The Asymptotical Equipartition Property of Supremus Typicality in the Weak Sense

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Outline

A Starting Example

Let $\{\alpha, \beta, \gamma\}$ be the state space of the Markov (i.i.d.) process with *transition matrix*

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}. \tag{1}$$

For

$$\mathbf{x} = (\alpha, \beta, \gamma, \alpha, \beta, \gamma, \alpha, \beta, \gamma), \tag{2}$$

it is easy to verify that $\mathbf{x}$ is a strongly Markov 5/12-typical sequence.

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it is easy to verify that $\mathbf{x}$ is a strongly Markov 5/12-typical sequence. However, the subsequence

$$\mathbf{x}_{\{\alpha,\gamma\}} = (\alpha, \gamma, \alpha, \gamma, \alpha, \gamma) \tag{3}$$

is no long a strongly Markov 5/12-typical sequence, because the *stochastic complement* [Mey89] $\mathbf{S}_{\{\alpha,\gamma\}} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and

$$\left| \frac{\text{the number of subsequence } (\alpha, \alpha)\text{'s in } \mathbf{x}_{\{\alpha,\gamma\}}}{6} - 0.5 \right| = |0 - 0.5| > \frac{5}{12}. \tag{4}$$

A.M.S. Random Processes (I)

Given a probability space $(\Omega, \mathscr{F}, \mu)$ and a measurable transformation $T : \Omega \to \Omega$ (not necessarily probability preserving), the tuple $(\Omega, \mathscr{F}, \mu, T)$ is called a *dynamical system* (or *ergodic system*).

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Let $X : \Omega \to \mathscr{X}$ (e.g. $\mathscr{X}$ is a finite set) be a measurable function. Then

$$\{X^{(n)}\} = \{X(T^n)\} \tag{5}$$

defines a *random process* with state space $\mathscr{X}$ and pdf/pmf

$$p\left(x^{(0)}, x^{(1)}, \cdots, x^{(n-1)}\right) = \mu\left(\bigcap_{i=0}^{n-1} T^{-i}\left(X^{-1}\left(x^{(i)}\right)\right)\right). \tag{6}$$

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Example 1

Given a random process $\{X^{(n)}\}$ with sample space. Let $\Omega = \prod_{i=-\infty}^{\infty} \mathscr{X}$, T be a time shift and X be the coordinate function

$$X : (\cdots, x^{(-1)}, x^{(0)}, x^{(1)}, \cdots) \mapsto x^{(0)}. \tag{7}$$

Define μ satisfying $\mu\left(\bigcap_{i=0}^{n-1} T^{-i}\left(X^{-1}\left(x^{(i)}\right)\right)\right) = p\left(x^{(0)}, x^{(1)}, \cdots, x^{(n-1)}\right)$.
By the Kolmogorov Extension Theorem, $\{X^{(n)}\} = \{X(T^n)\}$.

A.M.S. Random Processes (II)

$(\Omega, \mathscr{F}, \mu, T)$ is said to be *asymptotically mean stationary* (*a.m.s.*) [1] [GK80] if there exists a measure $\overline{\mu}$ on $(\Omega, \mathscr{F})$ satisfying

$$\overline{\mu}(B) = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m} \mu(T^{-i}B), \forall\, B \in \mathscr{F}. \tag{8}$$

[1] The a.m.s. condition is interesting because it is a sufficient and necessary condition for the Point-wise Ergodic Theorem to hold [GK80, Theorem 1].

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Obviously, if $(\Omega, \mathscr{F}, \mu, T)$ is stationary, i.e. $\mu(B) = \mu(T^{-1}B)$, then it is a.m.s.. In addition, $(\Omega, \mathscr{F}, \mu, T)$ is said to be ergodic if

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } \mu(B) = 1. \tag{9}$$

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The random process $\{X^{(n)}\} = \{X(T^n)\}$ is said to be *a.m.s.* (stationary/ergodic) if $(\Omega, \mathscr{F}, \mu, T)$ is a.m.s. (stationary/ergodic).

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Induced Transformations and Reduced Processes (I)

Definition 2

A dynamical system $(\Omega, \mathscr{F}, \mu, T)$ is said to be *recurrent* (*conservative*) if $\mu\left(B - \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} T^{-j}B\right) = 0, \forall\, B \in \mathscr{F}$.

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Given a recurrent system $(\Omega, \mathscr{F}, \mu, T)$ and $A \in \mathscr{F}$ ($\mu(A) > 0$), one can define a new transformation T_A on $(A_0, \mathscr{A}, \mu|_{\mathscr{A}})$, where $A_0 = A \cap \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} T^{-j}A$ and $\mathscr{A} = \{A_0 \cap B | B \in \mathscr{F}\}$, such that

$$T_A(x) = T^{\psi_A^{(1)}(x)}(x), \forall\, x \in A_0, \tag{10}$$

where

$$\psi_A^{(1)}(x) = \min\left\{i \in \mathbb{N}^+ | T^i(x) \in A_0\right\} \tag{11}$$

is the first return time function.

1. $(A_0, \mathscr{A}, \mu|_{\mathscr{A}}, T_A)$ forms a new dynamical system;
2. T_A is called an *induced transformation* of $(\Omega, \mathscr{F}, \mu, T)$ with respect to A [Kak43].

Induced Transformations and Reduced Processes (II)

Let $\{X^{(n)}\}$ be a random process with state space $\mathscr{X}$. A reduced process $\left\{X_{\mathscr{Y}}^{(k)}\right\}$ of $\{X^{(n)}\}$ with sub-state space $\mathscr{Y} \subseteq \mathscr{X}$ is defined to be $\left\{X_{\mathscr{Y}}^{(k)}\right\} = \left\{X^{(n_k)}\right\}$, where

$$n_k = \begin{cases} \min\{n \geq 0 | X^{(n)} \in \mathscr{Y}\}; & k = 0, \\ \min\{n > n_{k-1} | X^{(n)} \in \mathscr{Y}\}; & k > 0. \end{cases}$$

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Assume that $\{X^{(n)}\} = \{X(T^n)\}$ defined by $(\Omega, \mathscr{F}, \mu, T)$ and the measurable function $X : \Omega \to \mathscr{X}$, and let $A = X^{-1}(\mathscr{Y})$. It is easily seen that $\left\{X_{\mathscr{Y}}^{(k)}\right\}$ is essentially the random process $\left\{X\left(T_A^k\right)\right\}$ defined by the system

$$\left(A_0, A_0 \cap \mathscr{F}, \frac{1}{\mu(A)} \mu|_{A_0 \cap \mathscr{F}}, T_A\right) \tag{12}$$

and X.

Supremus Typicality in the Weak Sense

Let $\mathbf{x}_{\mathscr{Y}}$ be the subsequence of $\mathbf{x} = \left[x^{(1)}, x^{(2)}, \cdots, x^{(n)}\right] \in \mathscr{X}^n$ formed by all those $x^{(l)}$'s that belong to $\mathscr{Y} \subseteq \mathscr{X}$ in the original ordering. $\mathbf{x}_{\mathscr{Y}}$ is called a *reduced subsequence* of $\mathbf{x}$ with respect to $\mathscr{Y}$.

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Definition 3 (Supremus Typicality in the Weak Sense [HS14])

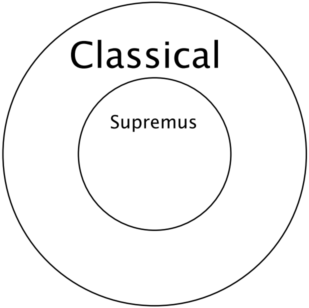
Let $\{X^{(n)}\}$ be a recurrent a.m.s. ergodic process with state space $\mathscr{X}$. A sequence $\mathbf{x} \in \mathscr{X}^n$ is said to be *Supremus ϵ-typical* with respect to $\{X^{(n)}\}$ for some $\epsilon > 0$, if

$$|\mathbf{x}_{\mathscr{Y}}| (H_{\mathscr{Y}} - \epsilon) < -\log p_{\mathscr{Y}}(\mathbf{x}_{\mathscr{Y}}) < |\mathbf{x}_{\mathscr{Y}}| (H_{\mathscr{Y}} + \epsilon), \forall\, \emptyset \neq \mathscr{Y} \subseteq \mathscr{X}, \quad (13)$$

where $p_{\mathscr{Y}}$ and $H_{\mathscr{Y}}$ are the joint distribution and entropy rate of the reduced process $\left\{X_{\mathscr{Y}}^{(k)}\right\}$ of $\{X^{(n)}\}$ with sub-state space $\mathscr{Y}$, respectively.

Asymptotical Equipartition Property

Designate $\mathcal{S}_\epsilon(n, \{X^{(n)}\})$ as the set of all Supremus ϵ-typical sequences with respect to $\{X^{(n)}\}$ in $\mathscr{X}^n$. Obviously, $\mathcal{S}_\epsilon(n, \{X^{(n)}\})$ is a subset of all classical ϵ-typical sequences [SW49].



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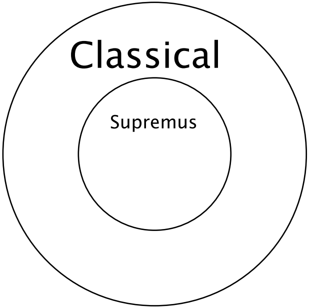

Theorem 4 (AEP of Weak Supremus Typicality [HS14])

In Definition 3, $\forall\ \eta > 0$, there exists some positive integer N_0, such that

$$\Pr\left\{\left[X^{(1)}, X^{(2)}, \cdots, X^{(n)}\right] \notin \mathcal{S}_\epsilon(n, \{X^{(n)}\})\right\} < \eta,$$

for all $n > N_0$.

Supporting Results

Theorem 5 ([HS13b])

If $(\Omega, \mathscr{F}, \mu, T)$ is recurrent a.m.s., then $(A_0, \mathscr{A}, \mu|_{\mathscr{A}}, T_A)$ is a.m.s. for all $A \in \mathscr{F}$ $(\mu(A) > 0)$.

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Theorem 6 (Shannon–McMillan–Breiman–Gray Theorem [GK80])

If

$$\{X^{(n)}\} = \{X(T^n)\}$$

is a.m.s. and ergodic, then the Shannon–McMillan–Breiman Theorem holds. In exact terms,

$$-\frac{1}{n}\log p(X^{(0)}, X^{(1)}, \cdots, X^{(n-1)}) \to H \text{ with probability } 1,$$

where H is the entropy rate of $\{X^{(n)}\}$.

The Proof (I)

Let $\mathbf{X} = \left[X^{(1)}, X^{(2)}, \cdots, X^{(n)}\right]$. Then

$$\left\{\mathbf{X} \notin \mathcal{S}_{\epsilon}(n, \{X^{(n)}\})\right\} = \bigcup_{\emptyset \neq \mathscr{Y} \subseteq \mathscr{X}} \left\{\mathbf{X}_{\mathscr{Y}} \notin \mathcal{T}_{\epsilon}(n, \{X_{\mathscr{Y}}^{(k)}\})\right\}. \tag{14}$$

Assume that $(\Omega, \mathscr{F}, \mu, T)$ and X are the recurrent a.m.s. ergodic system and the measurable function define $\{X^{(n)}\}$, i.e. $\{X^{(n)}\} = \{X(T^n)\}$. For any non-empty $\mathscr{Y} \subseteq \mathscr{X}$, we have that $\left\{X_{\mathscr{Y}}^{(k)}\right\} = \left\{X\left(T_A^k\right)\right\}$, where $A = X^{-1}(\mathscr{Y})$ and T_A is an induced transformation of $(\Omega, \mathscr{F}, \mu, T)$ with respect to A. Furthermore, Theorem 5 and [Aar97, Proposition 1.5.2] guarantee that

$$\left(A_0, A_0 \cap \mathscr{F}, \frac{1}{\mu(A)} \mu|_{A_0 \cap \mathscr{F}}, T_A\right), \tag{15}$$

is a.m.s. ergodic.

The Proof (II)

Consequently, the Shannon–McMillan–Breiman–Gray Theorem (Theorem 6) says

$$-\frac{1}{n}\log p_{\mathscr{Y}}\left(X_{\mathscr{Y}}^{(0)}, X_{\mathscr{Y}}^{(1)}, \cdots, X_{\mathscr{Y}}^{(n-1)}\right) \to H_{\mathscr{Y}}, \text{ with probability 1.}$$

This implies that there exists a positive integer $N_{\mathscr{Y}}$ such that

$$\Pr\left\{\mathbf{X}_{\mathscr{Y}} \notin \mathcal{T}_{\epsilon}(n, \{X_{\mathscr{Y}}^{(k)}\})\right\} < \frac{\eta}{2^{|\mathscr{X}|} - 1}, \forall\ n > N_{\mathscr{Y}}.$$

Let $N_0 = \max_{\emptyset \neq \mathscr{Y} \subseteq \mathscr{X}} N_{\mathscr{Y}}$. One easily concludes that

$$\Pr\left\{\mathbf{X} \notin \mathcal{S}_{\epsilon}(n, \{X^{(n)}\})\right\} < \eta, \forall\ n > N_0.$$

The statement is proved.

Thanks

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Thanks!

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