integral

$$\int_0^1 |\mathbf{x}-t(\mathbf{y}-\mathbf{x})-\mathbf{x}_k|^{-|\alpha|-1}dt.$$

Choosing coordinates in a plane containing $\mathbf{x}_k, \mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}_k = (0,0)$, $\mathbf{x} = (r_1, 0)$ and $\mathbf{y} = (r_2\cos(\theta), r_2\sin(\theta))$ with $\theta\in[0,\pi)$ we can write this integral as

$$\begin{aligned}&\int_0^1 (((1-t)r_1 - tr_2\cos\theta)^2 + t^2r_2^2\sin^2\theta)^{-\frac{|\alpha|+1}{2}}dt\\ &\quad= \int_0^1\left(\left(1-\frac{tr_2}{(1-t)r_1}\cos\theta\right)^2 + \frac{t^2r_2^2}{(1-t)^2r_1^2}\sin^2\theta\right)^{-\frac{|\alpha|+1}{2}}dt\\ &\quad= \frac{1}{r_1r_2^{|\alpha|}}\int_0^\infty \frac{(s+r_2/r_1)^{|\alpha|-1}}{((1+s\cos\theta)^2+s^2\sin^2\theta)^{\frac{|\alpha|+1}{2}}}ds\\ &\quad\le \frac{1}{r_1r_2^{|\alpha|}}\int_0^\infty \frac{(s+1)^{|\alpha|-1}}{((1-s)^2+2s(1+\cos\theta))^{\frac{|\alpha|+1}{2}}}ds\\ &\quad=: \frac{g(\theta)}{r_1r_2^{|\alpha|}}\end{aligned}$$

The integral $g(\theta)$ tends to infinity in the limit $\theta\to\pi$. However, this corresponds to $\mathbf{x}$ and $\mathbf{y}$ being far apart relative to their distance to the $\mathbf{x}_k$.

When θ is far from 0 we shall instead use the following bound which follows directly from the supremum bound in (B.1)

$$|D^\alpha_{\mathbf{x}_j}\chi^*_\varepsilon(\mathbf{x};\mathrm{x}') - D^\alpha_{\mathbf{x}_j}\chi^*_\varepsilon(\mathbf{y};\mathrm{x}')| \le C\varepsilon\sum_{k\neq j}\left[\frac{\mathbb{1}_{B^c_{e^{-2/\varepsilon}}(\mathbf{x}_k)}(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_k|^{|\alpha|}} + \frac{\mathbb{1}_{B^c_{e^{-2/\varepsilon}}(\mathbf{x}_k)}(\mathbf{y})}{|\mathbf{y}-\mathbf{x}_k|^{|\alpha|}}\right] \tag{B.3}$$

together with the fact that

$$|\mathbf{x}-\mathbf{y}| \ge \sin(\theta/2)\max\{|\mathbf{x}-\mathbf{x}_k|, |\mathbf{y}-\mathbf{x}_k|\}, \tag{B.4}$$

where θ is the angle between the vectors $\mathbf{y}-\mathbf{x}_k$ and $\mathbf{x}-\mathbf{x}_k$. Note that the bound in (B.3) does not capture the continuity of $D^\alpha\chi^*_\varepsilon$ and hence cannot be sufficiently accurate for our purposes when $|\mathbf{x}-\mathbf{y}|$ is small.

We are now ready to start estimating the H^s-seminorm of χ_ε^*. Using the same notation as in the $d=1$ case

$$\begin{aligned}\|\chi_\varepsilon^*\|^2_{\dot H^s_{\mathbf{x}_j}(\Omega)} &= \iint_{\Omega\times\Omega} \frac{|D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y}\\ &\leq 2\iint_{U\times\Omega} \frac{|D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y},\end{aligned}$$

where we used that $|D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{x};\mathrm{x}')| = 0$ for $\mathbf{x}\in U_1\cup U_2$, since $|\alpha|\geq 1$.

To bound the integral we use the estimates derived earlier. Recalling that in the case under consideration $|\alpha| = \frac{d-1}{2}$ the derived bounds tells us that

$$\begin{aligned}&\iint_{U\times\Omega} \frac{|D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{x};\mathrm{x}') - D^\alpha_{\mathbf{x}_j}\chi_\varepsilon^*(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y}\\ &\leq C\varepsilon^2\sum_{k\neq j}\iint_{U\times\Omega} \frac{\min\{\frac{g(\theta_k)^2}{|\mathbf{x}-\mathbf{x}_k|^{d-1}|\mathbf{y}-\mathbf{x}_k|^2}, \frac{g(\theta_k)^2}{|\mathbf{x}-\mathbf{x}_k|^2|\mathbf{y}-\mathbf{x}_k|^{d-1}}, \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{|\mathbf{x}-\mathbf{x}_k|^{d-1}} + \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{|\mathbf{y}-\mathbf{x}_k|^{d-1}}\}}{|\mathbf{x}-\mathbf{y}|^{d-1}} d\mathbf{x}d\mathbf{y},\end{aligned}$$

here θ_k denotes the angle between the vectors $\mathbf{x}-\mathbf{x}_k$ and $\mathbf{y}-\mathbf{x}_k$. For each fixed $\mathbf{x}$ we rewrite the integral over Ω in spherical coordinates around $\mathbf{x}_k$, oriented so that $\mathbf{x}$ is located at the south pole. With $R=|\mathbf{x}-\mathbf{x}_k|, r=|\mathbf{y}-\mathbf{x}_k|$ and θ_k as before, the integral becomes

$$\begin{aligned}&\iint_{U\times\Omega} \frac{\min\{\frac{g(\theta_k)^2}{R^{d-1}r^2}, \frac{g(\theta_k)^2}{R^2r^{d-1}}, \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{R^{d-1}} + \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{r^{d-1}}\}}{|\mathbf{x}-\mathbf{y}|^{d-1}} d\mathbf{x}d\mathbf{y}\\ &\leq \int_U\int_0^\infty\int_0^\pi\int_{\mathbb{S}^{d-2}} \frac{\min\{\frac{g(\theta_k)^2}{R^{d-1}r^2}, \frac{g(\theta_k)^2}{R^2r^{d-1}}, \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{R^{d-1}} + \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{r^{d-1}}\}}{((R-r\cos\theta_k)^2 + r^2\sin^2\theta_k|\hat\theta|^2)^{(d-1)/2}} r^{d-1}\sin^{d-2}\theta_k dr d\theta_k dS(\hat\theta)d\mathbf{x}\\ &= C\int_U\int_0^\infty\int_0^\pi \frac{\min\{\frac{g(\theta_k)^2}{R^{d-1}r^2}, \frac{g(\theta_k)^2}{R^2r^{d-1}}, \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{R^{d-1}} + \frac{|\mathbf{x}-\mathbf{y}|^{-2}}{r^{d-1}}\}}{((R-r\cos\theta_k)^2 + r^2\sin^2\theta_k)^{(d-1)/2}} r^{d-1}\sin^{d-2}\theta_k dr d\theta_k d\mathbf{x}.\end{aligned}$$

For $\theta\in[\pi/2,\pi]$ we use the bounds in (B.3), (B.4):

$$\begin{aligned}&\int_U\int_0^\infty\int_{\pi/2}^\pi \frac{R^{-d+1}+r^{-d+1}}{\sin^{d+1}(\theta_k/2)\max\{R,r\}^{d+1}} r^{d-1}\sin^{d-2}\theta_k dr d\theta_k d\mathbf{x}\\ &\qquad= \int_U\int_R^\infty\int_{\pi/2}^\pi \frac{R^{-d+1}+r^{-d+1}}{\sin^{d+1}(\theta_k/2)r^2}\sin^{d-2}(\theta_k)dr d\theta_k d\mathbf{x}\\ &\qquad\quad+ \int_U\int_0^R\int_{\pi/2}^\pi \frac{R^{-d+1}+r^{-d+1}}{\sin^{d+1}(\theta_k/2)R^{d+1}} r^{d-1}\sin^{d-2}(\theta_k)dr d\theta_k d\mathbf{x}\\ &\qquad\leq C\int_U\int_R^\infty \frac{R^{-d+1}+r^{-d+1}}{r^2}dr d\mathbf{x} + C\int_U\int_0^R \frac{R^{-d+1}+r^{-d+1}}{R^{d+1}} r^{d-1}dr d\mathbf{x}\\ &\qquad= C\int_U R^{-d}d\mathbf{x}\\ &\qquad\leq C\int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} R^{-1}dR = C\varepsilon^{-1}.\end{aligned}$$

Thus this part of the integral is $O(\varepsilon^{-1})$.

What remains is to bound the integral when $r \geq 0$ and $\theta_k \in [0, \pi/2)$. To accomplish this we shall use the bound for the difference of the derivatives derived earlier. Note that since $\theta_k < \pi/2$ we can replace the factor $g(\theta_k)$ by a constant without any loss. Using that $|\mathbf{x}-\mathbf{y}|^2 = R^2 + r^2 - 2rR\cos\theta \geq \max\{(R-r)^2, 2rR(1-\cos\theta_k)\}$ we for any fixed $\mu \in (0,1)$ find

$$\begin{aligned}
&\int_U \int_0^\infty \int_0^{\pi/2} \frac{\min\{\frac{1}{R^{d-1}r^2}, \frac{1}{R^2 r^{d-1}}\}}{((R - r\cos\theta_k)^2 + r^2\sin^2\theta_k)^{(d-1)/2}} r^{d-1}\sin^{d-2}\theta_k dr d\theta_k d\mathbf{x} \\
&\quad= \int_U \int_0^R \int_0^{\pi/2} ((R - r\cos\theta_k)^2 + r^2\sin^2\theta_k)^{-(d-1)/2} R^{-d+1} r^{d-3}\sin^{d-2}\theta_k dr d\theta_k d\mathbf{x} \\
&\quad\quad+ \int_U \int_R^\infty \int_0^{\pi/2} ((R - r\cos\theta_k)^2 + r^2\sin^2\theta_k)^{-(d-1)/2} R^{-2}\sin^{d-2}\theta_k dr d\theta_k d\mathbf{x} \\
&\quad\leq \int_U \int_0^R (R-r)^{-\mu} r^{(d-5+\mu)/2} R^{-(3d-3-\mu)/2} dr d\mathbf{x} \int_0^{\pi/2} \frac{\sin^{d-2}\theta_k}{(1-\cos\theta_k)^{(d-1-\mu)/2}} d\theta_k \\
&\quad\quad+ \int_U \int_R^\infty (r-R)^{-\mu} r^{-(d-1-\mu)/2} R^{-(d-3-\mu)/2} dr d\mathbf{x} \int_0^{\pi/2} \frac{\sin^{d-2}\theta_k}{(1-\cos\theta_k)^{(d-1-\mu)/2}} d\theta_k \\
&\quad\leq C \int_U (R^{-d} + R^{-d+3}) d\mathbf{x} \\
&\quad\leq C \int_{e^{-2/\varepsilon}}^{e^{-1/\varepsilon}} (R^{-1} + R^2) dR = C\varepsilon^{-1}.
\end{aligned}$$

Consequently, also this part of the integral is $O(\varepsilon^{-1})$ which completes the proof. ☐

Lemma B.2. *For all $s > 0$ it holds that $\mathcal{H}_W^{s,N}(\mathbb{R}^d) \subseteq \mathcal{H}_0^{s,N}(\mathbb{R}^d)$.*

Proof. Take $\Psi \in H^s(\mathbb{R}^{dN})$ s.t. $\int W_s|\Psi|^2 < \infty$ and let $\Psi_\varepsilon := \chi_\varepsilon \Psi$. Since Ψ_ε is supported away from $\mathbb{A}_\varepsilon := \mathbb{A} + B_\varepsilon(0)$ and thus may be approximated in $C_c^\infty(\mathbb{R}^{dN}\setminus\mathbb{A})$, it is sufficient to prove that $\|\Psi - \Psi_\varepsilon\|_{H^s(\mathbb{R}^{dN})} \to 0$ to conclude the lemma. We have by dominated convergence

$$\|\Psi - \Psi_\varepsilon\|^2_{L^2(\mathbb{R}^{dN})} \lesssim \int_{\mathbb{A}_\varepsilon \cap \mathbb{R}^{dN}} |1 - \chi_\varepsilon|^2 |\Psi|^2 \to 0,$$

while for $\alpha \neq 0$

$$D^\alpha_{\mathbf{x}_j}((1-\chi_\varepsilon)\Psi) = \sum_{0 \leq \beta \leq \alpha} D^\beta_{\mathbf{x}_j}(1-\chi_\varepsilon) D^{\alpha-\beta}_{\mathbf{x}_j}\Psi,$$

so for $s = m + \sigma$, $|\alpha| = m$, $0 \leq \sigma < 1$ (for $\sigma = 0$ we replace by L^2)

$$\begin{aligned}
\|\Psi - \Psi_\varepsilon\|_{\dot{H}^{s,N}(\mathbb{R}^d)} &\lesssim \sum_{j,\alpha} \|(1-\chi_\varepsilon) D^\alpha_{\mathbf{x}_j}\Psi\|_{\dot{H}^{\sigma,N}(\mathbb{R}^d)} + \sum_{j,\alpha} \sum_{0<\beta<\alpha} \|(D^\beta_{\mathbf{x}_j}\chi_\varepsilon)(D^{\alpha-\beta}_{\mathbf{x}_j}\Psi)\|_{\dot{H}^{\sigma,N}(\mathbb{R}^d)} \\
&\quad+ \sum_{j,\alpha} \|(D^\alpha_{\mathbf{x}_j}\chi_\varepsilon)\Psi\|_{\dot{H}^{\sigma,N}(\mathbb{R}^d)}.
\end{aligned}$$

We may estimate as in the proof of Lemma B.1,

$$\|(1-\chi_\varepsilon) D^\alpha \Psi\|^2_{\dot{H}^{\sigma,N}(\mathbb{R}^d)} \lesssim \sum_j \sum_{k \neq j} \int_{\mathbb{R}^{d(N-1)}} \|\Psi\|^2_{H^s_{\mathbf{x}_j}(B_\varepsilon(\mathbf{x}_k))} \to 0,$$

$$\begin{aligned}\|\Psi D^\alpha\chi_\varepsilon\|^2_{\dot H^\sigma_{\mathbf{x}_j}(\mathbb{R}^d)} &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|\Psi D^\alpha\chi_\varepsilon(\mathbf{x};\mathrm{x}')-\Psi D^\alpha\chi_\varepsilon(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}}\,d\mathbf{x}d\mathbf{y}\\ &\lesssim I_\alpha+\varepsilon^{-2|\alpha|-2}\sum_{k,l\neq j}\iint_{B_{2\varepsilon}(\mathbf{x}_k)\times B_{2\varepsilon}(\mathbf{x}_l)}|\Psi(\mathbf{x};\mathrm{x}')|^2|\mathbf{x}-\mathbf{y}|^{-d-2\sigma+2}d\mathbf{x}d\mathbf{y}\\ &\lesssim I_\alpha+\varepsilon^{-2|\alpha|-2}\sum_{k\neq j}\int_{B_{2\varepsilon}(\mathbf{x}_k)}|\Psi(\mathbf{x};\mathrm{x}')|^2\int_{B_{4\varepsilon}(\mathbf{x})}|\mathbf{x}-\mathbf{y}|^{-d-2\sigma+2}d\mathbf{y}\,d\mathbf{x}\\ &\lesssim I_\alpha+\varepsilon^{-2|\alpha|-2\sigma}\sum_{k\neq j}\int_{B_{2\varepsilon}(\mathbf{x}_k)}|\Psi(\mathbf{x};\mathrm{x}')|^2\,d\mathbf{x},\end{aligned}$$

where

$$I_\alpha=\sum_{k\neq j}\int_{B_{2\varepsilon}(\mathbf{x}_k)}|D^\alpha\chi_\varepsilon(\mathbf{x};\mathrm{x}')|^2\int_{B_{4\varepsilon}(\mathbf{x}_k)}\frac{|\Psi(\mathbf{x};\mathrm{x}')-\Psi(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}}d\mathbf{y}\,d\mathbf{x}.$$

For the highest-order derivatives $2|\alpha|=2s-2\sigma$:

$$\|\Psi D^\alpha\chi_\varepsilon\|^2_{\dot H^{\sigma,N}(\Omega)}\lesssim\int_{\Omega^{N-1}}I_\alpha+\varepsilon^{-2s}\int_{\triangle_{2\varepsilon}}|\Psi(\mathrm{x})|^2\,d\mathrm{x}\lesssim\int_{\Omega^{N-1}}I_\alpha+\int_{\triangle_{2\varepsilon}}W_s(\mathrm{x})|\Psi(\mathrm{x})|^2\,d\mathrm{x},$$

where the last term tends to zero as $\varepsilon\to0$ by dominated convergence.

For I_α we have that

$$\begin{aligned}I_\alpha&=\sum_{k\neq j}\int_{B_{2\varepsilon}(\mathbf{x}_k)}|D^\alpha\chi_\varepsilon(\mathbf{x};\mathrm{x}')|^2\int_{B_{4\varepsilon}(\mathbf{x}_k)}\frac{|\Psi(\mathbf{x};\mathrm{x}')-\Psi(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}}d\mathbf{y}\,d\mathbf{x}\\ &\lesssim\varepsilon^{-2|\alpha|}\sum_{k\neq j}\iint_{B_{4\varepsilon}(\mathbf{x}_k)\times B_{4\varepsilon}(\mathbf{x}_k)}\frac{|\Psi(\mathbf{x};\mathrm{x}')-\Psi(\mathbf{y};\mathrm{x}')|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\sigma}}d\mathbf{y}\,d\mathbf{x}\\ &\lesssim\varepsilon^{-2|\alpha|}\sum_{k\neq j}\|\Psi\|^2_{\dot H^\sigma_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}.\end{aligned}$$

By interpolation of Sobolev spaces and scaling we have for $C=C(d,\sigma,m)>0$

$$\|\Psi\|^2_{\dot H^\sigma_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\le C\varepsilon^{2m}\big(\|\Psi\|^2_{\dot H^s_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}+\|\Psi\sqrt{W_s}\|^2_{L^2_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\big),$$

and thus by dominated convergence

$$\int_{\mathbb{R}^{d(N-1)}}I_\alpha\lesssim\sum_{k\neq j}\int_{\mathbb{R}^{d(N-1)}}\big(\|\Psi\|^2_{\dot H^s_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}+\|\Psi\sqrt{W_s}\|^2_{L^2_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\big)\to0,$$

for $|\alpha|=s-\sigma$. Similarly, for the lower-order mixed terms

$$\begin{aligned}\|(D^{\alpha-\beta}_{\mathbf{x}_j}\Psi)(D^\beta_{\mathbf{x}_j}\chi_\varepsilon)\|_{\dot H^\sigma_{\mathbf{x}_j}(\mathbb{R}^d)}&\lesssim\varepsilon^{-2|\beta|}\sum_{k\neq j}\|D^{\alpha-\beta}_{\mathbf{x}_j}\Psi\|^2_{\dot H^\sigma_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\\ &\quad+\varepsilon^{-2|\beta|-2\sigma}\sum_{k\neq j}\|D^{\alpha-\beta}_{\mathbf{x}_j}\Psi\|^2_{L^2_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\\ &\lesssim\sum_{k\neq j}\big(\|\Psi\|^2_{\dot H^s_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}+\|\Psi\sqrt{W_s}\|^2_{L^2_{\mathbf{x}_j}(B_{4\varepsilon}(\mathbf{x}_k))}\big),\end{aligned}$$

which implies that also $\|(D^{\alpha-\beta}\Psi)(D^\beta\chi_\varepsilon)\|_{\dot H^{\sigma,N}(\mathbb{R}^d)}\to0$. $\square$

Lemma B.3. *For all $0 < 2s \le d$ it holds that $\mathcal{H}_0^{s,N}(\mathbb{R}^d) = \mathcal{H}^{s,N}(\mathbb{R}^d) = H^s(\mathbb{R}^{dN})$.*

Proof. As mentioned above, combining Lemma B.2 with the Hardy–Rellich inequality implies the claim when $0 < 2s < d$. For $2s = d$ we argue as follows.

It suffices to prove that $C_c^\infty(\mathbb{R}^{dN} \setminus \triangle)$ is dense in H^s, and moreover, using that $C_c^\infty(\mathbb{R}^{dN})$ is dense in H^s, it suffices to prove that if $\Psi \in C_c^\infty(\mathbb{R}^{dN})$ then $\Psi_\varepsilon := \chi_\varepsilon^* \Psi \to \Psi$ in H^s as $\varepsilon \to 0$. Clearly

$$\|\Psi - \Psi_\varepsilon\|^2_{L^2(\mathbb{R}^{dN})} \lesssim \int_{\triangle_\varepsilon \cap \operatorname{supp}\Psi} |1 - \chi_\varepsilon^*|^2 \to 0.$$

Moreover, by Lemma B.1 and arguing as in the proof of Lemma B.2

$$\|\Psi - \Psi_\varepsilon\|^2_{\dot{H}^{s,N}(\mathbb{R}^d)} \lesssim \varepsilon \to 0.$$

□

The above generalizes the case $d = 2$ and $s = 1$ where it is well known that hard-core bosons have non-extensive energy in the dilute limit [15] and thus that a Lieb–Thirring inequality of the type (2.1) cannot hold. See also [26] for generalizations with integer s.

References

[1] F. W. Carroll, *A polynomial in each variable separately is a polynomial*, Amer. Math. Monthly. **68** (1961), 42.

[2] I. Daubechies, *An uncertainty principle for fermions with generalized kinetic energy*, Commun. Math. Phys., 90 (1983), 511–520.

[3] F. J. Dyson, *Ground-state energy of a finite system of charged particles*, J. Math. Phys. **8** (1967), 1538–1545.

[4] F. J. Dyson and A. Lenard, *Stability of matter. I*, J. Math. Phys. **8** (1967), no. 3, 423–434, `doi`.

[5] R. L. Frank, D. Hundertmark, M. Jex, and P. T. Nam, *The Lieb–Thirring inequality revisited*, Preprint 2018, `arXiv:1808.09017`

[6] R. L. Frank and R. Seiringer, *Lieb–Thirring inequality for a model of particles with point interactions*, J. Math. Phys. **53** (2012), no. 9, 095201, 11, `doi`.

[7] M. Girardeau, *Relationship between systems of impenetrable bosons and fermions in one dimension*, J. Mathematical Phys. **1** (1960), 516–523.

[8] R. Hurri-Syrjänen and A. V. Vähäkangas, *On fractional Poincaré inequalities*, J. Anal. Math. **120** (2013), 85–104.

[9] A. Laptev, *Spectral inequalities for Partial Differential Equations and their applications*, Proceedings of ICCM2010 in Beijing, AMS/IP Studies in Advanced Mathematics 51 (2012), 629-643.

[10] S. Larson and D. Lundholm, *Exclusion bounds for extended anyons*, Arch. Ration. Mech. Anal. **227** (2018) 309–365, `doi`.

[11] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[12] E. H. Lieb and R. Seiringer, *The Stability of Matter in Quantum Mechanics*, Cambridge Univ. Press, 2010.

[13] E. H. Lieb and W. E. Thirring, *Bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett. **35** (1975), 687–689, `doi`.

[14] E. H. Lieb and W. E. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, pp. 269–303, Princeton Univ. Press, 1976.

[15] E. H. Lieb and J. Yngvason, *The ground state energy of a dilute two-dimensional Bose gas*, J. Statist. Phys. **103** (2001), no. 3-4, 509–526, `doi`.

[16] D. Lundholm, *Methods of modern mathematical physics: Uncertainty and exclusion principles in quantum mechanics*, Lecture notes, KTH, 2018, `arXiv:1805.03063`.

[17] D. Lundholm, P. T. Nam, and F. Portmann, *Fractional Hardy-Lieb-Thirring and related inequalities for interacting systems*, Arch. Ration. Mech. Anal. **219** (2016), no. 3, 1343–1382, `doi`.

[18] D. Lundholm, F. Portmann, and J. P. Solovej, *Lieb–Thirring bounds for interacting Bose gases*, Comm. Math. Phys. **335** (2015), no. 2, 1019–1056, `doi`.

[19] D. Lundholm and R. Seiringer, *Fermionic behavior of ideal anyons*, Lett. Math. Phys. **108** (2018) 2523–2541, `doi`.

[20] D. Lundholm and J. P. Solovej, *Hardy and Lieb–Thirring inequalities for anyons*, Comm. Math. Phys. **322** (2013), 883–908, `doi`.

[21] D. Lundholm and J. P. Solovej, *Local exclusion principle for identical particles obeying intermediate and fractional statistics*, Phys. Rev. A **88** (2013), 062106, `doi`.

[22] D. Lundholm and J. P. Solovej, *Local exclusion and Lieb–Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré **15** (2014), 1061–1107, `doi`.

[23] P. T. Nam, *Lieb–Thirring inequality with semiclassical constant and gradient error term*, J. Funct. Anal. **274** (2018), 1739-1746.

[24] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[25] M. Rumin, *Balanced distribution-energy inequalities and related entropy bounds* , Duke Math. J. **160** (2011), 567–597.

[26] E. C. Svendsen, *The effect of submanifolds upon essential self-adjointness and deficiency indices* , J. Math. Anal. Appl. **80** (1981), 551–565.

[27] D. Yafaev, *Sharp Constants in the Hardy–Rellich Inequalities*, J. Funct. Anal. **168** (1999), 121–144.

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