Supremus Typicality

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Outline

An Observation from the Classic Typicality (I)

Classic Asymptotically Equipartition Property (AEP): Given a randomly generated (w.r.t. some stationary ergodic process of sample space $\mathscr{X}$, say $\{0, 1, 2\}$) sequence X^n, say

00102 - 02210 - 02112 - 01022 - 01102,

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in probability ϵ close to 1 that X^n is classic ϵ-typical (in the strong or weak sense) for large enough n.

Let $\emptyset \neq A \subseteq \mathscr{X}$. Define $Y_A^{(l)} = X^{(T_{A,l})}$ where

$$T_{A,l} = \begin{cases} \inf\left\{j > 0 | X^{(j)} \in A\right\}; & l = 1, \\ \inf\left\{j > T_{A,l-1} | X^{(j)} \in A\right\}; & l > 1, \\ \sup\left\{j < T_{A,l+1} | X^{(j)} \in A\right\}; & l < 1. \end{cases}$$

Let $k = \max\{j | X^{(j)} \in A\}$. The property that the reduced sequence Y_A^k of X^n, say

$$0010 \text{ - } 010 \text{ - } 011 \text{ - } 010 \text{ - } 0110 \text{ when } A = \{0, 1\},$$

is also typical in probability close to 1 is important. Can one make such a claim in general for all non-empty subset A of $\mathscr{X}$?

An Observation from the Classic Typicality (II)

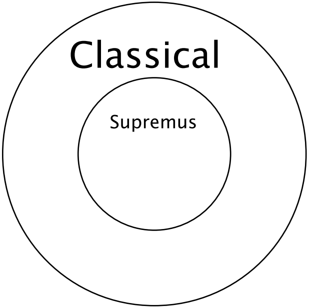
Let $\{\alpha, \beta, \gamma\}$ be the state space of the Markov (i.i.d.) process with transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}. \tag{1}$$

For

$$\mathbf{x} = (\alpha, \beta, \gamma, \alpha, \beta, \gamma, \alpha, \beta, \gamma), \tag{2}$$

it is easy to verify that $\mathbf{x}$ is a strongly Markov 5/12-typical sequence.



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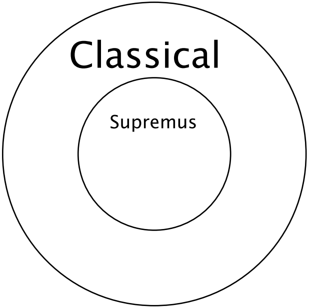

However, the subsequence

$$\mathbf{x}_{\{\alpha,\gamma\}} = (\alpha, \gamma, \alpha, \gamma, \alpha, \gamma) \qquad (3)$$

is no long a strongly Markov 5/12-typical sequence, because the stochastic complement [Mey89] $\mathbf{S}_{\{\alpha,\gamma\}} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and

$$\left| \frac{\text{the number of subsequence } (\alpha, \alpha)\text{'s in } \mathbf{x}_{\{\alpha,\gamma\}}}{6} - 0.5 \right| = |0 - 0.5| > \frac{5}{12}. \qquad (4)$$

The Technical Issues

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2. How is (the stochastic properties of) $\{Y_A^{(l)}\}$ described mathematically that allows the analysis to be carried out?
3. Is there an ergodic theorem associated with $\{Y_A^{(l)}\}$?

Supremus Typicality in the Strong Sense

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2. If $\{X^{(j)}\}$ is irreducible Markov with transition matrix $\mathbf{P}$, then $\{Y_A^{(l)}\}$ is also irreducible Markov. Moreover, the stochastic complement

$$\mathbf{S}_A = \mathbf{P}_{A,A} + \mathbf{P}_{A,A^c}\left(\mathbf{1} - \mathbf{P}_{A^c,A^c}\right)^{-1}\mathbf{P}_{A^c,A}$$

of $\mathbf{P} = \begin{bmatrix} \mathbf{P}_{A,A} & \mathbf{P}_{A,A^c} \\ \mathbf{P}_{A^c,A} & \mathbf{P}_{A^c,A^c} \end{bmatrix}$ is the transition matrix of $\{Y_A^{(l)}\}$ [Mey89].

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3. Ergodic theorem of irreducible Markov chain [Nor98, Theorem 1.10.2].

AEP of Supremus Typical in the Strong Sense: In probability ϵ close to 1, all reduced sequences of a randomly generated sequence of an irreducible Markov chain is Supremus ϵ-typical for large enough n.

Supremus Typicality in the Weak Sense: Backgrounds

1. probability space $(\Omega, \mathscr{F}, \mu)$.
2. measurable transformation $T : \Omega \to \Omega$ (not necessarily probability preserving).
3. dynamical system $(\Omega, \mathscr{F}, \mu, T)$.
4. Let $X : \Omega \to \mathscr{X}$ ($\mathscr{X}$ is always assumed to be finite from now on) be a measurable function. $\{X^{(j)}\} = \{X(T^j)\}$ defines a random process with state space $\mathscr{X}$.
5. $(\Omega, \mathscr{F}, \mu, T)$ is said to be asymptotically mean stationary (a.m.s.) [1] [GK80] if there exists a measure $\overline{\mu}$ on $(\Omega, \mathscr{F})$ satisfying
$$\overline{\mu}(B) = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m} \mu(T^{-i}B), \forall\, B \in \mathscr{F}.$$
6. $(\Omega, \mathscr{F}, \mu, T)$ is said to be ergodic if $T^{-1}B = B \implies \mu(B) = 0$ or $\mu(B) = 1$.
7. The random process $\{X^{(j)}\} = \{X(T^j)\}$ is said to be a.m.s. (ergodic) if $(\Omega, \mathscr{F}, \mu, T)$ is a.m.s. (ergodic).

[1] The a.m.s. condition is interesting because it is a sufficient and necessary condition for the Point-wise Ergodic Theorem to hold [GK80, Theorem 1].

Supremus Typicality in the Weak Sense: Induced Transformations

Given a recurrent system $(\Omega, \mathscr{F}, \mu, T)$ and $\mathcal{A} \in \mathscr{F}$ $(\mu(\mathcal{A}) > 0)$, one can define the induced transformation [Kak43] $T_{\mathcal{A}}$ on $(\mathcal{A}_0, \mathscr{A}, \mu|_{\mathscr{A}})$, where $\mathcal{A}_0 = \mathcal{A} \cap \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} T^{-j}\mathcal{A}$ and $\mathscr{A} = \{\mathcal{A}_0 \cap \mathcal{B} | \mathcal{B} \in \mathscr{F}\}$, such that

$$T_{\mathcal{A}}(x) = T^{\psi_{\mathcal{A}}^{(1)}(x)}(x), \forall\, x \in \mathcal{A}_0,$$

where

$$\psi_{\mathcal{A}}^{(1)}(x) = \min\left\{ i \in \mathbb{N}^+ \middle| T^i(x) \in \mathcal{A}_0 \right\}$$

is the first return time function.

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is the first return time function.
One can verify that $(\mathcal{A}_0, \mathscr{A}, \frac{1}{\mu(\mathcal{A}_0)}\mu|_{\mathscr{A}}, T_{\mathcal{A}})$ forms a new dynamical system. Moreover ...

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1. If $\{X^{(j)}\} = \{X(T^j)\}$ is recurrent, then $\{Y_A^{(l)}\} = \left\{X\left(T_{\mathcal{A}}^l\right)\right\}$ with $\mathcal{A} = X^{-1}(A)$, and $\{Y_A^{(l)}\}$ is recurrent.

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4. If $\{X^{(j)}\} = \{X(T^j)\}$ is a.m.s. and ergodic, then the Shannon–McMillan–Breiman (SMB) Theorem holds. In exact terms,

$$-\frac{1}{n}\log p(X^{(0)}, X^{(1)}, \cdots, X^{(n-1)}) \to H \text{ with probability } 1,$$

where H is the entropy rate of $\{X^{(l)}\}$ [GK80].

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Where is it used

For i.i.d. sources,

SW scheme [SW73]	field linear scheme [Eli55, Csi82]	non-field ring linear scheme [HS12]
	dominate SW for encoding binary sum $x \oplus y$ over $\mathbb{Z}_2$ [KM79]	dominate field linear for encoding some functions e.g. $x + 2y + 3z$ over $\mathbb{Z}_4$ [HS12]

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To generalized results in [HS12] to the non-i.i.d. scenarios, the argument based on classic typicality does not lead to a conclusion that is accessible and easy to analyse.

non-field ring linear scheme for i.i.d. sources [HS12]	non-field ring linear scheme for irreducible Markov sources [HS13]
classic typicality	Supremus typicality

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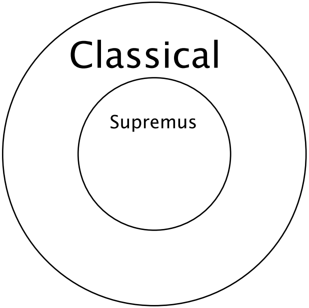

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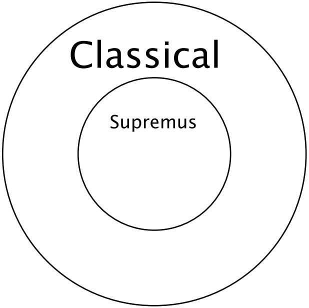

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2. Apart from the ergodic theorem, it takes the self-iterating properties of the random process into account;
3. Self-iterating properties:
    1. reduced chains of an irreducible Markov chain are irreducible Markov [Mey89];
    2. reduced processes of a recurrent (a.m.s. and ergodic, resp.) random process are also recurrent (a.m.s. and ergodic, resp.) [HS14, Aar97].

Thanks

♫ ♬

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